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Mean field and ϵ-expansion study of spin glasses*

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A Landau-Ginzburg phenomenological free energy for the Edwards and Anderson spin-glass model when there is competition between spin-glass and ferromagnetic ordering is developed. This free energy obtained with the use of the $n \to 0$ replication procedure is analyzed using mean-field theory and the ϵ expansion. Critical exponents for the ferromagnetic–spin-glass multicritical point are calculated in $6-\epsilon$ dimensions. For Ising systems, $\nu = 1/2 + (2/3)\epsilon$ and $\phi = 1 + (1/2)\epsilon$. For XY and Heisenberg systems, these exponents are complex. This result is not fully understood. The Harris-Plischke-Zuckermann model for amorphous magnetism is shown to have an Ising-like spin-glass fixed point in high enough dimension.

I. INTRODUCTION

Quenched spin systems with random exchange interactions, J, taking on both positive and negative values are believed to exhibit a low-temperature ordered phase called a spin-glass phase. This phase is characterized[1] by a vanishing average moment, $[\langle \vec{S} \rangle]_{av} = 0$ but a non-vanishing moment squared, $[\langle \vec{S} \rangle \cdot \langle \vec{S} \rangle]_{av}$. Here $\langle \vec{S} \rangle$ refers to the thermodynamic average and $[\ ]_{av}$ refers to an average over the ensemble of random exchanges. Most experimentally investigated spin glasses are dilute substitutional magnetic alloys such as CuMn or AuFe.[2,3] Here the positive and negative values of the exchange interactions are produced by the sinusoidal oscillations of the Rudderman-Kittel-Kasuya-Yoshida interaction. A long-range oscillating interaction is difficult to treat theoretically. Therefore, arguing that it is the random sign and not the long-range nature of the exchange interaction that is essential to produce a spin-glass state, Edwards and Anderson[1] introduced a model for a spin glass with short-range random exchange. They analyzed this model by a clever generalization of the replication procedure[4] in which the spin-glass order parameter $[\langle \vec{S} \rangle \cdot \langle \vec{S} \rangle]_{av}$ appeared as the average of the product of the spins in two different replicas. As usual,[4] to reproduce the quenched random averages, the number, n, of replicas is allowed to go to zero. Sherrington and Kirkpatrick[5] following previous work[6] analyzed a generalization of this model and produced an "exact solution" within the context of the replication procedure. This calculation produced a complete phase diagram with ferromagnetic, spin-glass, and paramagnetic phases. Following this, the authors and A. B. Harris[7] introduced a Landau-Ginzburg continuum formulation of the Edwards-Anderson model for the special case when the average exchange $[J]_{av}$ is zero. This permitted a calculation of critical exponents in $6-\epsilon$ dimensions. This paper is an extension of the work of Ref. 7 to include $[J]_{av} \neq 0$. This generalization produces a mean-field phase diagram similar to that found by Sherrington and Kirkpatrick[5] and permits a $(6-\epsilon)$-dimensional calculation of the critical exponents for the multicritical point where there is simultaneous spin-glass and ferromagnetic ordering.

It is now clear that the $n = 0$ replica procedure leads to many puzzling problems. First the solution which produces the ordered spin-glass state maximizes rather than minimizes the free energy in the $n = 0$ limit. Second the "exact" solution of Sherrington and Kirkpatrick[5] yields a clearly unphysical negative entropy at zero temperature. These problems are presumably associated with the interchange of the $n \to 0$ and the thermodynamic (number of sites $N \to \infty$) limits. For any finite N, the $n \to 0$ limit reproduces exactly $[\ln Z]_{av}$ as can be seen, for example, term by term in a high-temperature series. Thus, if n is allowed to go to zero before N goes to infinity, the correct answer will be obtained. If $N \to \infty$ first, then $n \to 0$, problems may arise which could presumably change the equilibrium state from a free-energy minimum to a free-energy maximum for example. Recent work on mean-field theories for the spin-glass transition which do not rely on the replication procedure[8-10] indicate that that procedure is satisfactory in the disordered phase and in the critical regime but that it breaks down at zero temperature. There is, thus, some hope that the $n = 0, \epsilon$ expansion will give the correct critical behavior. For the Ising spin glass, we find no reason to doubt this. On the other hand, for two- and three-component (XY and Heisenberg) spin systems, we find the disquieting result that the thermal critical exponent at the ferromagnetic–spin-glass multicritical point is complex. Whether this result is due to a mistreatment of systems with more than one spin component or whether it is a result of the

inadequacies of the replication procedure is unclear. Recently another model for a spin-glass transition has surfaced.[11-13] In this model, *site* randomness is described by a variable $\epsilon(\vec{x})$ at each site $\vec{x}$ that can take on values ± 1. The "spin-glass" order parameter is then $[\langle \epsilon(\vec{x})\, \vec{S}(\vec{x})\rangle]_{av}$. An unambiguous mean-field theory for this model can be developed,[12] and critical exponents become nonclassical below four dimensions.[14] It is our belief that site randomness of the above type is fundamentally different from the bond randomness discussed here.[14] We believe that the spin-glass order parameter introduced by Edwards and Anderson and used here is the correct one for the bond problem. When both bound and site randomness is present, there will be competition among ordering of $\langle \vec{S}(\vec{x})\rangle$, $[\langle \epsilon(\vec{x})\vec{S}(\vec{x})\rangle]_{av}$, and $[\langle \vec{S}(\vec{x})\rangle \cdot \langle \vec{S}(\vec{x})\rangle]_{av}$. A more complete treatment of this question is in preparation.[14]

The bond spin-glass problem has recently been treated using discrete versions of the renormalization group[15] for random systems.[16] Grinstein *et al.*[17] have treated the random one-dimensional Ising model and have found a spin-glass fixed probability distribution (i.e., a probability distribution which does not change under the renormalization group at zero temperature). Similar results have been found by Young and Stinchcombe[18] who also carry out a decimation[15(b)] renormalization calculation on a two-dimensional square lattice. They find no spin-glass fixed distribution in low dimensions but argue that the higher connectivity of the three-dimensional lattice would probably produce a spin-glass transition. Monte Carlo calculations[19] in two and three dimensions also provide strong indications that there is a spin-glass transition. The above, plus evidence from the nonreplication mean-field calculation,[8-10] indicate that the bond random system can have a spin-glass transition. It seems probable to us that the $n=0$ calculations presented here give a correct description of the critical properties of the ferromagnetic and spin-glass transitions for the Ising model, at least.

This paper is divided into five sections. Section II introduces the model. Section III treats Landau mean-field theory for the model introduced in Sec. II. Section IV calculates the exponents for the spin-glass–ferromagnetic multicritical point in $6-\epsilon$ dimensions. Section V shows that the Harris-Plishke-Zuckerman model[20] for an amorphous ferromagnet has an Ising-like spin-glass fixed point in $6-\epsilon$ dimensions.

II. DEVELOPMENT OF THE MODEL

Consider a d-dimensional lattice with lattice sites $\vec{x}$ consisting of two sublattices a and b. At each site $\vec{x}$, there is an m-component classical spin $\vec{S}(\vec{x})$. Sublattice spins $\vec{S}_a(\vec{x})$ and $\vec{S}_b(\vec{x})$ are defined as follows:

$$\vec{S}_a(\vec{x}) = \begin{cases} \vec{S}(\vec{x}) & \text{if } \vec{x} \text{ belongs to sublattice } a, \\ 0 & \text{otherwise;} \end{cases}$$
$$\vec{S}_b(\vec{x}) = \begin{cases} \vec{S}(\vec{x}) & \text{if } \vec{x} \text{ belongs to sublattice } b, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then $\vec{S}(\vec{x}) = \vec{S}_a(\vec{x}) + \vec{S}_b(\vec{x})$. The spin Hamiltonian for short-range exchange can now be expressed in terms of the sublattice spins

$$\mathcal{H} = -\sum_{\vec{x},\vec{\delta}} J(\vec{x}, \vec{x}+\vec{\delta})\, \vec{S}_a(\vec{x}) \cdot \vec{S}_b(\vec{x}+\vec{\delta}) \ . \tag{2.2}$$

$\vec{x}$ runs over all sites in the a sublattice and $\vec{\delta}$ is a nearest-neighbor vector (positive or negative) connecting the two sublattices. Thus the pair $(\vec{x}, \vec{x}+\vec{\delta})$ uniquely defines a bond and only appears once in the sum. $J(\vec{x}, \vec{x}+\vec{\delta})$ is the exchange integral associated with that bond. The bonds are random variables with a probability distribution $P(\{J\})$.

The free energy of a quenched random system is calculated by averaging the free energy over the random bond variables[21]

$$F = [F(\{J\})]_{av} \equiv \int d\{J\}\, P(\{J\})\, F(\{J\}) \ , \tag{2.3}$$

where

$$\beta F(\{J\}) = -\ln Z(\{J\}) \ , \tag{2.4}$$

where $\beta = 1/kT$ and $Z(\{J\})$ is the partition function for a particular distribution of J's. To calculate $F(\{J\})$, we evaluate the partition function of the system replicated n times[4]

$$Z^n(\{J\}) = \mathrm{Tr}_s\, e^{-\beta \mathcal{H}^{(n)}} \ , \tag{2.5}$$

where

$$\mathcal{H}^{(n)} = -\sum_{\alpha=1}^{n} \sum_{\vec{x},\vec{\delta}} J(\vec{x}, \vec{x}+\vec{\delta})\, \vec{S}_a^{\alpha}(\vec{x}) \cdot \vec{S}_b^{\alpha}(\vec{x}+\vec{\delta}) \ . \tag{2.6}$$

The free energy is then given by

$$\beta F = -\lim_{n\to 0} \frac{1}{n} \left([Z^n(\{J\})]_{av} - 1 \right) \ . \tag{2.7}$$

Assuming that each bond is statistically independent, the average over $\{J\}$ in Eq. (2.7) can be carried out formally. The result is

$$[Z^n]_{av} = \mathrm{Tr}_s\, e^{-\beta \tilde{\mathcal{H}}^{(n)}} \ , \tag{2.8}$$

where

$$\beta \tilde{\mathcal{H}}^{(n)} = -\sum_{p=1}^{\infty} \sum_{\vec{x},\vec{\delta}} C_p \left(\beta \sum_{\alpha=1}^{n} \vec{S}_a^{\alpha}(\vec{x}) \cdot \vec{S}_b^{\alpha}(\vec{x}+\vec{\delta}) \right)^p \ , \tag{2.9}$$

where C_p is the pth cumulant of the independent bond probability distribution

$$C_1 = [J]_{\rm av}, \quad C_2 = \tfrac{1}{2}([J^2]_{\rm av} - [J]^2_{\rm av}) \ . \tag{2.10}$$

The case with $C_1 = 0$ yields a spin-glass transition and has already been studied.[7] In this paper, we will consider the general case in which $C_1 \neq 0$. When $C_2 = 0$, there is a ferromagnetic phase transition when $C_1 > 0$ with $T_c \sim C_1$ and an antiferromagnetic transition when $C_1 < 0$ with $T_N \sim |C_1|$. When $C_1 = 0$, there is a spin-glass transition with $T_f \sim \sqrt{C_2}$. When both C_1 and C_2 are present, all three phases can occur, depending on the relative values of C_1 and C_2. We therefore wish to express our free energy in terms of the ferromagnetic, antiferromagnetic, and spin-glass order parameters

$$\vec{S}^{\alpha}(\vec{x}) = \vec{S}^{\alpha}_a(\vec{x}) + \vec{S}^{\alpha}_b(\vec{x}) \ , \tag{2.11}$$

$$\vec{n}^{\alpha}(\vec{x}) = \vec{S}^{\alpha}_a(\vec{x}) - \vec{S}^{\alpha}_b(\vec{x}) \ , \tag{2.12}$$

$$q^{\alpha\beta}_{ij}(\vec{x}) = [S^{\alpha}_{ai}(\vec{x}) S^{\beta}_{aj}(\vec{x}) + S^{\alpha}_{bi}(\vec{x}) S^{\beta}_{bj}(\vec{x})](1 - \delta^{\alpha\beta}) \ . \tag{2.13}$$

In terms of these variables, the Hamiltonian becomes

$$\begin{aligned}\beta\tilde{\mathcal{H}}^{(n)} = &-\frac{\beta}{4}\sum_{\vec{x},\vec{\delta}} C_1 \sum_{\alpha} [\vec{S}^{\alpha}(\vec{x})\cdot\vec{S}^{\alpha}(\vec{x}+\vec{\delta}) \\ &\qquad - \vec{n}^{\alpha}(\vec{x})\cdot\vec{n}^{\alpha}(\vec{x}+\vec{\delta})] \\ &-\beta^2 \sum_{\vec{x},\vec{\delta}} C_2 \sum_{\alpha\neq\beta} q^{\alpha\beta}_{ij}(\vec{x}) q^{\alpha\beta}_{ij}(\vec{x}) + \cdots \ .\end{aligned} \tag{2.14}$$

In order to develop a continuum theory, we express the partition function in terms of order-parameter densities[22]

$$Z = \int d\{M\}\, d\{N\}\, d\{Q\}\, e^{-\beta\mathfrak{F}[M,N,Q]} \ , \tag{2.15}$$

where

$$\beta\mathfrak{F}[M,N,Q] = -S[M,N,Q] + \beta\tilde{\mathcal{H}}^{(n)} \tag{2.16}$$

and

$$\begin{aligned}e^{S[M,N,Q]} = {\rm Tr}_s\Big[&\delta\Big(\vec{M}^{\alpha}(\vec{r}) - \sum_{\vec{x}} \vec{S}^{\alpha}(\vec{x})\,\delta(\vec{x}-\vec{r})\Big) \\ &\times\delta\Big(\vec{N}^{\alpha}(\vec{r}) - \sum_{\vec{x}} \vec{n}^{\alpha}(\vec{x})\,\delta(\vec{x}-\vec{r})\Big) \\ &\times\delta\Big(Q^{\alpha\beta}_{ij}(\vec{r}) - \sum_{\vec{x}} q^{\alpha\beta}_{ij}(\vec{x})\,\delta(\vec{x}-\vec{r})\Big) \\ &\times P(\{\vec{S}^{\alpha}(\vec{x})\})\Big] \ ,\end{aligned} \tag{2.17}$$

where $\vec{r}$ is a continuous space variable and $P(\{\vec{S}^{\alpha}(\vec{x})\})$ is the statistical weighting factor for the original spins. We now proceed in the usual way.[23] We eliminate momentum components outside a sphere of radius Λ in reciprocal space and rescale variables to obtain a phenomenological free energy which can be expressed as follows:

$$\mathfrak{F} = \mathfrak{F}_M + \mathfrak{F}_N + \mathfrak{F}_Q + \mathfrak{F}_{MQ} + \mathfrak{F}_{NQ} + \mathfrak{F}_R \ , \tag{2.18}$$

where

$$\begin{aligned}\beta\mathfrak{F}_M = \frac{1}{2}\int d^d r\Big[&r_M \sum_{\alpha} \vec{M}^{\alpha}(\vec{r})\cdot\vec{M}^{\alpha}(\vec{r}) \\ &+ \sum_{\alpha} \nabla\vec{M}^{\alpha}(\vec{r})\cdot\nabla\vec{M}^{\alpha}(\vec{r}) \\ &+ 2u_1\Big(\sum_{\alpha} \vec{M}^{\alpha}(\vec{r})\cdot\vec{M}^{\alpha}(\vec{r})\Big)^2 \\ &+ 2v_1 \sum_{\alpha} [\vec{M}^{\alpha}(\vec{r})\cdot\vec{M}^{\alpha}(\vec{r})]^2\Big] \ ,\end{aligned} \tag{2.19a}$$

$$\begin{aligned}\beta\mathfrak{F}_N = \frac{1}{2}\int d^d r\Big[&r_N \sum_{\alpha} \vec{N}^{\alpha}(\vec{r})\cdot\vec{N}^{\alpha}(\vec{r}) \\ &+ \sum_{\alpha} \nabla\vec{N}^{\alpha}(\vec{r})\cdot\nabla\vec{N}^{\alpha}(\vec{r}) \\ &+ 2u_2\Big(\sum_{\alpha} \vec{N}^{\alpha}(\vec{r})\cdot\vec{N}^{\alpha}(\vec{r})\Big)^2 \\ &+ 2v_2 \sum_{\alpha} [\vec{N}^{\alpha}(\vec{r})\cdot\vec{N}^{\alpha}(\vec{r})]^2\Big] \ ,\end{aligned} \tag{2.19b}$$

$$\begin{aligned}\beta\mathfrak{F}_Q = &\frac{1}{4}\int d^d r[r_Q\ {\rm Tr}Q^2(\vec{r}) \\ &+ {\rm Tr}\nabla Q(\vec{r})\cdot\nabla Q(\vec{r}) + 4w{\rm Tr}Q^3(\vec{r})] \\ &+ \int d^d\vec{r}\Big(u_Q[{\rm Tr}Q^2(\vec{r})]^2 + v_{Q1}\,{\rm Tr}Q^4(\vec{r}) \\ &+ \sum_{i=2} v_{Qi} f_i(Q(\vec{r}))\Big) \ ,\end{aligned} \tag{2.19c}$$

$$\beta\mathfrak{F}_{MQ} = w_1 \int d^d r\, Q^{\alpha\beta}_{ij}(\vec{r}) M^{\alpha}_i(\vec{r})\, M^{\beta}_j(\vec{r}) \ , \tag{2.19d}$$

$$\beta\mathfrak{F}_{NQ} = w_2 \int d^d r\, Q^{\alpha\beta}_{ij}(\vec{r})\, N^{\alpha}_i(\vec{r})\, N^{\beta}_j(\vec{r}) \ , \tag{2.19e}$$

where the summation convention on repeated indices is understood,

$$\nabla\vec{M}^{\alpha}\cdot\nabla\vec{M}^{\alpha} = \sum_{j=1}\sum_{i=1} \nabla_j M^{\alpha}_i \nabla_j M^{\alpha}_i$$

and ${\rm Tr}\vec{Q}^2 = Q^{\alpha\beta}_{ij} Q^{\alpha\beta}_{ij}$ and where $f_i(Q(\vec{r}))$ for $i \geq 2$ represents other fourth-order invariants that can be formed from $Q(\vec{r})$. The sign of w is chosen to be positive to ensure a second rather than a first-order spin-glass transition. $\mathfrak{F}_R$ includes all higher-order terms not explicitly included in Eqs. (2.19). In particular, it includes N^2M^2 couplings. For simplicity, we will ignore $\mathfrak{F}_R$ in our model. This is consistent with the renormalization group because N^2M^2 terms will be irrelevant since it is impossible for both N and M to be critical at the same time.[24] r_M, r_N, and r_Q change signs as a function of temperature

$$r_M = a_M(T - T_M) \ , \tag{2.20a}$$

$$r_N = a_N(T + T_M) \ , \tag{2.20b}$$

$$r_Q = a_Q(T - T_f) \ . \tag{2.20c}$$

T_M stands for the magnetic transition temperature and T_f for the spin-glass freezing temperature. It is obvious from Eq. (2.15) that T_M is proportional to C_1 and T_f is proportional to $\sqrt{C_2}$. C_1 can, of course, be positive or negative. In general, the coefficients w_1 and w_2 are positive. For a simple cubic lattice, $a_M = a_N$, $w_1 = w_2$, $u_1 = u_2$, and $v_1 = v_2$.

III. MEAN-FIELD THEORY

In this section, we will analyze the mean-field theory for the free energy of Eqs. (2.18) and (2.19) in some detail, indicating wherever possible ambiguities inherent in the $n = 0$ treatment presented here. Some general properties of the phase diagram which we represent in the (T_M, T) plane in Fig. 1 are immediately evident. It is clear that the free energy in the two colorable lattices we are considering here is invariant under the transformation $C_1 \to -C_1$ $(T_M \to -T_M)$ and $M \rightleftharpoons N$. This says that the phase diagram is symmetric about the line $C_1 = 0$ $(T_M = 0)$ with a ferromagnetic phase for sufficiently large C_1 and a corresponding antiferromagnetic phase for sufficiently large and negative C_1. If we neglect M^2N^2 couplings, we can obtain the phase diagram for $C_1 > 0$ by considering $\mathcal{F}_M + \mathcal{F}_Q + \mathcal{F}_{MQ}$ only. The phase diagram for $T_M < 0$ is obtained by the interchange $M \rightleftharpoons N$. If M^2N^2 terms are included, no qualitative changes in the results presented here will occur.

In order to study transitions with both ferromagnetic and spin-glass order parameters, we have to allow for the fact that a nonzero magnetization leads to a nonzero $Q_{ij}^{\alpha\beta}$. In an ordered magnetic state, the magnetization in each replica should be identical. We, therefore, write

$$M_i^\alpha = M e_i \ , \tag{3.1}$$

where e_i is an m-component unit vector and

$$Q_{ij}^{\alpha\beta} = [Q_\perp(\delta_{ij} - e_i e_j) + Q_\parallel e_i e_j](1 - \delta^{\alpha\beta}) \ . \tag{3.2}$$

In terms of these variables, the free energy becomes

$$\begin{aligned} \beta\mathcal{F}/n\Omega = {} & \tfrac{1}{2} r_M M^2 + (n u_1 + v_1) M^4 - HM + (n-1) \\ & \times \{ \tfrac{1}{4} r_Q [(m-1)Q_\perp^2 + Q_\parallel^2] \\ & - w(n-2)[(m-1)Q_\perp^3 + Q_\parallel^3] \\ & - w_1 Q_\parallel M^2 + u_Q n(n-1)[(m-1)Q_\perp^2 + Q_\parallel^2]^2 \\ & + v_{Q1}(n^2 - 3n + 3)[(m-1)Q_\perp^4 + Q_\parallel^4]\} \ , \end{aligned} \tag{3.3}$$

where we have added an external magnetic field H, where Ω is the volume, and where we have set

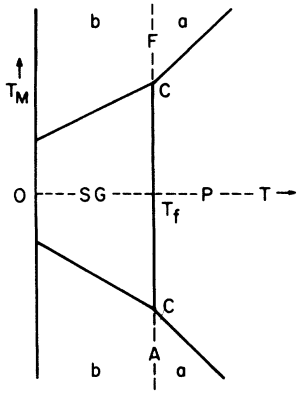

FIG. 1. Phase diagram showing paramagnetic (P), spin-glass (SG), ferromagnetic (F), and antiferromagnetic (A) phases. For $m \geq 2$, the ferromagnetic and antiferromagnetic phases are further subdivided into regions a and b. In region F_a, $m \neq 0$ and $Q_\perp = 0$; in region F_b, both M and $Q_\perp$ are nonzero. Similar statements apply to regions A_a and A_b.

$v_{Qi} = 0$ for $i \geq 2$. Note that u_1 and u_Q play no role in stabilizing the system when $n = 0$ and will be omitted from future calculations. The coefficient of v_{Q1} is positive when $n = 0$, ensuring that the terms in the curly brackets reach a minimum in the pure spin-glass phase ($M = 0$). Since we will be interested only in the minimum with $Q > 0$ which is independent of the value of v_{Q1} for small values of the order parameter,[7,25] we will also ignore v_{Q1} in what follows.

The extrema of Eq. (3.3) in the $n = 0$ limit are determined by

$$\frac{\partial}{\partial M}\frac{\beta\mathcal{F}}{n\Omega} = (r_M + 4v_1 M^2 + 2w_1 Q_\parallel)M = H \ , \tag{3.4a}$$

$$\frac{\partial}{\partial Q_\parallel}\frac{\beta\mathcal{F}}{n\Omega} = -\left(\frac{r_Q Q_\parallel}{2} + 6wQ_\parallel^2 - w_1 M^2\right) = 0 \ , \tag{3.4b}$$

$$\frac{\partial}{\partial Q_\perp}\frac{\beta\mathcal{F}}{n\Omega} = -(m-1)\left(\frac{r_Q Q_\perp}{2} + 6wQ_\perp^2\right) = 0 \ . \tag{3.4c}$$

These equations with $H = 0$ produce the phase diagram shown in Fig. 1, with regions P, SG, F, and A defined as follows.

A. P: paramagnetic phase

M, $Q_\parallel$, and $Q_\perp$ are all zero. This region is defined by r_M *and* $r_Q > 0$. The point C where $r_M = r_Q = 0$; i.e., where $T_M = T_f$, is a special multicritical point where both Q and M have divergent critical fluctuations. Throughout this regime, the spin-glass susceptibility χ_Q and the magnetic susceptibility χ_M satisfy

$$\chi_Q^{-1} = \tfrac{1}{2} r_Q \ , \tag{3.5a}$$

$$\chi_M^{-1} = r_M \ . \tag{3.5b}$$

Thus $\gamma_M = \gamma_Q = 1$. The free energy and specific heat are zero in the mean-field theory and the correlation length exponents are $\nu_Q = \nu_M = \frac{1}{2}$.

B. SG: spin-glass phase

We first consider the case with $H = 0$. In this phase, $M = 0$, $r_Q < 0$, and

$$Q_{\parallel} = Q_{\perp} = -r_Q/6w \tag{3.6}$$

as reported previously. This says that $\beta_Q = 1$. The susceptibilities are

$$\chi_Q^{-1} = -\tfrac{1}{2} r_Q \ , \tag{3.7a}$$

$$\chi_M^{-1} = r_M - (w_1/6w) r_Q \ . \tag{3.7b}$$

Thus, $\gamma_Q' = 1$. The magnetic susceptibility diverges when $r_M = (w_1/6w) r_Q$. This defines the boundary between the spin-glass and ferromagnetic phase (regions SG and F in Fig. 1). This boundary is a straight line passing through C and satisfying

$$\Delta T_M = [1 - (w_1/6w)(a_Q/a_M)] \Delta T \ , \tag{3.8}$$

where $\Delta T_M = T_M - T_f$ and $\Delta T = T - T_f$. The exponent describing the divergence of χ_M as the SG-F boundary is crossed is unity. We will not bother to introduce a symbol for this exponent. Note that χ_Q is finite along this boundary except at the point C.

The entropy and specific heat suffer from the same ambiguity as the free energy itself. If the sign of the $(n-1)$ factor is changed to produce a free-energy minimum rather than a free-energy maximum in the spin-glass phase, the entropy s and specific heat C_v per spin component are positive at least for small r_Q:

$$s = \frac{1}{4} \frac{T}{(12w)^2} a_Q r_Q^2, \quad C_v = \frac{T^2}{2(12w)^2} a_Q^2 r_Q \ . \tag{3.9}$$

This corresponds to a specific-heat exponent $\alpha_Q = -1$.

It is evident from Eqs. (3.5b) and (3.7b) that the magnetic susceptibility has a cusp at the spin-glass paramagnetic boundary when $H = 0$:

$$\frac{d\chi_M}{dT}(T = T_f^+) = -a_M^{-1}(T_M - T_f)^{-2} \ , \tag{3.10a}$$

$$\frac{d\chi_M}{dT}(T = T_f^-) = -a_M^{-1}\left(a_M - \frac{w_1}{6w} a_Q\right)(T_M - T_f)^{-2} . \tag{3.10b}$$

An external magnetic field will smear out this cusp as can be seen by solving Eqs. (3.4) for χ_M when $H \neq 0$:

$$\chi_M = \frac{r_M - (w_1/12w) r_Q + (w_1/12w)(r_Q^2 + 96 w w_1 H^2/r_M^2)^{-1/2}}{[r_M - (w_1/12w) r_Q + (w_1/12w)(r_Q + 96 w w_1 H^2/r_M^2)^{1/2}]^2} . \tag{3.11}$$

This function is plotted in Fig. 2 for three values of H. Note the rounding and shift of the maximum for $H \neq 0$. These results are in accord with experiment[2,3] and previous theories.[1,5]

C. F: ferromagnetic phase

In this phase, both M and $Q_{\parallel}$ are nonzero. If $m \geq 2$, $Q_{\perp}$ completely decouples from $Q_{\parallel}$ and M and this region divides into subregions F_a and F_b in which $Q_{\perp} = 0$ and $Q_{\perp} \neq 0$, respectively. The behavior of $Q_{\perp}$ as the F_a-F_b boundary is crossed is identical to its behavior as the boundary P-SG is crossed and will not be discussed further. In region F, M and $Q_{\parallel}$ satisfy

$$M^2 = (1/w_1)(\tfrac{1}{2} r_Q Q_{\parallel} + 6w Q_{\parallel}^2) \ , \tag{3.12}$$

$$Q_{\parallel} = (w_1/24 v_1 w)\{-A + [A^2 - (24 v_1 w/w_1) r_M]^{1/2}\} \ , \tag{3.13}$$

where $A = w_1 + (v/w_1) r_Q$. First note that $Q_{\parallel}$ and thus M is trivially zero when $r_M = 0$. This says that there is a continuous transition from the paramagnetic to ferromagnetic phase as expected. Next note that $\frac{1}{2} r_Q + 6w Q_{\parallel} = 0$ when $r_M = (w_1/6w) r_Q$. Thus M^2 goes to zero from the ferromagnetic side along the same line that χ_M diverges on the spin-glass

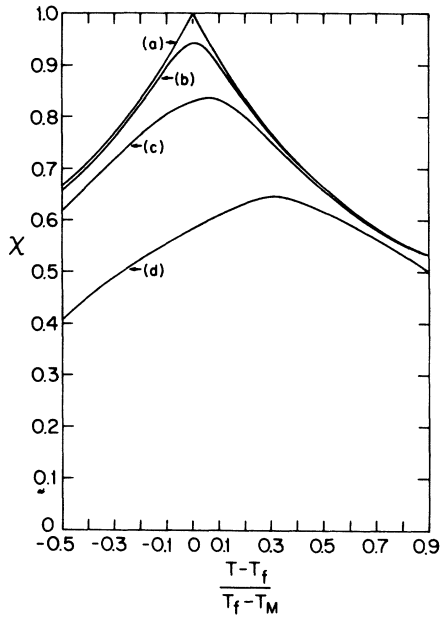

FIG. 2. Magnetic susceptibility χ in an external magnetic field when the P-SG boundary is crossed. $w_1 a_Q / 12 w a_M$ is chosen to be 1. χ is in units of $1/a_M(T_f - T_M)$. (a) Free field magnetic susceptibility; it shows a cusp at $T = T_f$. (b)–(d) are magnetic susceptibilities for three different values of $\Delta^2 = 96 w w_1 H^2 / a_Q^2 a_M^2 (T_f - T_M)^2$, $\Delta^2 = 0.001$, 0.01, 0.1, respectively. Cusp in (a) is rounded and there is a shift of the maximum.

side. Later, we will see that χ_M on the ferromagnetic side diverges along the same line and there is no discontinuity in the entropy as this line is crossed. Thus, the ferromagnetic–spin-glass transition is second order.

Let us first investigate the magnetic transition far from the point C, i.e., when r_Q is not near zero. In this case, if $r_M \ll w_1 A^2/24 v_1 w$, we have

$$Q_{\parallel} = -\tfrac{1}{2}(r_M/A) \ ,$$

$$M^2 = -\frac{1}{4w_1}\frac{r_M r_Q}{A} = -\frac{r_M}{4v_1}\left(1 - \frac{w_1}{w_1 + (v/w_1)r_Q}\right). \tag{3.14}$$

Thus, the fluctuations of $Q_{\parallel}$ merely renormalize the fourth-order coupling constant v_1. This transition is the familiar ferromagnetic transition in a disordered system,[4(b),16,26] and we will not consider it further here.

The point C is a type of multicritical point which we will now investigate in more detail. In the vicinity of C, both r_Q and r_M are near zero, and a simultaneous expansion in these two variables is necessary. First, we have

$$Q_{\parallel} = -\tfrac{1}{2}(r_M/w_1) - 2(v/w_1)M^2 \ , \tag{3.15a}$$

$$M^2 = -\tfrac{1}{4}(r_M/w_1^2)[r_Q - 6(w/w_1)r_M] \ . \tag{3.15b}$$

From this, we can immediately see that $M^2 \sim (\Delta T)^2$ and $Q_{\parallel} \sim -\Delta T$ if $\Delta T_M = 0$, and $M^2 \sim (\Delta T_M)^2$ and $Q_{\parallel} \sim \Delta T_M$ if $\Delta T = 0$. If we define ordering exponents β_{M1} and β_{Q1} and crossover exponents ϕ_{M1} and ϕ_{Q1} via

$$Q = (\Delta T)^{\beta_{M1}} f\left(\frac{|\Delta T|}{|\Delta T_M|^{\phi_{Q1}}}\right)$$

and

$$M = (\Delta T)^{\beta_{M1}} f\left(\frac{|\Delta T|}{|\Delta T_M|^{\phi_{Q1}}}\right),$$

we find that $\beta_{Q1} = \beta_{M1}$ and $\phi = \phi_{Q1} = \phi_{M1} = 1$ in mean-field theory:

The susceptibilities in the vicinity of C are

$$\chi'_Q = [\tfrac{1}{2}r_Q - 6(w/w_1)r_M + \tfrac{1}{2}w_1^2/v_1]^{-1} \ , \tag{3.16a}$$

$$\chi'_M = \frac{1}{2}\frac{r_Q - 12(w/w_1)r_M}{4w_1^2 M^2} \ . \tag{3.16b}$$

Thus, $\gamma_{M1} = 1$. Note that χ_M diverges along all of the boundaries of F. χ_Q on the other hand does not diverge anywhere. Along the P-F boundary it equals $[+\tfrac{1}{2}r_Q + \tfrac{1}{2}(w_1^2/v_1)]^{-1}$, and along the SG-$F$ boundary, it equals $[-\tfrac{1}{2}r_Q + \tfrac{1}{2}(w_1^2/v_1)]^{-1}$. Thus χ_Q^{-1} has a discontinuity of $\tfrac{1}{2}(w_1^2/v)$ across all boundaries into region F. There are also mixed M-Q susceptibilities in region F which we do not quote here.

The sign of the entropy and specific heat again depend on the sign of $(n-1)$ in front of $\mathcal{F}_Q$. ($\mathcal{F}_{MQ} + \mathcal{F}_M \equiv 0$ to the order we are considering in the vicinity of C.) If we again choose $(n-1)$ to be $+1$, we obtain

$$s = \frac{T}{2}\frac{r_M}{w_1^2}\left[a_M\left(\frac{1}{4}r_Q - \frac{w}{w_1}r_M\right) + \frac{1}{2}r_M\left(\frac{1}{4}a_Q - \frac{w}{w_1}a_M\right)\right], \tag{3.17a}$$

$$C_v = \frac{3w^2 a_M^2}{2w_1^2}T^2\left[\left(1 - \frac{w_1}{6w}\frac{a_2}{a_M}\right)\Delta T_M - \left(1 - \frac{1}{4}\frac{w_1}{w}\frac{a_Q}{a_M}\right)\Delta T\right]. \tag{3.17b}$$

It is easy to see that s reduces to $\tfrac{1}{4}[T/(12w)^2]a_Q r_Q^2$ along the line $r_M = (w_1/6w)r_Q$ indicating a second-order spin-glass to ferromagnetic transition.

At this point, it is worth considering whether there are any constraints that the above analysis places on the values of the potential in the problem. M^2 must be positive throughout region F. This is only possible if the line $r_Q = 6(w/w_1)r_M$ lies below the line $r_M = 0$ for $T > T_f$. This implies $-\infty < 1 - (w_1/6w)(a_Q/a_M) < 1$ which is satisfied as long as all of the potentials are positive. In particular, this implies $w_1 > 0$ since we have chosen $w > 0$ in order to have a second-order spin-glass transition. If we require C_v [Eq. (3.17)] to be positive, there is a further constraint that

$$0 < (w_1/6w)(a_Q/a_M) < \tfrac{1}{2}$$

or

$$\tfrac{1}{2} < 1 - (w_1/6w)(a_Q/a_M) < 1 \ .$$

The phase diagram shown in Fig. 1 satisfies this constraint.

We close this section with the observation that both β_{Q1} and β_{M1} are one near C. Thus the critical dimensionality below which mean-field theory is expected to no longer be valid satisfies the Josephson-Rushbrook relation $d_c\nu = 2\beta + \gamma = 3$, or $d_c = 6$. In Sec. IV, we will calculate critical exponents for this transition in $6 - \epsilon$ dimensions.

IV. ϵ EXPANSION

In this section we analyze the Hamiltonian [Eq. (3.1)] using renormalization-group theory. Again we restrict ourselves to the case $C_1 > 0$ and neglect the antiferromagnetic order parameter $\vec{N}(\vec{r})$. We first Fourier transform the variables $\vec{M}(\vec{r})$, $Q_{ij}^{\alpha\beta}(\vec{r})$, and introduce a spherical Brillouin zone of radius Λ. The Fisher-Wilson recursion relations[27] are obtained by integrating over fluctuations with wave vectors between Λ/b and Λ. ($b > 1$), rescaling all lengths by the factor b^{-1}, and $\vec{M}(\vec{r})$, $Q_{ij}^{\alpha\beta}$ by factors $b^{(d+2-\eta_1)/2}$, $b^{(d+2-\eta_2)/2}$, respectively. We get the following recursion relations:

$$r_M' = b^{2-\eta_1}\{r_M - 4(n-1)mw_1^2[A(0) - K_d(r_M + r_Q)\ln b]\}\ , \tag{4.1}$$

$$r_Q' = b^{2-\eta_2}\{r_Q - 36(n-2)mw^2[A(0) - K_d 2r_Q \ln b] - 4w_1^2[A(0) - 2K_d r_M \ln b]\}\ , \tag{4.2}$$

$$w' = b^{\epsilon/2-3\eta_2/2}(w + \{36[(n-3)m+1]w^3 + \tfrac{4}{3}w_1^3\}K_d \ln b)\ , \tag{4.3}$$

$$w_1' = b^{\epsilon/2-\eta_2/2-\eta_1}\{w_1 + 4[w_1 + 3(n-2)mww_1^2]K_d \ln b\}\ , \tag{4.4}$$

$$\eta_1 = \tfrac{4}{3}(n-1)mw_1^2 K_d\ , \tag{4.5}$$

$$\eta_2 = \tfrac{1}{3}[3b(n-2)mw^2 + 4w_1^2]K_d\ , \tag{4.6}$$

where

$$A(0) = \int_{\Lambda/b}^{\Lambda} \frac{d^6q}{(2\pi)^6}\frac{1}{q^2}\ , \quad K_d = \frac{1}{(2\pi)^6}\Omega_d\ ,$$

where Ω_d is the solid angle in d dimension and $\epsilon = 6-d$. Equations (4.1)–(4.6) are then solved to find the fixed points. In the limit $n \to 0$, we find: (i) a Gaussian fixed point with $w^* = w_1^* = 0$; (ii) a spin-glass fixed point with $(w^*)^2 = \epsilon[36K_6(4m-2)]^{-1}$, $w_1^* = 0$, and (iii) a new fixed $w^* = w_n^*$, $w_1^* = w_{1n}^*$ and its complement with $w^* = -w_n^*$, $w_1^* = -w_{1n}^*$. The spin-glass fixed point was studied in Ref. 7 and has critical exponents $\nu = \frac{1}{2} + 5m\epsilon/12(2m-1)$ and $\eta = -m\epsilon/3(2m-1)$. The new fixed point corresponds to the point C in the mean-field phase diagram. Values for w_n^* and w_{n1}^* for $m = 1$, 2, and 3 are listed in Table I. Equations (4.1) and (4.2) yield two exponents λ_{t1} and λ_{t2} ($<\lambda_{t1}$) which are related to the correlation length and crossover exponents via $\nu^{-1} = \lambda_{t1}$ and $\varphi = \nu\lambda_{t2}$. Equations (4.3) and (4.4) yield the stability exponents λ_1 and λ_2. For $m = 1$, the fixed point C is stable with respect to changes in w and w_1, and all exponents are real with the following values:

$$\nu = \tfrac{1}{2} + \tfrac{2}{3}\epsilon\ , \tag{4.7a}$$

$$\varphi = 1 + \tfrac{1}{2}\epsilon\ , \tag{4.7b}$$

$$\eta_1 = \eta_2 = -\tfrac{1}{3}\epsilon\ , \tag{4.7c}$$

$$\lambda_1 = -\epsilon, \quad \lambda_2 = -\tfrac{5}{3}\epsilon\ . \tag{4.7d}$$

For $m = 2$ and 3, λ_{t1} and λ_{t2} are complex and are listed along with λ_1, λ_2, η_1, and η_2 in Table II. This result is puzzling, difficult to interpret, and may cast further doubt on the applicability of the Edwards and Anderson replica procedure to the spin-glass transition, especially in XY and Heisenberg systems. Complex exponents have been found previously in random dipolar systems,[28] but to our knowledge, this is the first system for which the thermal eigenvalues are complex. Wallace and Zia[29] have pointed out that there will be real eigenvalues whenever the recursion relations in differential form can be expressed in terms of a Riemanian metric times a gradient of a potential. It is easy to verify that it is impossible to find a metric for $m = 2$ and 3 for the r equations with only positive eigenvalues in agreement with Wallace and Zia.[29]

TABLE I. Fixed point values for w and w_1 at the non-Gaussian fixed point for $m = 1, 2, 3$.

	$m=1$	$m=2$	$m=3$
$W_n^*\left(\frac{Kd}{\epsilon}\right)^{1/2}$	$\frac{1}{6}$	0.0839	0.0614
$W_{n1}^*\left(\frac{Kd}{\epsilon}\right)^{1/2}$	$\frac{1}{2}$	0.3032	0.2373

V. TOPOLOGICAL DISORDER

The spin-glass state can also be induced by topological disorder. We begin with a model introduced by Harris, Plischke, and Zuckermann[19] to describe transitions in amorphous $TbFe_2$ and YFe_2. In this model, each magnetic ion is subjected to a local anisotropy field of random orientation. It is clear how this model might produce a spin-glass state. For sufficiently strong anisotropy, the spins follow the local anisotropy axis. Since this axis has a random orientation, $[\langle\vec{S}(\vec{x})\rangle]_{av} = 0$. At low enough temperature, however, $[\langle\vec{S}(\vec{x})\rangle \cdot \langle\vec{S}(\vec{x})\rangle]_{av}$ is nonzero. The Harris-Plischke-Zuckermann Hamiltonian assuming constant exchange is

$$H = -\sum_{\vec{x},\vec{\delta}} J\vec{S}(\vec{x})\cdot\vec{S}(\vec{x}+\vec{\delta}) - D_0\sum_{\vec{x}}[\vec{S}(\vec{x})\cdot\hat{e}(\vec{x})]^2\ , \tag{5.1}$$

where D_0 represents the strength of uniaxial anisotropy and $\hat{e}(\vec{x})$ is a unit vector pointing in the

TABLE II. Exponents for the $m = 2$ and $m = 3$ non-Gaussian fixed points.

	$\lambda_{t1,2} - 2$	λ_1	λ_2	η_1	η_2
$m=2$	$-(1.1501 \pm 0.3247i)\epsilon$	$-\epsilon$	-1.079ϵ	-0.2451ϵ	-0.2149ϵ
$m=3$	$-(0.9407 \pm 0.2539i)\epsilon$	$-\epsilon$	-0.8686ϵ	-0.2253ϵ	-0.1960ϵ

direction of local (random) unaxial axis. Using the standard procedure of replicating the Hamiltonian [Eq. (5.1)] n times, Aharony[30] has shown that Eq. (5.1) leads to an effective average Hamiltonian

$$\begin{aligned}\beta\mathcal{H}_{\text{eff}} = &-\beta J \sum_{\vec{x},\vec{\delta},\alpha} \vec{S}^{\alpha}(\vec{x})\cdot\vec{S}^{\alpha}(\vec{x}+\vec{\delta}) \\ &-\beta m^{-1}D_0 \sum_{\vec{x},\alpha} S^{\alpha}(\vec{x})\cdot\vec{S}^{\alpha}(\vec{x}) \\ &+\beta^2 \frac{D_0^2}{m^2(m+2)} \sum_{\alpha,\beta,\vec{x}} S_i^{\alpha}(\vec{x}) S_j^{\beta}(\vec{x}) S_i^{\alpha}(\vec{x}) S_j^{\beta}(\vec{x}) \\ &-\frac{\beta^2 D_0^2}{m(m+2)} \sum_{\alpha,\beta,\vec{x}} S_i^{\alpha}(\vec{x}) S_i^{\beta}(\vec{x}) S_j^{\alpha}(\vec{x}) S_j^{\beta}(\vec{x}) \ . \end{aligned} \tag{5.2}$$

There are two different symmetry fourth-order forms in Eq. (5.2) which break the degeneracy between the isotropic and anisotropic parts of the spin-glass order parameter $q_{ij}^{\alpha\beta}$ in Eq. (2.13). Therefore, we decompose $q_{ij}^{\alpha\beta}$ into two parts

$$q_{ij}^{\alpha\beta} = q_{1ij}^{\alpha\beta} + q_{2ij}^{\alpha\beta} \ , \tag{5.3}$$

where

$$q_{1ij}^{\alpha\beta} = q_1 \delta_{ij}(\delta^{\alpha\beta} - 1) \ , \tag{5.4a}$$

$$q_{2ij}^{\alpha\beta} = q_{2ij}(\delta^{\alpha\beta} - 1) \ , \tag{5.4b}$$

where q_{2ij} is symmetric and traceless in i and j. Substituting Eqs. (5.3) and (5.4) into Eq. (5.2), we obtain

$$\begin{aligned}\beta\mathcal{H} = &-\beta J \sum_{\vec{x},\vec{\delta},\alpha} \vec{S}^{\alpha}(\vec{x})\cdot\vec{S}^{\alpha}(\vec{x}+\vec{\delta}) - \beta m^{-1}D_0 \sum_{\vec{x},\alpha} \vec{S}^{\alpha}(\vec{x})\cdot\vec{S}^{\alpha}(\vec{x}) - \beta^2 D_0^2 \frac{1}{m(m+2)}\left(1-\frac{1}{m}\right)\sum_{\vec{x}} q_{1ij}^{\alpha\beta}(\vec{x}) q_{1ij}^{\alpha\beta}(\vec{x}) \\ &+\frac{\beta^2 D_0^2}{m^2(m+2)} \sum_{\vec{x}} q_{2ij}^{\alpha\beta}(\vec{x}) q_{2ij}^{\alpha\beta}(\vec{x}) + \beta^2 D_0^2 \frac{1}{m(m+2)}\left(1-\frac{1}{m}\right)\sum_{\alpha\vec{x}} [\vec{S}^{\alpha}(\vec{x})\cdot\vec{S}^{\alpha}(\vec{x})]^2 \ . \end{aligned} \tag{5.5}$$

There are now three fields in the problem: $\vec{S}^{\alpha}(\vec{x})$, $q_{1ij}^{\alpha\beta}(\vec{x})$, and $q_{2ij}^{\alpha\beta}(\vec{x})$. It is clear that the local interaction between $q_{2ij}^{\alpha\beta}(\vec{x})$ is repulsive. Thus, $q_{2ij}^{\alpha\beta}$ does not order and is noncritical or irrelevant. If we assume that the spin-glass state is more stable than the ferromagnetic state, both $\vec{S}^{\alpha}$ and $q_{2ij}^{\alpha\beta}$ can be removed. We then introduce an order-parameter density $Q_{1ij}^{\alpha\beta}(\vec{r})$, as in Sec. II and remove high momentum degrees of freedom to obtain an effective free-energy density

$$\beta\mathcal{F} = \int [\tfrac{1}{4} r \, \mathrm{Tr} Q_1^2 + \mathrm{Tr}(\nabla Q_1)^2] + w \int \mathrm{Tr} Q_1^3 \ , \tag{5.6}$$

where $r \sim T - T_f$ and

$$T_f^2 \sim \frac{D_0^2}{(m+2)m}\left(1-\frac{1}{m}\right) \ . \tag{5.7}$$

This free energy is identical to that considered in Ref. 7 for a system with Ising symmetry. Thus, the Harris-Plischke-Zuckermann model can have an Ising spin-glass transition if D_0 is sufficiently large. Aharony[23] failed to find a stable ferromagnetic fixed point for this system in $4-\epsilon$ dimensions. It is possible that the flow away from the ferromagnetic fixed points is to the above spin-glass fixed point. It is interesting to note that the magnetic susceptibility of the amorphous magnet shows the same cusp as a spin glass.[31]

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