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Composite solitons and magnetic resonances in superfluid ^{3}He-A †

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Making use of the Ginzburg-Landau free energy describing the textures in superfluid ^{3}He-A, detailed properties of a composite soliton (which involves both $\hat{l}$ and $\hat{d}$ vectors) are studied. It is shown that $\hat{d}$ solitons (or magnetic solitons) are unstable against formation of composite solitons in an open system. The stability of the composite soliton as well as the magnetic resonance associated with them are considered.

I. INTRODUCTION

In superfluid ^{3}He-A, a general texture associated with the condensate is described in terms of two unit vectors $\hat{l}$ and $\hat{d}$, where $\hat{l}$ designates the direction of the symmetry axis of the energy gap, while $\hat{d}$ the direction of the spin component. The nuclear dipole interaction between ^{3}He atoms provides the coupling between these two vectors, which favors either parallel or antiparallel alignment. In a recent letter,[1] we have shown that among planar solutions in superfluid ^{3}He-A, the composite soliton is the domain wall with the lowest energy, where both $\hat{l}$ and $\hat{d}$ vectors are involved. In fact, we will see later that pure $\hat{d}$ solitons (or magnetic solitons)[2,3] are unstable against formation of the composite solitons in an open system. The decay rate of the $\hat{d}$ soliton is controlled by the orbital viscosity predicted theoretically by Cross and Anderson[4] and measured by Paulson, Krusius, and Wheatley.[5]

In contrast to the pure $\hat{d}$ soliton, the $\hat{l}$ vector provides a trapping potential for the $\hat{d}$ vector in the composite soliton, giving rise to nonvanishing frequency shifts in magnetic resonances. These shifts are (i) in general, smaller than the normal shifts Ω_A predicted by Leggett, and (ii) different for transverse and longitudinal resonance. We believe that the shifts due to the twist composite soliton are responsible for the longitudinal satellite observed by the Orsay-Saclay group[6] and more recently by Gould and Lee.[7] We note also, that unlike the ordinary resonance, the widths of these satellites are mainly determined by the spin-diffusion tensor and therefore depend only weakly on temperature. The calculation of the frequency shifts as well as widths are described in detail in Secs. III and IV. We have also studied the stability of all the planar structures so far described against distortions of both $\hat{l}$ and $\hat{d}$ vectors. This is done by ignoring the external constraints and assuming that the instability is driven only by the dipolar interaction energy. For the pure $\hat{d}$ soliton (uniform $\hat{l}$), we find two distinct instabilities, one of which corresponds to the rotation of the $\hat{l}$ vector in the same plane as the $\hat{d}$ vector (and thus leading to the formation of a composite soliton), and the other one corresponds to the rotation of $\hat{l}$ off the original $(\hat{l},\hat{d})$ plane. The growth rate for the former instability is found to be larger and therefore the pure $\hat{d}$ soliton is unstable against formation of a composite soliton. The composite soliton itself, on the other hand, is found to be completely stable. These calculations are described in Sec. III.

In the following analysis, we make use of the texture free energy as given by Ambegaokar *et al.*[8,9] This is given in its most general form,

$$F_{\text{kin}}=\frac{1}{2}\int d^3r\,(K_1\partial_i A_{\mu i}\partial_j A^*_{\mu j}+K_2\partial_i A_{\mu j}\partial_i A^*_{\mu j}+K_3\partial_i A_{\mu j}\partial_j A^*_{\mu i})\,, \qquad (1)$$

where $A_{\mu i}$ $(\mu, i=1,2,3)$ are nine complex order parameters describing the triplet P-wave condensate of superfluid ^{3}He. Here, the subscript μ refers to the spin components while i, refers to the orbital components of the condensate.

In the Ginzburg-Landau regime, Eq. (1) is further simplified, as we have

$$K_1=K_2=K_3=K\equiv\frac{6}{5}\,\frac{N}{8m^*}\,\frac{7\zeta(3)}{(2\pi T_c)^2}\ , \qquad (2)$$

where $\zeta(3)=1.202,\ldots,N$ is the density of the ^{3}He atom, and m^* is the effective mass of the quasiparticle. If we further limit ourselves to the A phase, $A_{\mu i}$ is given by

$$A_{\mu i}=\hat{d}_\mu\vec{\Delta}_i,\quad \vec{\Delta}_i=(\Delta_0/\sqrt{2})(\hat{\delta}^1_i+i\hat{\delta}^2_i)\,, \qquad (3)$$

where $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{l}$ $(\equiv\hat{\delta}^1\times\hat{\delta}^2)$ are the orthogonal triad and $\hat{d}$ is a unit vector describing the spin component. Then, in terms of $\hat{d}$ and $\vec{\Delta}$, Eq. (1) in the Ginzburg-Landau regime is given as

$$\begin{aligned}F_{\text{kin}}=\tfrac{1}{2}K\int d^3r\,\big(&3|\vec{\nabla}\cdot\vec{\Delta}|^2+|\vec{\nabla}\times\vec{\Delta}|^2+2|\vec{\Delta}\cdot\vec{\nabla}\hat{d}|^2\\&+|\vec{\Delta}|^2(|\vec{\nabla}\cdot\hat{d}|^2+|\vec{\nabla}\times\hat{d}|^2)\\&+\vec{\nabla}\{2[(\vec{\nabla}\cdot\vec{\Delta})\vec{\Delta}^*-\vec{\Delta}(\vec{\nabla}\cdot\vec{\Delta}^*)]\\&+|\vec{\Delta}|^2[(\hat{d}\cdot\vec{\nabla})\hat{d}-\hat{d}(\vec{\nabla}\cdot\hat{d})]\}\big)\,. \qquad (4)\end{aligned}$$

The last two terms in the above equation give rise to surface energy. However, in the following, we will be concerned with situations where the surface energy is negligible and we will omit the surface energy. As already pointed out, in order to have a stable planar structure, we need a symmetry breaking energy, which is provided by the dipolar interaction energy

$$E_d = -\tfrac{1}{2}\chi_N \Omega_A^2 (\hat{l}\cdot\hat{d})^2 , \tag{5}$$

where χ_N is the spin susceptibility and Ω_A is the longitudinal resonance frequency. The question of texture is handled then by considering $F = F_{\text{kin}} + E_d$. In order to discuss dynamics of $\hat{l}$ and $\hat{d}$ vectors, we have to supplement the above free energy with the time-dependent terms. In the case of the spin dynamics, this is most conveniently done with the Lagrangian formulation of the spin dynamics as described previously by one of us.[10] In the case of the orbital dynamics, we write down a dynamic equation, which incorporates the ideas of Cross and Anderson.[4]

II. COMPOSITE SOLITON

In this section we study the general planar structure which involves both $\hat{l}$ and $\hat{d}$ fields. We assume that a static field H is applied in the z direction. This field is assumed to be large enough so that the $\hat{d}$ vector in equilibrium lies in the x-y plane. We can therefore take

$$\begin{aligned} \hat{d} &= \sin\psi\, \hat{x} + \cos\psi\, \hat{y} , \\ \hat{l} &= \sin\chi\, \hat{x} + \cos\chi\, \hat{y} , \end{aligned} \tag{6}$$

where ψ and χ are unknown functions of position to be determined later. $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the Cartesian unit vectors. The vector order parameter consistent with the above $\hat{l}$ is then given by

$$\vec{\Delta} = (\Delta_0/\sqrt{2}) e^{i\Phi}(-\cos\chi\, \hat{x} + \sin\chi\, \hat{y} + i\hat{z}) , \tag{7}$$

where Φ is an additional scalar function to be determined. Substituting Eqs. (6) and (7) into Eqs. (4) and (5), we have

$$\begin{aligned} F = \tfrac{1}{2} A \int d^3r \,[\, &|\vec{\nabla}\chi|^2 + 2(\sin\chi\chi_x + \cos\chi\chi_y)^2 \\ &+ 4|\vec{\nabla}\Phi|^2 - 2(\sin\chi\Phi_x + \cos\chi\Phi_y)^2 + 4|\vec{\nabla}\psi|^2 \\ &- 2(\sin\chi\psi_x + \cos\chi\psi_y)^2 + 2\chi_z(\sin\chi\Phi_x + \cos\chi\Phi_y) \\ &- 6(\sin\chi\chi_x + \cos\chi\chi_y)\Phi_z + 4\xi_\perp^{-2}\sin^2(\chi-\psi)] , \end{aligned} \tag{8}$$

where $A = \tfrac{1}{2}K\Delta_0^2 = \tfrac{1}{4}\chi_N C_\perp^2$, $\xi_\perp = C_\perp/\Omega_A$, the dipolar coherence length, and $C_\perp$ is the spin-wave velocity with the propagation vector perpendicular to $\hat{l}$. If we take $\chi = \Phi = 0$, the magnetic solitons discussed previously are obtained by minimizing Eq. (8). In particular, if we assume further that ψ depends only on z, we have

$$\psi = 2\tan^{-1}[\exp(z/\xi_\perp)] \tag{9}$$

and the energy per unit surface,

$$f^{\hat{d}} = F/\sigma_3 = 2\chi_N \Omega_A^2 \xi_\perp = 8A\xi_\perp^{-1} . \tag{10}$$

Let us now consider the planar solution by assuming that $\chi = \chi(s)$, $\psi = \psi(s)$, and $\Phi = \Phi(s)$, with $s = \hat{k}\cdot\vec{x}$ and $\hat{k}^2 = 1$. Then Eq. (8) reduces to

$$\begin{aligned} F/\sigma(\hat{k}) = \tfrac{1}{2} A \int ds\, [&(1+2a^2)\chi_s^2 + (4-2a^2)\Phi_s^2 \\ &+ (4-2a^2)\psi_s^2 - 4\hat{k}_3 a\chi_s\Phi_s \\ &+ 4\xi_\perp^{-2}\sin^2(\chi-\psi)] , \end{aligned} \tag{11}$$

with $a = \sin\chi\, k_1 + \cos\chi\, k_2$, where $\sigma(\hat{k})$ is the surface with the normal vector $\hat{k}$. Here, the subscripts s on χ, Φ, and ψ imply their derivative with respect to s. We can eliminate Φ_s from Eq. (11) by

$$\Phi_s = 2\hat{k}_3 a\chi_s(4-2a^2)^{-1} . \tag{12}$$

Substituting this back to Eq. (11), we have

$$\begin{aligned} \frac{F}{\sigma(\hat{k})} = \tfrac{1}{2} A \int ds \{&[1 + 2a^2 - 2\hat{k}_3^2 a^2 (2-a^2)^{-1}]\chi_s^2 \\ &+ 2(2-a^2)\psi_s^2 + 4\xi_\perp^{-2}\sin^2(\chi-\psi)\} . \end{aligned} \tag{13}$$

Since the planar structure with the lowest energy is given in the case $\hat{k}_1 = \hat{k}_2 = 0$ and $\hat{k}_3 = \pm 1$ (i.e., pure twist solution), we will concentrate in the following on this particular case. Introducing new variables u and v by $u = \chi + 4\psi$ and $v = \chi - \psi$, Eq. (13) is rewritten as (for $\hat{k} = \hat{z}$)

$$f^c_{\text{twist}} = \frac{F}{\sigma_z} = \tfrac{1}{2} A \int dz \left(\tfrac{1}{5}u_z^2 + \tfrac{4}{5}v_z^2 + 4\xi_\perp^{-2}\sin^2 v\right) , \tag{14}$$

which allows a solution with

$$u = \text{const.}, \quad \text{and } \tan\tfrac{1}{2}v = \exp(\sqrt{5}z/\xi_\perp) . \tag{15}$$

The corresponding free energy is given by

$$f^c_{\text{twist}} = \frac{1}{\sqrt{5}} f^{\hat{d}} , \tag{16}$$

where $f^{\hat{d}}$ has been already defined in Eq. (10).

The energy of the composite soliton is smaller than that of d soliton by a factor of $\sqrt{5}$. The source of this reduction is the fact that energies of the dipole energy generated textures go like $2\chi_N\Omega_A^2$ (the dipole energy stored in the texture) multiplied by the width of the domain wall. Since the composite system is narrower than the pure $\hat{d}$ soliton by a factor of $\sqrt{5}$, the corresponding surface energy is reduced by a factor of $\sqrt{5}$. When $\hat{k}_\perp [\equiv (\hat{k}_1^2 + \hat{k}_2^2)^{1/2}]$ is not large, we can calculate the corresponding surface energy by pertubation starting from the case $\hat{k}_\perp = 0$. The result is given by[11]

$$\frac{F}{\sigma(\hat{k})} = f^c_{\text{twist}}\left[1+\frac{7}{40}\left(1-\frac{25}{39}\cos\frac{\pi}{5}\right)\hat{k}^2_\perp + O(\hat{k}^4_\perp)\right]$$
$$\simeq f^c_{\text{twist}}(1+0.084k^2_\perp)\,. \qquad (17)$$

Therefore, in the case of the composite soliton, the twist solution given in Eq. (15) is the one with the lowest energy.

Unlike the d soliton, the composite soliton has mass current given by $\vec{v}_{\text{curl}}$,

$$\vec{v}_{\text{curl}} = (1/8m)(\vec{\nabla}\times\hat{l}) = (1/10m)\vec{\nabla}_z\hat{l} \qquad (18)$$

for the pure twist case. The spatical average of the $\vec{v}_{\text{curl}}$ does not vanish, but gives

$$\langle\vec{v}_{\text{curl}}\rangle = \int dz\, \vec{v}_{\text{curl}}$$
$$= \frac{1}{8m}\left[(1+\cos\tfrac{1}{5}\pi)\hat{x} + (\sin\tfrac{1}{5}\pi)\hat{y}\right]. \qquad (19)$$

The associated mass current compensates $\frac{4}{5}$ of the dipole force in the texture. The remaining $\frac{1}{5}$ is compensated by the spin current associated with inhomogeneous $\hat{d}$ vector. Furthermore, the present soliton carries an intrinsic angular momentum

$$\vec{L} = \rho_s \int d^3r(\vec{r}\times\vec{v}_{\text{curl}})$$
$$= \frac{\sigma_3\rho_s\xi_\perp}{8\sqrt{5}\,m}\left[(1-\cos(\tfrac{1}{5}\pi)\hat{y} - (\sin\tfrac{1}{5}\pi)\hat{x}\right](c-2\ln 2)\,,$$

with

$$c = 10\int_0^1 du\,\frac{1-u^9}{1-u^{10}} = 9.846\,, \qquad (20)$$

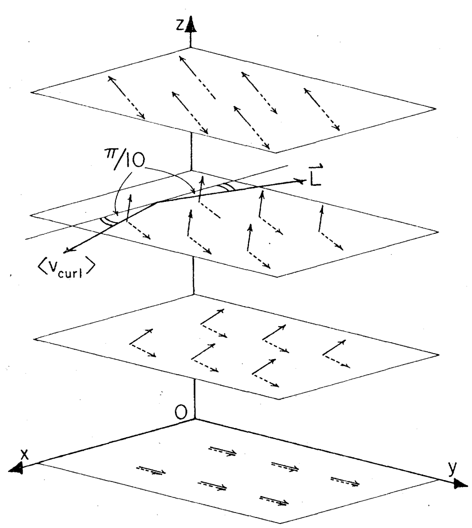

FIG. 1. The pure twist composite soliton is shown schematically. The solid arrows indicate the direction of the $\hat{l}$ vector, while the dashed arrows represent the $\hat{d}$ vector. The curl velocity $\vec{v}_{\text{curl}}$ and the angular momentum $\vec{L}$ of the texture are also shown.

where ρ_s is the superfluid density parallel to $\hat{l}$, and σ_3 is the surface area of the soliton. The configuration of $\hat{l}$, $\hat{d}$, $\langle\vec{v}_{\text{curl}}\rangle$, and $\vec{L}$ for the composite soliton are shown schematically in Fig. 1.

III. FLUCTUATIONS

In this section we shall study the fluctuations of the $\hat{l}$ and $\hat{d}$ vectors around their equilibrium positions. As the equilibrium configurations, we shall consider the pure $\hat{d}$ soliton and the composite soliton. The free energy is expanded in powers of small displacements from the equilibrium positions. The term linear and quadratic in fluctuations we call fluctuation free energy. Absence of the linear term is the necessary condition for the stability of the equilibrium configuration. The quadratic term in fluctuation may be diagonalized. The eigenvalues associated with pure $\hat{d}$ fluctuation determine the extra resonance frequencies associated with the texture in nuclear magnetic resonance. Furthermore, the eigenvalues associated with $\hat{l}$ and $\hat{d}$ fluctuation provide the sufficient condition for the stability of the equilibrium configuration; the equilibrium configuration is stable if all eigenvalues are non-negative.

For simplicity, we will consider the planar textures with the normal vector in the z direction. Furthermore, we shall assume that all fluctuations have only z dependence. For the fluctuations, the restriction on z dependence can be easily removed. However, the most interesting fluctuations, of course, occur only in the z direction. For a general configuration of $\hat{l}$ and $\hat{d}$ vectors, we take

$$\hat{l} = (\sin\chi\,\hat{x} + \cos\chi\,\hat{y})\cos\phi + \sin\phi\,\hat{z}\,,$$
$$\hat{d} = (\sin\psi\,\hat{x} + \cos\psi\,\hat{y})\cos\theta + \sin\theta\,\hat{z}\,,$$

and

$$\vec{\Delta} = \Delta_0/\sqrt{2}\,[-(\cos\chi + i\sin\phi\sin\chi)\hat{x} + (\sin\chi - i\sin\phi\cos\chi)\hat{y} + i\cos\phi\hat{z}]\,. \qquad (21)$$

The corresponding free energy is given by

$$\frac{F}{\sigma_3} = \tfrac{1}{2}A\int dz\big((1+2\sin^2\phi)\phi_z^2 + (1+\sin^2\phi)\chi_z^2 + 2(1+\cos^2\phi)(\theta_z^2+\cos^2\theta\psi_z^2) + 4\xi_\perp^{-2}\{1-[\sin\phi\sin\theta + \cos\phi\cos\theta\cos(\chi-\psi)]^2\}\big)\,. \qquad (22)$$

A. Fluctuations about the $\hat{d}$ soliton

First, we shall consider the fluctuations about a pure $\hat{d}$ soliton. The equilibrium values then correspond to $\theta_0 = \phi_0 = \chi_0 = 0$ and $\psi_0 = 2\tan^{-1}[\exp(z/\xi_\perp)]$. The fluctuation free energy can be split into two terms:

$$\delta F = \delta F(\chi,\psi) + \delta F(\phi_0\theta)\,,$$

$$\frac{\delta F(\chi,\psi)}{\sigma_3} = \tfrac{1}{2}A\int dz\left[-8\xi_\perp^{-2}\sin\psi_0\cos\psi_0\chi + \chi_z^2 + 4f_z^2 + 4\xi_\perp^{-2}(1-2\sin^2\chi_0)(\chi-f)^2\right], \tag{23a}$$

and

$$\frac{\delta F(\phi,\theta)}{\sigma_3} = \tfrac{1}{2}A\int d_z\left\{\phi_z^2 + 4\theta_z^2 + 4\xi_\perp^{-2}\left[(1-2\sin^2\chi_0)\theta^2 + (1-\tfrac{3}{2}\sin^2\chi_0)\phi^2 - 2\cos\chi_0\phi\theta\right]\right\}, \tag{23b}$$

where $f=\psi-\psi_0$, the fluctuation in ψ.

First we shall consider the eigenvalues associated with the pure $\hat{d}$ fluctuation, which are related to the nuclear magnetic resonance frequencies associated with the $\hat{d}$ soliton, as we will see later. By setting $\chi=\phi=0$ in Eqs. (21a) and (21b), we derive the eigenequations

$$\lambda f = -\xi_\perp^2 f_{zz} + (1-2\sin^2\chi_0)f \tag{24}$$

and exactly the same equation for θ. Here, we have normalized λ, so that λ is a dimensionless constant. Equation (24) has one bound state, with the eigenvalue $\lambda=0$ and $f\propto \operatorname{sech}(z/\xi_\perp)$. Since the eigenvalues for ψ and θ modes are related to the shifts in the magnetic resonance frequencies in the longitudinal and the transverse configurations, respectively, this predicts that $\hat{d}$ solitons produce resonances at unshifted frequencies. Physically, these modes correspond to sliding and rotation of the $\hat{d}$ soliton over uniform $\hat{l}$ texture.

Next, let us consider more general fluctuations involving both $\hat{l}$ and $\hat{d}$ vectors. The associated eigenvalues determine the stability or instability of the $\hat{d}$ soliton. The fluctuation energy $\delta F(\chi,\psi)$ in Eq. (21a) is diagonalized by introducing new variables u and v by

$$u=\chi+4f \quad\text{and}\quad v=\chi-f\,,$$

$$\frac{\delta F(\chi,\psi)}{\sigma_3} = \tfrac{1}{2}A\int dz\left\{-\tfrac{8}{5}\xi_\perp^{-2}\tanh\frac{z}{\xi_\perp}\operatorname{sech}\frac{z}{\xi_\perp}(u-v) + \tfrac{1}{5}u_z^2 + \tfrac{4}{5}v_z^2 + 4\xi_\perp^{-2}\times\left[1-2\operatorname{sech}^2(z/\xi_\perp)\right]v^2\right\}. \tag{25}$$

We note that existence of the linear term in the fluctuation already suggests the instability of the pure $\hat{d}$ soliton (in fact, it is the necessary condition for the local instability). For the moment, neglecting the linear term, which gives rise to inhomogeneous terms, the eigenequations are given by

$$\lambda_u u = -\tfrac{1}{20}\xi_\perp^2 u_{zz}$$

and

$$\lambda_v v = -\tfrac{1}{5}\xi_\perp^2 v_{zz} + \left[1-2\operatorname{sech}^2(z/\xi_\perp)\right]v\,. \tag{26}$$

The u mode has a plane-wave solution with eigenvalue

$$\lambda_u = \tfrac{1}{20}(\xi_\perp k)^2,\quad u\propto e^{ikz}\,, \tag{27}$$

while the v mode has two bound states (or discrete eigenvalues),

$$\lambda_{v1} = \tfrac{1}{10}(\sqrt{41}-10)\simeq -0.4597\,,$$

and with

$$v\propto\left[\operatorname{sech}(z/\xi_\perp)\right]^{(\sqrt{41}-1)/2}, \tag{28}$$

$$\lambda_{v2} = \tfrac{3}{2}(\sqrt{41}/5-1)\simeq 0.4209\,,$$

with

$$v\propto\tanh(z/\xi_\perp)\left[\operatorname{sech}(z/\xi_\perp)\right]^{(\sqrt{41}-3)/2}. \tag{29}$$

We note that the λ_{v1} is negative. This implies that the $\hat{d}$ soliton is intrinsically unstable. We will see later that $|\lambda_{v1}|$ is proportional to the decay rate of $\hat{d}$ soliton, the end product of this decay is very likely to be the composite soliton discussed in Sec. II. The fluctuation free energy $\delta F(\phi,\theta)$, on the other hand, cannot be diagonalized by simple transformation. Therefore we have to solve the coupled equations for θ and ϕ;

$$\lambda\phi = -\tfrac{1}{4}\xi_\perp^2\phi_{zz} + \left\{\left[1-\tfrac{3}{2}\operatorname{sech}^2\left(\frac{z}{\xi_\perp}\right)\right]\phi - \tanh\left(\frac{z}{\xi_\perp}\right)\theta\right\},$$

$$\lambda\theta = -\xi_\perp^2\theta_{zz} + \left\{\left[1-2\operatorname{sech}^2\left(\frac{z}{\xi_\perp}\right)\right]\theta - \tanh\left(\frac{z}{\xi_\perp}\right)\phi\right\}. \tag{30}$$

These simultaneous equations have been solved by variational methods taking as the basis two mutually orthogonal functions $(\operatorname{sech}z/\xi_\perp)^\nu$ and $(\operatorname{sech}z/\xi_\perp)^\mu\tanh z/\xi_\perp$. Note that the even ϕ couples to the odd θ, and the odd ϕ to the even θ only. The two eigenvalues thus found are

$$\lambda(\phi,\theta)_1 = -0.3406$$
$$\text{for } (\phi,\theta) = (\tanh(z/\xi_\perp)X^{0.77}, X^{0.695})$$

and

$$\lambda(\phi,\theta)_2 = -0.139$$
$$\text{for } (\phi,\theta) = (X, \tanh(z/\xi_\perp)X^{0.57}), \tag{31}$$

respectively, where $X = \mathrm{sech}(z/\xi_\perp)$. Again, we have negative eigenvalues, indicating the additional instability of $\hat{d}$ soliton. These instabilities correspond to the fluctuations of $\hat{l}$ and $\hat{d}$ vectors outside of the original plane they are defined. However, the v_1 mode has the lowest eigenvalue, and the initial instability is certainly dominated by this mode, implying the likely formation of the composite solition. In Sec. V, we will determine the decay rate of the $\hat{d}$ soliton. The additional instability of the $\hat{d}$ soliton associated with ϕ, θ may be suppressed by applying a strong magnetic field along the z axis, which will provide an additional potential energy for the $\hat{d}$ vector.

B. Fluctuations about the composite soliton

Second, we shall study the fluctuation about the composite soliton. The equilibrium configuration is now given by

$$\phi_0 = \theta_0 = 0, \quad \chi_0 + 4\psi_0 = 0,$$

and

$$\chi_0 - \psi_0 = v_0 = 2\tan^{-1}[\exp(\sqrt{5}z/\xi_\perp)] .$$

Then the fluctuation free energy (20) is again decomposed into two parts;

$$\delta F = \delta F(\chi, \psi) + \delta F(\phi, \theta) ,$$

where

$$\frac{\delta F(\chi,\psi)}{\sigma_3} = \tfrac{1}{2}A\int_{-\infty}^{\infty} dz\left[g_{zz}^2 + 4f_{zz}^2 + 4\xi_\perp^{-2}\left(1 - 2\,\mathrm{sech}^2\frac{\sqrt{5}z}{\xi_\perp}\right)(g-f)^2\right] \tag{32a}$$

and

$$\frac{\delta F(\phi,\theta)}{\sigma_3} = \tfrac{1}{2}A\int_{-\infty}^{\infty} dz\left\{\phi_z^2 + 4\theta_z^2 + 4\xi_\perp^{-2}\left[\left(1 - \tfrac{3}{10}\,\mathrm{sech}^2\frac{\sqrt{5}z}{\xi_\perp}\right)\phi^2 + \left(1 - \tfrac{6}{5}\,\mathrm{sech}^2\frac{\sqrt{5}z}{\xi_\perp}\right)\theta^2 - 2\tanh\left(\frac{\sqrt{5}z}{\xi_\perp}\right)\theta\phi\right]\right\} \tag{32b}$$

with $g = \chi - \chi_0$ and $f = \psi - \psi_0$.

First of all, we note that Eqs. (31a) and (31b) do not contain any linear term in fluctuations. This implies that the composite soliton is in fact stable, if all eigenvalues associated with fluctuations are non-negative. We will see later that the all eigenvalues are indeed non-negative for the composite soliton. First let us consider pure $\hat{d}$ fluctuation by letting $g = \phi = 0$. Then the corresponding eigenequations are given by

$$\lambda_f f = -\xi_\perp^2 f_{zz} + [1 - 2\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]f$$

and (33)

$$\lambda_\theta\theta = -\xi_\perp^2\theta_{zz} + [1 - \tfrac{6}{5}\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]\theta$$

for f and θ mode, respectively. Both equations have one bound state each with the eigenvalues,

$$\lambda_f = \tfrac{1}{2}(\sqrt{65} - 7) \simeq 0.5311 ,$$

$$f \propto [\mathrm{sech}(\sqrt{5}z/\xi_\perp)]^{(\sqrt{13}/5 - 1)/2}$$

and

$$\lambda_\theta = \tfrac{4}{5}, \quad \theta \propto [\mathrm{sech}(\sqrt{5}z/\xi_\perp)]^{1/5} \tag{34}$$

These eigenvalues are related to the shifts in the longitudinal and transverse magnetic resonance frequencies associated with the composite soliton.

We will now study the eigenvalues associated simultaneous fluctuation of $\hat{l}$ and $\hat{d}$ vectors. We have again diagonalized $\delta F(\chi, \psi)$ [Eq. (31a)] as

$$\frac{\delta F(\chi,\psi)}{\sigma_3} = \tfrac{1}{2}A\int_{-\infty}^{\infty} dz\{\tfrac{1}{5}u_z^2 + \tfrac{4}{5}v_z^2 + 4\xi_\perp^{-2} \times [1 - 2\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]v^2\} , \tag{35}$$

$u = g + 4f$ and $v = g - f$.

The resulting eigenequations are given by

$$\lambda_u u = -\tfrac{1}{20}\xi_\perp^2 u_{zz}$$

and

$$\lambda_v v = -\tfrac{1}{5}\xi_\perp^2 v_{zz} + [1 - 2\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]v . \tag{36}$$

The u mode has a plane-wave solution, while the v mode has one bound state with $\lambda_v = 0$.

$$\lambda_u = \tfrac{1}{20}(\xi_\perp k)^2 \quad \text{with} \quad u \propto e^{ikz}$$

and (37)

$$\lambda_v = 0 \quad \text{with} \quad v \propto \mathrm{sech}(\sqrt{5}z/\xi_\perp)$$

The latter gapless mode describes the sliding motion of the composite system. For the ϕ, θ mode the eigenequation is given by

$$\begin{aligned}\lambda_\phi\phi &= -\tfrac{1}{4}\xi_\perp^2\phi_{zz} + [1 - \tfrac{3}{10}\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]\phi - \tanh(\sqrt{5}z/\xi_\perp)\theta ,\\ \lambda_\theta\theta &= -\xi_\perp^2\theta_{zz} + [1 - \tfrac{6}{5}\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]\theta - \tanh(\sqrt{5}z/\xi_\perp)\phi .\end{aligned} \tag{38}$$

The simultaneous equations are solved variationally, as before. We find that Eq. (37) has two gapless modes [i.e., $\lambda(\phi,\theta)=0$].

$$(\phi,\theta)\simeq(1,\tanh(\sqrt{5}z/\xi)),$$
$$(\phi,\theta)\simeq(\tanh(\sqrt{5}z/\xi_\perp),1). \quad (39)$$

Although these solutions are non-normalizable, they can be considered as the limits of normalizable solutions. These modes describe the rotation of the composite system as a whole.

Therefore, we conclude that all eigenvalues associated with the fluctuations around the composite soliton are non-negative. Thus, the composite system is stable, at least locally. Furthermore, the present soliton carries a conserved quantity $\Delta v=v(\infty)-v(-\infty)$, as in the case of $\hat{d}$ soliton. Therefore, composite soliton is stable locally, as well as globally. However, it should be borne in mind that the stability of the composite system is established only within the framework of $F=F_{\text{kin}}+F_d$.

IV. MAGNETIC RESONANCE

The magnetic resonance in the presence of a composite soliton arises from the motion of $\hat{d}$ vector about its equilibrium position. This motion takes place at a time scale much faster than the characteristic time scale of motion for the $\hat{l}$ vector. Therefore the $\hat{l}$ texture in the composite soliton acts as a static trapping potential for the $\hat{d}$ motion. Consequently, there exist characteristic frequencies associated with the oscillation of the entire $\hat{d}$ structure in a potential provided by the $\hat{l}$ texture. This frequency, being of the order of the usual dipolar shift Ω_A, is directly observable in a nuclear magnetic resonance experiment. In particular, the longitudinal resonance frequency obtained here appears to account for the unusual satellites observed by Avenel *et al.*[6] and more recently by Gould and Lee.[7]

In order to consider the magnetic resonance, we have to introduce the dynamic terms associated with the motion of $\hat{d}$ vector into the theory. This is most easily achieved in terms of the Lagrangian formulation of the spin dynamics.[11] The dynamics are described in terms of Lagrangian $L=T-V$, where δF discussed in Sec. III provides the potential energy. The kinetic energy T is given in terms of Eulerian angles (α,β,γ) describing the rotation of the $\hat{d}$ vector as

$$T=\tfrac{1}{2}\chi_n\int d^3r[\alpha_t^2+\beta_t^2+\gamma_t^2+2\alpha_t\gamma_t\cos\beta -2\omega_0(\alpha_t+\gamma_t\cos\beta)], \quad (40)$$

where suffix t implies the time derivative, χ_n is the spin susceptibility as before and $\omega_0=\gamma_0 H$ is the Larmor frequency associated with the static magnetic field along the z axis. The induced magnetization is related to α_t, β_t, and γ_t by $\vec{M}=-\gamma_0\vec{\omega}$:

$$\omega_x=-\beta_t\sin\alpha+\gamma_t\cos\alpha\sin\beta,$$
$$\omega_y=\beta_t\cos\alpha+\gamma_t\sin\alpha\sin\beta, \quad (41)$$
$$\omega_z=\alpha_t+\gamma_t\cos\beta.$$

To make contact with the results of the previous section, we have to express ψ and θ in terms of α, β, and γ. Noting the fact that the starting configuration (i.e., $\alpha=\beta=\gamma=0$) corresponds to the $\hat{d}$ vector along the y axis ($\hat{d}=\hat{y}$), we can determine the d vector after the rotation $R(\alpha,\beta,\gamma)$:

$$\hat{d}=-(\sin\alpha\cos\gamma+\cos\alpha\cos\beta\sin\gamma)\hat{x} +(\cos\alpha\cos\gamma-\sin\alpha\cos\beta\sin\gamma)\hat{y} +\sin\beta\sin\gamma\hat{z}. \quad (42)$$

Comparing Eq. (42) with Eq. (21), we find that

$$\psi=-\alpha,\quad \theta=\gamma \text{ with } \beta=\tfrac{1}{2}\pi.$$

With this identification, the Lagrangian equation of motion in terms of small fluctuations around the composite soliton is given

$$f_{tt}-c^2 f_{zz}+\Omega_A^2[1-2\,\text{sech}^2(\sqrt{5}z/\xi_\perp)]f=0,$$
$$\theta_{tt}+\omega_0\beta_{1t}-C^2\theta_{zz}+\Omega_A^2[1-\tfrac{6}{5}\,\text{sech}^2(\sqrt{5}z/\xi_\perp)]\theta=0,$$
$$\beta_{1tt}-\omega_0\theta_t=0, \quad (43)$$

where

$$f_1=\psi-\psi_0 \text{ and } \beta_1=\beta-\tfrac{1}{2}\pi.$$

The ψ modes appear in the longitudinal resonance, while the θ modes appear in the transverse resonance. The resonance frequencies are given by

$$\omega_l^2=\lambda_f\Omega_A^2,\quad \omega_t^2=\omega_0^2+\lambda_\theta\Omega_A^2, \quad (44)$$

where λ_f and λ_θ are given by Eq. (34). The resonances are shifted from the main resonance and appear as satellites. In terms of actual numbers we have

$$\frac{\omega_l}{\Omega_A}=(\lambda_f)^{1/2}\cong 0.729$$

and (45)

$$(\omega_t^2-\omega_0^2)^{1/2}/\Omega_A=(\lambda_\theta)^{1/2}\cong 0.8944.$$

About a year ago, Avenel *et al.*[6] reported observation of a satellite peak in their longitudinal resonance experiment in superfluid ^{3}He-A. The satellite frequency was reported to be $\omega_l/\Omega_A\simeq 1/\sqrt{2}$. More recently an extensive study of these satellites, in both the longitudinal and the transverse case has been reported by Gould and Lee.[7] Their results are summarized as

$$\omega_l/\Omega_A = 0.74 - 0.35(1 - T/T_c)$$

and (46)

$$(\omega_t^2 - \omega_0^2)^{1/2}/\Omega_A = 0.835 .$$

The agreement for the longitudinal case is remarkable, if we ignore the small temperature dependence. Even this can be qualitatively understood in terms of the temperature-dependent Fermi-liquid corrections as discussed by Cross.[12] On the other hand, in the transverse case the agreement is not satisfactory. Furthermore, the observed shift normalized by Ω_A appears to be independent of temperature, which is rather strange, if we assume the transverse satellites arise from the composite soliton discussed here.

In the present calculation, the satellite frequencies are calculated for the same texture. On the other hand, in the Gould-Lee experiment, the transverse resonance is done by holding the direction of the rf field fixed, but rotating the static magnetic field by $\frac{1}{2}\pi$. A possibility is that this results in an entirely different texture for the longitudinal and the transverse experiments. We, however, predict that if the transverse experiment is done in the same texture (this seems most easily achieved by holding the static field, but rotating the rf field), a transverse satellite would be observed at the frequency given by Eq. (44). Furthermore, the frequency would have a similar temperature dependence as the longitudinal satellite.

Since the satellites arise from localized spin-wave states, the widths of the above satellites are much broader than the ordinary resonance, mainly due to the spin-diffusion term. The spin-diffusion coefficients, which give rise to additional broadening, are easily incorporated into Eq. (43) as

$$f_{tt} - c^2 f_{zz} - D f_{tzz} + \Omega_A^2[1 - 2\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]f = 0 ,$$

$$\theta_{tt} + \omega_0\beta_{1t} - C^2\theta_{zz} - D\theta_{tzz} + \Omega_A^2[1 - \tfrac{6}{5}\,\mathrm{sech}^2(\sqrt{5}z/\xi_\perp)]\theta = 0 ,$$

$$\beta_{1,tt} - \omega_0\theta_t = 0 , \qquad (47)$$

where D is an appropriate component of the spin-diffusion tensor in ^{3}He-A. Substituting the eigenfunctions in the absence of the spin diffusion, the half widths of the satellites are given by

$$\Delta\omega_t = \tfrac{1}{2}D\langle|f_z|^2\rangle/\langle|f|^2\rangle = 0.145 D\xi_\perp^{-2}$$

and (48)

$$\Delta\omega_t = \tfrac{1}{2}D\langle|\theta_z|^2\rangle/\langle|\theta|^2\rangle = 0.07 D\xi_\perp^2$$

for the longitudinal and the transverse satellites, respectively.

Before concluding this section, we point out that the present analysis applies also for pure $\hat{d}$ soliton. In this case, since the corresponding eigenvalues vanish, we have

$$\omega_l = 0 \text{ and } \omega_t = \omega_0$$

(i.e., unshifted resonance frequencies) for satellites.[3,13] The linewidths are $\frac{1}{6}$(in the units of $D\xi_\perp^{-2}$).

V. INSTABILITY OF THE $\hat{d}$ SOLITON

We have seen already in Sec. III that the pure $\hat{d}$ soliton is unstable against fluctuations of $\hat{l}$ and $\hat{d}$ vectors. In this section, we will estimate the characteristic time associated with the decay of $\hat{d}$ soliton. Since the decay of $\hat{d}$ soliton is closely related to the formation of the composite soliton, we may identify this time tentatively with the necessary time for the formation of the composite soliton.

As we will see, the characteristic time is controlled by the orbital viscosity of Cross and Anderson (for $1 - T/T_c \gg 10^{-5}$), which is given roughly by $10^{-2}\,(1 - T/T_c)^{1/2}$ sec. This is much longer than the time taken by the $\hat{d}$ soliton to lose its velocity due to the spin-diffusion effect[2] ($\sim 10^{-4}$ sec). Therefore, we can envision the formation of the composite soliton in three stages: First the $\hat{d}$ solitons are created magnetically (e.g.., by turning off a step-type magnetic field), then the $\hat{d}$ solitons move out into space from their points of creation but stop somewhere within the time scale of 10^{-4} sec. Finally the $\hat{d}$ soliton relaxes into a composite soliton within the time scale of $10^{-2}\,(1 - T/T_c)^{1/2}$ sec.

We formulate the question of instability in terms of long time scale equations of motion for the $\hat{l}$ and $\hat{d}$ vectors. This permits us to express the growth rate of the instability in terms of eigenvalues of the fluctuation free energy (those eigenvalues that are negative). In a complete calculation, one would have to include the final state as the natural by product of the equations of motion and that would require solving a nonlinear equation of motion. What we present here is to be considered as an estimate, and we believe a good estimate, for the decay rate of $\hat{d}$ soliton into a composite one.

The equations of motion are derived using the Lagrangian formulation as discussed in Sec. IV for the spin dynamics, supplemented by spin diffusion term for the $\hat{d}$ vector and the Cross Anderson term for the orbital viscosity. In as much as we are interested only in long time scales, we neglect the inertial terms. The equation for χ and Ψ are then

$$-\mu\frac{\partial\chi}{\partial t} = \frac{\partial}{\partial\chi}\,\delta F(\chi, \psi) ,$$

$$\chi_n D\psi_{tzz} = \frac{\partial}{\partial\Psi}\,\delta F(\chi, \psi) , \qquad (49)$$

where μ is the orbital viscosity introduced by Cross and Anderson and D is the spin-diffusion coefficient.

Similar equations can be derived for ϕ, θ, as well. However, for the instability, we are interested only in the lowest negative eigenvalue which corresponds to the motion of $v = \chi - \psi$ [see Eq. (24)]. Therefore, we will restrict ourselves to the motion of the v variable alone. It is also apparent that the instability in $\chi - \psi$ points in the direction of formation of composite soliton as the final state. Equation (46) can be rewritten in terms of the v variable as

$$-\frac{\partial}{\partial t}\left(16\tau v - \frac{D}{C^2}\xi_\perp^2 v_{zz}\right) = -\tfrac{1}{5}\xi_\perp^2 v_{zz} + \left(1 - 2\,\mathrm{sech}^2\frac{z}{\xi_\perp}\right)v - \mathrm{sech}\frac{z}{\xi_\perp}\tanh\frac{z}{\xi_\perp}, \tag{50}$$

where $\tau = \mu\,\xi_\perp^2/4A$ is the $\hat{l}$-vector relaxation time in a uniform system as measured by Paulson, Krusius, and Wheatley[5] ($\tau = t_{1/2}/\ln 2$). If we analyze Eq. (47) in terms of the eigenfunctions of Eq. (28) {i.e., for $v(z) \propto e^{-\Gamma_v t}[\mathrm{sech}(z/\xi_\perp)]^{(\sqrt{41}-1)/2}$} the growth rate is found to be (see Appendix)

$$\Gamma_v = 25\lambda_{v1}/(16\tau + 1.1398D/C_\perp^2)\,. \tag{51}$$

Substituting the measured spin-diffusion constant[14] at $T = T_c$, the spin-diffusion damping is given by (at $P = 27$ bar)

$$1.1398D/C_\perp^2 = 5.3\times10^{-9}(1 - T/T_c)^{-1}\ \mathrm{sec}\,, \tag{52}$$

where $C_\perp(t)$ is taken from Wheatley's review.[15] The spin-diffusion damping is much faster than the orbital damping for $1 - T/T_c \gtrsim 10^{-5}$. Therefore, the decay of the $\hat{d}$ soliton is dominated by the orbital relaxation and we have, except in this narrow temperature range,

$$\begin{aligned}\Gamma_v &= -0.718\tau^{-1}\\ &\cong -63(1 - T/T_c)^{-1/2}\ \mathrm{sec}^{-1} \text{ at } P = 29 \text{ bar}\,.\end{aligned} \tag{53}$$

Here, τ is related to $t_{1/2}$ determined by Paulson *et al.*[5] by $\tau = t_{1/2}/\ln 2$.

In the above consideration, we have ignored the ϕ, θ motion entirely, in spite of the presence of negative eigenvalues. There are two reasons for doing so. First, of course, the initial instability is governed by the lowest eigenvalue: both the eigenvalues of ϕ, θ motion are higher. Second, in the presence of an external field in the z direction, the $\hat{d}$ vector is clamped in the x-y plane and instability can be completely inhibited.

The present analysis of instability assumes that the $\hat{l}$ vector is completely free to move around. This freedom may be lost if liquid ^{3}He is confined between two narrow channels with width comparable to $\xi_\perp$, the spatial extension of $v(z)$ mode. Then the pure $\hat{d}$ soliton becomes completely stable. The details of the effect of confinement will be discussed elsewhere.

VI. CONCLUDING REMARKS

We have shown that in an open system (i) the $\hat{d}$ solitons created magnetically are unstable and decay into the composite soliton within time scale of $10^{-2}(1 - T/T_c)^{1/2}$ sec; (ii) the composite soliton is stable against small fluctuations in $\hat{d}$ and $\hat{l}$ vectors; (iii) the satellite frequencies in magnetic resonance associated with the composite soliton are determined. In particular, the predicted longitudinal satellite frequency is in good agreement with recent experiments by Gould and Lee.[9] On the other hand, for the transverse satellite frequency, the agreement is not good; and (iv) as a by product we show also that the satellite frequencies associated with pure $\hat{d}$ soliton are unshifted. In particular, in a narrow channel with the width of 10^{-3} cm, it is very likely that the $\hat{d}$ solitons may be stabilized. In this case, the detection of unshifted Larmor frequency in the transverse experiment will be favorable for the identification of the $\hat{d}$ soliton.

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APPENDIX: GROWTH RATE OF THE INSTABILITY ASSOCIATED WITH PURE $\hat{d}$ SOLITON

Substituting into Eq. (50)

$$v(z,t) = Ne^{-\Gamma_v t}[\mathrm{sech}(z/\xi_\perp)]^{(\sqrt{41}-1)/2}\,, \tag{A1}$$

where N is the normalization constant:

$$N = \xi_\perp^{-1/2}(2)^{1-\sqrt{41}/2}[B(\sqrt{41} - 1/2, \sqrt{41} - 1/2)]^{-1/2} \tag{A2}$$

and $B(x,y)$ is Euler's beta function, we have

$$(\Gamma_v/25)[16\tau v(z,0) - D\Omega_A^{-2} v_{3z}(z,0)] = \lambda_{v1} v(z,0)\,. \tag{A8}$$

Here, we dropped the inhomogeneous term in the right-hand side of (A3), since this term is orthogonal to $v(z,0)$. [i.e., the inhomogeneous term does not affect the initial growth rate of dominant stability we are interested in here.] Then, multiplying $v(z,0)$ on the both sides of (A3) and integrating over z, we obtain

$$\Gamma_v = 25\lambda_{v1}/(16\tau + D\Omega_A^{-2}\langle v_z^2\rangle)\,, \tag{A4}$$

where

$$\langle v_z^2\rangle = \int_{-\infty}^{\infty} dz\, v_z^2(z,0) \Big/ \int_{-\infty}^{\infty} dz v^2(z,0) = \xi_\perp^{-2}\tfrac{1}{2}(21/\sqrt{41} - 1) \tag{A5}$$

Putting them together, we obtain Eq. (48) in the text.

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