PHYSICAL REVIEW B VOLUME 16, NUMBER 3 1 AUGUST 1977

Asymptotic superparamagnetic time constants for cubic anisotropy. II. Negative anisotropy constant

I. Eisenstein and A. Aharoni
Department of Electronics, The Weizmann Institute of Science, Rehovoth, Israel
(Received 16 June 1976)

We consider superparamagnetic particles with cubic anisotropy where the anisotropy constant is negative. It is found that in the limit of a high barrier for a magnetization reversal, *three* (and *only* three) time constants are needed in order to describe the relaxation rate of an ensemble of such particles. It is further shown that their ratios tend towards 1:2:3. Asymptotic expressions for this limit are obtained theoretically that depend exponentially on the barrier height. Numerical computations in a limited region of barrier height give reasonably close results for all practical purposes. But the asymptotic conditions of a high barrier are not as yet realized in the region where the computations are carried. Still, the 1:2:3 ratios between the time constants seem to be confirmed by the numerical results.

I. INTRODUCTION

This study deals with the asymptotic behavior of the time constants associated with superparamagnetic particles with cubic anisotropy where the anisotropy constant is *negative*, i.e., the easy axis is along ⟨111⟩. This paper uses the same assumptions and techniques used in the previous paper[1] (to be denoted by EA from now on) which dealt with a positive cubic anisotropy, i.e., the easy axis along ⟨100⟩. Moreover, the same notations will be adopted throughout except when otherwise stated. In particular, Secs. I, II, and III A of EA are followed very closely in the present paper. In the following we shall take the anisotropy energy density $F(\theta,\phi)$ to be given by

$$F(\theta,\phi) = -\tfrac{1}{4}K(\sin^2 2\theta + \sin^4\theta \sin^2 2\phi), \quad \text{with } K>0\,. \tag{1}$$

Note that with this definition the μ_i's and α [Eqs. (6) and (7) of EA] are *positive*.

The same correction of the numerical calculations made in EA is applicable here too. The results of similar computations are plotted in Fig. 1, where the reduced time constants μ_1, μ_2, μ_3, and $\mathrm{Re}(\mu_4)$ are shown as functions of the parameter α. At $\alpha=12$, μ_4 becomes complex (with μ_5 as its complex conjugate) and it is not plotted for larger values of α. On the other hand, μ_1, μ_2, and μ_3 are real in the region considered, and they seem to behave according to the same law. The computations were carried out up to $\alpha=23$, since larger α's require excessive computer time. In the following sections we shall derive asymptotic relations between the μ_i's and asymptotic expressions as $\alpha\to\infty$. For this, the same assumptions as in Sec. III A of EA will be used.

II. ASYMPTOTIC RELATIONS BETWEEN TIME CONSTANTS

Considering $F(\theta,\phi)$ of (1), the minima and maxima of the anisotropy potential have exchanged places with respect to the potential used in EA. We now have eight equivalent minima situated along the directions

$$\left.\begin{array}{l}\theta_1=\cos^{-1}(3^{-1/2})\\ \theta_2=\pi-\theta_1\end{array}\right\}\ \phi=\tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{5}{4}\pi,\ \tfrac{7}{4}\pi\,. \tag{2}$$

It is convenient to view them as residing at the corners of a cube and to divide them into four groups as shown in Fig. 2: M_1—any one of the minima that for convenience is taken along the [111] direction; M_2—the three minima nearest to M_1; M_3—the three minima next nearest to M_1; M_4—the farthest minimum from M_1. In this picture it may be said that the maxima of F sit at the centers of the cube faces and the saddles at the middle of the sides.

Considering, as in EA, an ensemble of n uniformly magnetized particles with their n representative points on the unit sphere in the limit $\alpha\to\infty$, we denote by n_1, n_2, n_3, and n_4, the *total* number of representative points in each of the groups of minima M_1, M_2, M_3, and M_4, respectively. We require that at any time t,

$$n_1+n_2+n_3+n_4=n\,, \tag{3}$$

and impose the initial conditions

$$n_1(t=0)=n\,, \tag{4a}$$

$$n_2(t=0)=n_3(t=0)=n_4(t=0)=0\,. \tag{4b}$$

Let ν be the probability per unit time for a point to pass from one minimum to *any* one of the three

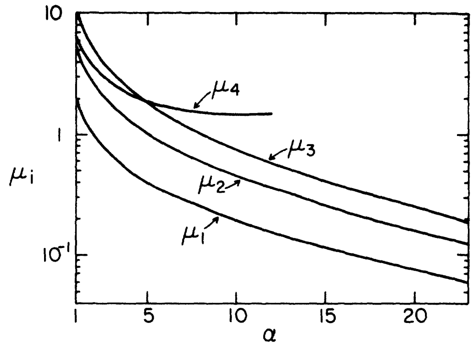

FIG. 1. Reduced numerically computed time constants μ_1, μ_2, μ_3, and Re(μ_4) [Eqs. (6) and (2) of EA] as functions of the reduced barrier energy α [Eqs. (7) and (1b) of EA] for superparamagnetic particles with cubic anisotropy whose energy density is given by (1). For $\alpha \geqslant 12$, μ_4 is complex. For $\alpha \to \infty$ it is just μ_1, μ_2, and μ_3 that determine the relaxation rate.

surrounding nearest minima. As in EA, we assume that a *direct* transfer to a farther minimum is not possible. This leads to the time-rate equations:

$$\dot{n}_1 = \tfrac{1}{3}\nu n_2 - \nu n_1 , \tag{5a}$$

$$\dot{n}_2 = \nu n_1 + \tfrac{2}{3}\nu n_3 - \nu n_2 , \tag{5b}$$

$$\dot{n}_3 = \nu n_4 + \tfrac{2}{3}\nu n_2 - \nu n_3 , \tag{5c}$$

$$\dot{n}_4 = \tfrac{1}{3}\nu n_3 - \nu n_4 , \tag{5d}$$

where because of (3) one of them is dependent on the others.

A solution to Eqs. (3)-(5) that is of the form of (2) in EA is

$$n_1 = \tfrac{1}{8} n(1 + 3e^{-p_1 t} + 3e^{-p_2 t} + e^{-p_3 t}) , \tag{6a}$$

$$n_2 = \tfrac{3}{8} n(1 + e^{-p_1 t} - e^{-p_2 t} - e^{-p_3 t}) , \tag{6b}$$

$$n_3 = \tfrac{3}{8} n(1 - e^{-p_1 t} - e^{-p_2 t} + e^{-p_3 t}) , \tag{6c}$$

$$n_4 = \tfrac{1}{8} n(1 - 3e^{-p_1 t} + 3e^{-p_2 t} - e^{-p_3 t}) , \tag{6d}$$

where

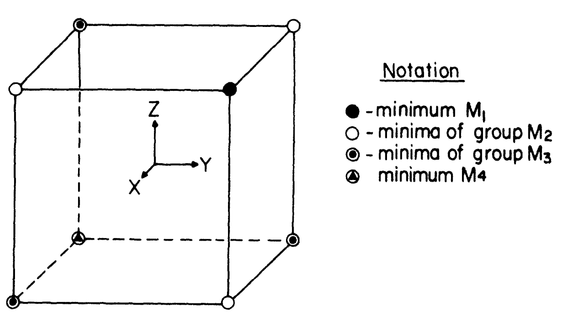

FIG. 2. Schematic picture of the minima of the anisotropy potential $F(\theta, \phi)$ [Eq. (1)] as residing at the corners of a cube that is bounded by the unit sphere. The subdivision of the minima into four groups is also shown.

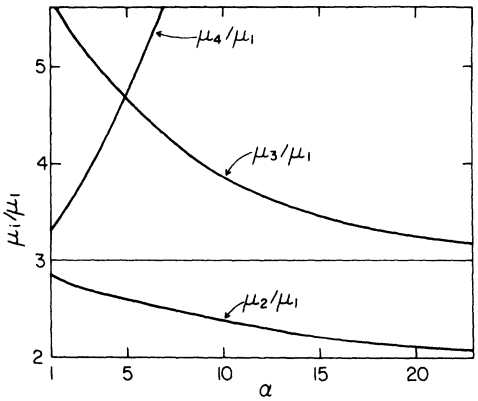

FIG. 3. Numerically computed ratios of the parameters μ of Fig. 1.

$$p_1 = \tfrac{2}{3}\nu , \quad p_2 = 2p_1 , \quad p_3 = 3p_1 . \tag{7}$$

In other words we have the prediction that as $\alpha \to \infty$, *three* (and *only* three) time constants are needed in order to describe the relaxation process and that the ratio between them is as given by (7). Still, as in EA, the remanent magnetization depends on time through p_1 only. In Fig. 3 we show the ratios $\mu_2/\mu_1 (= p_2/p_1)$, $\mu_3/\mu_1 (= p_3/p_1)$, and $\mu_4/\mu_1 (= p_4/p_1)$, as computed from the numerical results. The first two ratios seem to converge to (7), while the third one keeps growing up as predicted here.

III. ASYMPTOTIC CONSIDERATIONS $\alpha \to \infty$

A. Coordinate transformation

In the following we shall exploit the method used in EA for obtaining asymptotic expressions for time constants as $\alpha \to \infty$. For this it is convenient to transform by rotation from the (xyz) coordi-

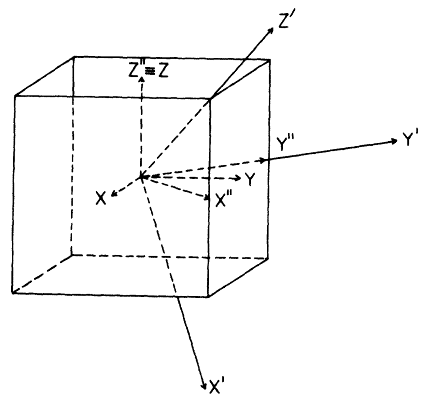

FIG. 4. Transformation from the (xyz) to the $(x'y'z')$ system [Sec. III A]. It is accomplished in two steps, (a), a rotation by an angle of $\frac{1}{4}\pi$ about the z axis that transforms (xy) into $(x''y'')$, (b), a rotation by an angle of $\cos^{-1}(3^{-1/2})$ about the y'' axis that transforms $(x''z)$ into $(x'z')$ where $y' = y''$.

nate system into a system $(x'y'z')$ where the z' axis coincides with the old [111] direction and the x' axis lies in the old $x=y$ plane. The transformation shown in Fig. 4 is accomplished in two steps: (a) a rotation by an angle of $\frac{1}{4}\pi$ about the z axis that transforms (xyz) into $(x''y''z'')$, as shown in Fig. 4 $(z''=z)$; (b) a rotation by an angle of $\cos^{-1}(3^{-1/2})$ about the y'' axis that transforms $(x''y''z'')$ into $(x'y'z')$, as shown in the figure $(y'=y'')$. The matrix of the complete transformation equals the ordered product of the matrices corresponding to the individual transformations. However, we are interested in the inverse transformation that, after carrying out the calculation, reads

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} . \tag{8}$$

Let us denote by α_1', α_2', and α_3' the direction cosines of $\vec{M}$ in the $(x'y'z')$ system (α_1' along x', etc.). Then the same matrix of Eq. (8) gives their law of transformation to the direction cosines α_1, α_2, and α_3 of $\vec{M}$ in the (xyz) system. Writing the anisotropy potential F [Eq. (1)] in terms of α_1, α_2, and α_3,

$$F=-K(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)\,, \tag{9}$$

we use (8) to express it in the new primed system

$$F=-2^{1/2}K\alpha_1'\alpha_3'(\tfrac{1}{3}\alpha_1'^2-\alpha_2'^2)+\tfrac{1}{4}(1-\alpha_3'^2)^2+\tfrac{1}{3}\alpha_3'^4\,. \tag{10}$$

Here and in the following F will denote the anisotropy *potential* and not a special *function* in a certain coordinate system. However we shall use notations like $F(\theta,\phi)$ that show the system in which F is expressed.

Denoting by ψ and χ the polar and azimuthal angles, respectively, in the primed (x',y',z') system, we have

$$\alpha_1'=\sin\psi\cos\chi\,, \tag{11a}$$

$$\alpha_2'=\sin\psi\sin\chi\,, \tag{11b}$$

$$\alpha_3'=\cos\psi\,. \tag{11c}$$

Substituting (11) into (10) we obtain an expression for F in terms of ψ and χ,

$$F(\psi,\chi)=-K[(2^{1/2}/3)\sin^3\psi\cos\psi\cos3\chi + \tfrac{1}{4}\sin^4\psi+\tfrac{1}{3}\cos^4\psi]\,. \tag{12}$$

The locations of the minima of F as arranged according to the groups of Fig. 2 are at the following (ψ,χ) values:

$$\begin{aligned} &(M_1):\ \psi=0,\ \chi=0;\\ &(M_2):\ \psi=2\sin^{-1}(3^{-1/2}),\ \chi=0,\tfrac{2}{3}\pi,\tfrac{4}{3}\pi;\\ &(M_3):\ \psi=2\cos^{-1}(3^{-1/2}),\ \chi=\tfrac{1}{3}\pi,\pi,\tfrac{5}{3}\pi;\\ &(M_4):\ \psi=\pi,\ \chi=0\,. \end{aligned} \tag{13}$$

The locations of the saddles between the groups of minima are at:

$$\begin{aligned} &(M_1-M_2):\ \psi=\sin^{-1}(3^{-1/2}),\ \chi=0,\tfrac{2}{3}\pi,\tfrac{4}{3}\pi;\\ &(M_2-M_3):\ \psi=\tfrac{1}{2}\pi,\ \chi=\tfrac{1}{6}\pi,\tfrac{3}{6}\pi,\tfrac{5}{6}\pi,\tfrac{7}{6}\pi,\tfrac{9}{6}\pi,\tfrac{11}{6}\pi;\\ &(M_3-M_4):\ \psi=\pi-\sin^{-1}(3^{-1/2}),\ \chi=\tfrac{1}{3}\pi,\pi,\tfrac{5}{3}\pi, \end{aligned} \tag{14}$$

and the locations of the maxima are at

$$\begin{aligned} &\psi=\cos^{-1}(3^{-1/2}),\ \chi=\tfrac{1}{3}\pi,\pi,\tfrac{5}{3}\pi;\\ &\psi=\pi-\cos^{-1}(3^{-1/2}),\ \chi=0,\tfrac{2}{3}\pi,\tfrac{4}{3}\pi\,. \end{aligned} \tag{15}$$

B. Asymptotic expressions $\alpha\to\infty$

As in Sec. III of EA we assume quasistationary conditions with a divergenceless current density $\vec{J}$ between the minima, namely,

$$\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi J_\psi)+\frac{1}{\sin\psi}\frac{1}{\partial\chi}J_\chi=0\,. \tag{16}$$

(Here and in the following we use expressions from EA where the polar and azimuthal angles θ and ϕ there should be replaced by ψ and χ here.) Integrating over χ from $-\frac{1}{3}\pi$ to $\frac{1}{3}\pi$, and using the three-fold symmetry of the system and of the initial conditions about the z' axis, we obtain

$$\frac{\partial}{\partial\psi}\int_{-\pi/3}^{\pi/3}J_\psi\sin\psi\,d\chi=-J_\chi\Big|_{-\pi/3}^{\pi/3}=0\,. \tag{17}$$

Hence,

$$\int_{-\pi/3}^{\pi/3}J_\psi\sin\psi\,d\chi=\tfrac{1}{3}I=\text{const.}, \tag{18}$$

where I is the *total* current of representative points flowing out of minimum M_1, or

$$I=-\dot{n}_1\,. \tag{19}$$

As in EA, we use the following relations, implied by (12):

$$F(\psi,\chi)=F(\psi,-\chi)\,, \tag{20a}$$

$$\frac{\partial F}{\partial\chi}(\psi,\chi)=-\frac{\partial F}{\partial\chi}(\psi,-\chi)\,, \tag{20b}$$

and the symmetry of the initial conditions. We shall also assume that

$$W(\psi,\chi)=W(\psi,-\chi) \tag{21}$$

in order to drop the term

$$\int_{-\pi/3}^{\pi/3} \frac{\partial F}{\partial \chi} W \, d\chi = 0 \tag{22}$$

which appears in (18) by virtue of the expression for J_ψ [Eq. (14a) in EA]. Finally we obtain

$$\frac{I}{6k'} = -\int_0^{\pi/3} e^{-\beta F} \frac{\partial}{\partial \psi} (e^{\beta F} W) \sin\psi \, d\chi \, . \tag{23}$$

Using the mean value theorem of integral calculus [see Eqs. (33) and (34) in EA],

$$\frac{I}{6k'} = -\sin\psi \frac{\partial}{\partial \psi} (e^{\beta F} W)_{\psi, \chi=\zeta} \int_0^{\pi/3} e^{-\beta F} d\chi \, , \tag{24}$$

where

$$0 \le \zeta = \zeta(\psi) \le \tfrac{1}{3}\pi \, . \tag{25}$$

Substituting (12) in (24) and carrying out the integration we get

$$\frac{I}{2k'} = -\pi \sin\psi \frac{\partial}{\partial \psi} (e^{\beta F} W)_{\psi, \chi=\zeta} \exp[\alpha(\tfrac{1}{4} \sin^4\psi + \tfrac{1}{3}\cos^4\psi)] \times I_0\left(\frac{\alpha 2^{1/2}}{3} \sin^3\psi \cos\psi\right) \, , \quad 0 \le \zeta \le \tfrac{1}{3}\pi \tag{26}$$

where I_0 is the modified Bessel function of the first kind and zeroth order. Rearranging terms and integrating over ψ we obtain

$$(e^{\beta F} W) \Big|_{\psi_1, \zeta(\psi_1)}^{\psi_2, \zeta(\psi_2)} = -\frac{I}{2\pi k'} \int_{\psi_1}^{\psi_2} G(\psi) H(\psi) \, d\psi \, , \tag{27}$$

where

$$G(\psi) = \frac{\exp(\frac{1}{3}\alpha 2^{1/2} \sin^3\psi \cos\psi)}{\sin\psi \, I_0(\frac{1}{3}\alpha 2^{1/2} \sin^3\psi \cos\psi)} \, , \tag{28}$$

$$H(\psi) = \exp[-\alpha(\tfrac{1}{3} 2^{1/2} \sin^3\psi \cos\psi + \tfrac{1}{4}\sin^4\psi + \tfrac{1}{3}\cos^4\psi)] \, . \tag{29}$$

In the limit $\alpha \to \infty$ we choose ψ_1 close enough to zero and ψ_2 close enough to (and smaller than) $2\sin^{-1}(3^{-1/2})$ [that is, close to the minimum M_1 at $\psi = 0$ and the minimum of group M_2 at ($\psi = 2\sin^{-1}(3^{-1/2})$, $\chi = 0$), respectively] such that a thermal equilibrium obtains in $(0, \psi_1)$ and in ($\psi_2, 2\sin^{-1}(3^{-1/2})$; $\chi \lesssim \psi_1$). Still, ψ_1 is assumed to obey $e^{-\beta F_1} \gg e^{-\beta F(\psi_1, \chi)} \gg e^{-\beta F_2}$ where F_1 and F_2 are the values of F at the minimum M_1 and at the saddle separating ψ_1 from ψ_2, respectively. Analogous assumption holds for ψ_2. Now for a small argument, $I_0(z) \sim \exp(z) \sim 1$, and for a large argument, $I_0(z) \sim e^z/(2\pi z)^{1/2}$. Hence, $H(\psi)$ changes much more rapidly than $G(\psi)$ in (ψ_1, ψ_2) as $\alpha \to \infty$. Moreover, most of the contribution to (27) comes from the vicinity of the maximum of $H(\psi)$ at $\psi = \sin^{-1}(3^{-1/2})$ (where a saddle of F exists at $\chi = 0$). Since $H(\psi) > 0$ and $G(\psi)$ is continuous in (ψ_1, ψ_2), we use the mean value theorem of integral calculus to take $G(\psi)$ out of the integral sign, and *only then* approximate $\psi_1 \sim 0$ and $\psi_2 \sim 2\sin^{-1}(3^{-1/2})$. We obtain

$$(e^{\beta F} W) \Big|_{\psi_1, \zeta_1}^{\psi_2, \zeta_2} = -\frac{I}{2\pi k'} G(\xi) \int_0^{2\sin^{-1}(3^{-1/2})} H(\psi) \, d\psi \, , \tag{30}$$

where

$$0 < \xi < 2\sin^{-1}(3^{-1/2}) \, , \quad \zeta_i = \zeta(\psi_i) \tag{31}$$

and

$$\xi \to \sin^{-1}(3^{-1/2}) \quad \text{for} \quad \alpha \to \infty \, . \tag{32}$$

In order to evaluate the integral in (30) we note that

$$H(\psi) = e^{-\beta F(\psi, \chi=0)} \tag{33}$$

so that the path of integration over ψ with the limits $(0, 2\sin^{-1}(3^{-1/2}))$ corresponds in the original coordinate system to a path of integration over θ with the limits $(\cos^{-1}(3^{-1/2}), -\cos^{-1}(3^{-1/2}))$ and with $\phi = \frac{1}{4}\pi$. This can be readily seen by looking at Figs. 2 and 3. Formally we can make the transformation

$$\psi = \theta + \cos^{-1}(3^{-1/2}) \, , \tag{34}$$

substitute it in $H(\psi)$ [Eq. (29)] in (30) and obtain

$$(e^{\beta F} W) \Big|_{\psi_1, \zeta_1}^{\psi_2, \zeta_2} = -\frac{I}{2\pi k'} G(\xi) \times \int_{\cos^{-1}(3^{-1/2})}^{-\cos^{-1}(3^{-1/2})} \exp[-\tfrac{1}{4}\alpha(\sin^2 2\theta + \sin^4\theta)] \, d\theta \, , \tag{35}$$

which can be brought to the form

$$(e^{\beta F} W) \Big|_{\psi_1, \zeta_1}^{\psi_2, \zeta_2} = -\frac{I}{\pi k'} G(\xi) e^{-\alpha/3} \times \int_{\pi/2}^{-\cos^{-1}(3^{-1/2})} \exp[\tfrac{1}{4} 3\alpha(1/3 - \cos^2\theta)^2] \, d\theta \, . \tag{36}$$

Defining a change of variable,

$$u \equiv (\tfrac{1}{3} - \cos^2\theta)^2 \, , \tag{37}$$

we obtain, after some manipulations,

$$(e^{\beta F} W) \Big|_{\psi_1, \zeta_1}^{\psi_2, \zeta_2} = -\frac{I}{4\pi k'} G(\xi) e^{-\alpha/3} \times \int_0^{1/9} D(u) \left(\frac{\exp(\frac{3}{4}\alpha u)}{[u(\frac{1}{9} - u)]^{1/2}}\right) du \, , \tag{38}$$

where

$$D(u) \equiv (\tfrac{1}{3} + u^{1/2})^{1/2} / (\tfrac{2}{3} + u^{1/2})^{1/2} \, . \tag{39}$$

We again use the mean value theorem of integral calculus in order to take $D(u)$ out of the integral sign in (38). Doing this and carrying out the integration over the remaining term, we obtain

$$(e^{\beta F}W)\Big|_{\psi_1,\zeta_1}^{\psi_2,\zeta_2} = -\frac{I}{4k'}G(\xi)D(\omega)\,e^{-7\alpha/24}I_0(\tfrac{1}{24}\alpha)\,, \quad (40)$$

where

$$0 \le \omega \le \tfrac{1}{9}\,. \quad (41)$$

Actually, in all our manipulations from (27) on we could have kept $G(\psi)$ under the integral sign and take it out only now together with $D(u)$. (So that only now the integration limits would have been approximated by their final values.) If we take this course, then ξ and ω are not independent but are related through (34) and (37),

$$\omega = \{\tfrac{1}{3} - \cos^2[\xi + \cos^{-1}(3^{-1/2})]\}^2\,. \quad (42)$$

When $\alpha \to \infty$, we use (32) and find

$$\omega \to \tfrac{1}{9} \text{ for } \alpha \to \infty\,. \quad (43)$$

This is consistent with (38) and (40) since in the limit $\alpha \to \infty$ $D(u)$ is nearly a constant compared with the remaining term in the integrand in (38), and most of the contribution to the integral comes from the vicinity of $u = \frac{1}{9}$.

In order to evaluate the left-hand side in (40) we look at times $t \sim 0$ so that $W(\psi_2, \zeta_2) \sim 0$ (that is, $n_2 \sim 0$). We replace I by $-\dot{n}_1$ [Eq. (19)] and use the same method used in EA to evaluate $W(\psi_1, \zeta_1)$. We find

$$(e^{\beta F}W)\Big|_{\psi_1,\zeta_1}^{\psi_2,\zeta_2} = -\frac{2\alpha}{3\pi}\,n_1 e^{-\alpha/3}\,, \quad t \sim 0\,. \quad (44)$$

Substituting (44) into (40) we obtain

$$\dot{n}_1 = -\left(\frac{8k'\alpha}{3\pi G(\xi)D(\omega(\xi))}\,\frac{\exp(-\frac{1}{24}\alpha)}{I_0(\frac{1}{24}\alpha)}\right)n_1\,. \quad (45)$$

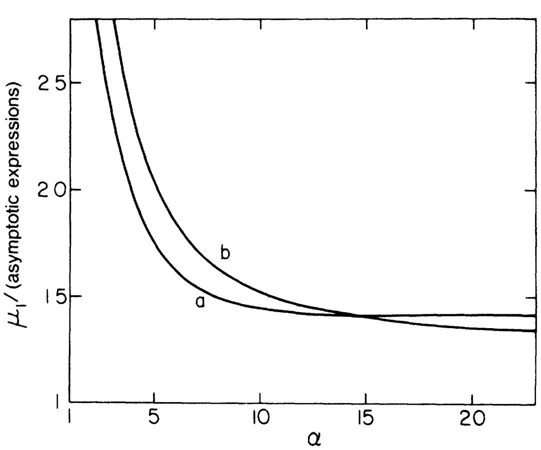

FIG. 5. Ratio of the numerically computed time constant μ_1 [Eqs. (6) and (2) of EA] to the asymptotic expressions (a), $(16/9\pi\times 2^{1/2})\,I_0(\frac{2}{27}\alpha)\exp(-\frac{1}{24}\alpha)/\exp(\frac{2}{27}\alpha)I_0(\frac{1}{24}\alpha)$ [Eq. (47)], (b), $(2^{1/2}\times 2/3\pi)e^{-\alpha/12}$ [Eq. (48)]. The curves are plotted as functions of the reduced barrier energy α [Eqs. (7) and (1b) of EA].

Comparing (45) with (5a) at $t=0$, we see that the expression within the curly brackets in (45) should equal $\nu = (3/2)p_1$ [by (7)]. Using the relations [Eqs. (6), (15), and (16) of EA]

$$\mu_1 = (2M_s/\gamma_0 K)p_1\,, \quad k' = \gamma_0/2M_s\beta\,, \quad (46)$$

and substituting (32), we obtain

$$\mu_1 \underset{\alpha\to\infty}{\sim} \frac{16}{9\pi\sqrt{2}}\,\frac{I_0(\frac{2}{27}\alpha)}{\exp(\frac{2}{27}\alpha)}\,\frac{\exp(-\frac{1}{24}\alpha)}{I_0(\frac{1}{24}\alpha)}\,. \quad (47)$$

Using $I_0(z) \underset{z\to\infty}{\sim} e^z/(2\pi z)^{1/2}$ we get further

$$\mu_1 \underset{\alpha\to\infty}{\sim} \frac{2\times 2^{1/2}}{3\pi}e^{-\alpha/12}\,. \quad (48)$$

This expression resembles Eq. (50) of EA in that the preexponential factor does not depend on α, unlike the uniaxial case. The analog of (53) of EA is

$$\mu_1 \underset{\alpha\to\infty}{\sim} [4\times 2^{1/2}/3\pi(\gamma_0\eta M_s + 1/\gamma_0\eta M_s)]\exp(-\tfrac{1}{12}\alpha) \quad (49)$$

that was also independently derived (with a different preexponential factor) by Smith and de Rozario.[2] In Fig. 5 we show the ratios of the numerically computed μ_1 to (47) and (48). As in EA they may oscillate before converging into their asymptotic value.

IV. DISCUSSION

The discussion at the end of EA related to the validity of the asymptotic formulas already at the region of α considered there apply here too. In

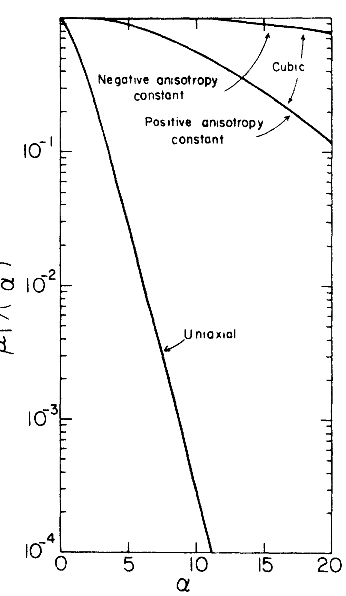

FIG. 6. Ratio $\mu_1/(2/\alpha)$ as a function of α for the anisotropies as marked on the figure.

particular it may be concluded that in the region considered here, the approximations $\alpha \to \infty$ and $\xi = \sin^{-1}(3^{-1/2})$ are not as yet *a priori* applicable though they give reasonably good practical results. This causes part of the large deviation of the curves in Fig. 5 from the value 1, but part of it is certainly due to the assumption (21). As in EA, (49) can be obtained by assuming $\eta \gg 1/\gamma_0 M_s$ instead of (21). The numerical results, however, are obtained for $\eta = 1/\gamma_0 M_s$.

Considering the expression for E_B [Eqs. (1) in EA], it is obvious that for a given value of α, asymptotic conditions are realized much earlier for a uniaxial anisotropy than for cubic anisotropy, and in the latter case, much earlier for a positive anisotropy constant than for a negative one. For example, a value of $\alpha = 24$ corresponds to $E_B = 24kT$ in the uniaxial case and to $E_B = 6$ and $2kT$ in the cubic cases.

An indication to the applicability of the approximation $\alpha \to \infty$ may be the departure from the validity of the approximation[2]

$$\mu_1 \underset{\alpha \to 0}{\sim} 2/\alpha \tag{50}$$

for all three cases. In Fig. 6 we show the ratio $\mu_1/(2/\alpha)$ for the three cases. It is seen that in case of a cubic anisotropy with a negative constant this approximation is excellent up to $\alpha \sim 10$. For a positive constant, such a good agreement is obtained only at $\alpha < 3$ and for the uniaxial case (50) is valid only below $\alpha = 0.1$.

The difference in the rate of approach to the asymptotic ($\alpha \to \infty$) expressions between the uniaxial case and the cubic cases does not lie only with the barrier height. The asymptotic expressions in the cubic cases do not only assume a high barrier but also assume that the current of representative points on the unit sphere is limited to the very bottom of the potential valleys that connect the minima through the saddle points. In the uniaxial case there is no such a limitation. This fact too gives its contribution to the slow rate of realizing asymptotic conditions in the cubic cases.

We may also add that the different arrangement of the minima in each of the three cases is the reason for the different number of time constants that should be taken into account for a higher barrier. It is only *one* for uniaxial anisotropy and it is *two* and *three* for cubic anisotropy with a positive and negative constant, respectively.

ACKNOWLEDGMENTS

We are grateful to Dr. D. A. Smith and F. A. de Rozario for sending us a preprint of their work before publication.

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