Real-space renormalization of the self-avoiding walk by a linear transformation

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We show how the Niemeijer–Van Leeuwen real-space renormalization can be used to study the self-avoiding walk on a lattice. We first establish that the de Gennes–des Cloizeaux equivalence between this problem and an n-component spin system (as $n \to 0$) remains valid if one takes discrete-valued cubically symmetric spins. A transformation for the self-avoiding walk is then obtained by letting n tend to zero in the transformation for the spin system. In the $n \to 0$ limit the interaction constants of the spin system are shown to correspond to the weights of the elementary segments of a self-avoiding walk. A general technique for practical renormalization calculations in the $n \to 0$ limit is given. As an example we consider a linear transformation depending upon two parameters, and apply it to a triangular lattice in second-order cumulant approximation. The results agree well with data from other sources. Our findings concerning the parameter dependence of linear renormalization transformations confirm and extend those of Bell and Wilson based on the Gaussian model.

I. INTRODUCTION

A few years ago Niemeijer and Van Leeuwen[1] introduced a new way of implementing the Wilson renormalization-group theory. These authors renormalized an Ising spin-$\frac{1}{2}$ Hamiltonian by grouping the spins together into cells and associating with each cell a new "renormalized" spin. The renormalization transformation is then the mapping of the interactions between the original spins onto those between the new spins. A study of this transformation yields the physical properties in which one is interested. These properties are usually the critical exponents but may also be the complete equation of state.[2] The Niemeijer–Van Leeuwen (NvL) method has the advantage that it can be applied to a system with fixed dimensionality. Thus it circumvents the ϵ expansion, which for dimensions below four becomes progressively imprecise. It offers furthermore the conceptual advantage of using the spatial representation of the Hamiltonian. Since the NvL method was introduced, many authors have been concerned with further investigating its properties and applying it to a variety of spin and related lattice systems. For a detailed description of the method and references to applications, our Refs. 3 and 4 may be consulted. The purpose of this paper is to show how the NvL method can be extended to study the properties of the self-avoiding walk on a lattice, and we present results obtained by applying it to a two-dimensional triangular lattice. The feasibility of this approach was already reported in a short communication.[5] For a detailed description of the self-avoiding walk problem and the various ways it has been studied heretofore we refer to the review articles by Domb[6] and Fried,[7] and the books by Yamakawa[8] and Flory.[9]

Our method is based upon the equivalence known to exist between the self-avoiding walk (or excluded volume) problem and an n-component spin system in the limit of zero components ($n \to 0$). This equivalence was pointed out by de Gennes,[10] who considered a single self-avoiding walk, and generalized by des Cloizeaux[11] to a system of self-avoiding walks (a solution of polymer chains) that are also mutually exclusive. Modified derivations of the de Gennes–des Cloizeaux equivalences were given by Bowers and McKerrel,[12] Emery,[13] Fisher and Jasnow,[14] and, in a particularly transparent way, by Sarma.[15] All these authors used continuous, spherically symmetric spins. Due to this equivalence, results obtained for n-component spin systems (such as critical exponents) directly apply to the corresponding quantities in the excluded volume problem if one sets $n = 0$. In this way, e.g., the ϵ expansion[16] has led to estimates of the critical exponents associated with the excluded volume problem (see Sec. VI).

In this paper we show how the NvL method can yield results that are complementary to those of the ϵ expansion and those obtained by more classical methods.[6-9] The NvL method requires that summations on small numbers of spin variables can be carried out exactly. Since such is not the case for the usual non-Gaussian spins,[12-16] we are led to use discrete-valued n-component spins with cubic symmetry.[17] For $n = 1$ these reduce to spin-$\frac{1}{2}$ Ising spins. In Sec. II we define a spin Hamiltonian and show how for $n \to 0$ the excluded volume problem is recovered. The connection with the n-component Potts model[18] is mentioned. In Sec. III we set up a classification of spin operators, based on their behavior in the limit $n \to 0$, which will serve throughout later sections. In Secs. IV and V we show how one can construct a renormalization

transformation for the self-avoiding walk by letting n tend to zero in a transformation for the spin system. The behavior of the $n=0$ transformation near its fixed point determines the properties of the self-avoiding walk. It is found that for $n=0$ the transformation takes place between the weights of the elementary fragments of certain self-avoiding walks. These walks are of a more general type than the original one, and determine the embedding of the self-avoiding walk into a larger class of excluded volume problems.

We apply our theory to the renormalization of a two-dimensional triangular lattice. We use a so-called linear transformation which depends upon two free parameters. In Sec. VI we study a cumulant expansion to second order, involving seven different interaction constants. The resulting approximate values for the critical exponents and the connective constant of the self-avoiding walk are compared to values obtained by other methods and found in good agreement. Our results also shed light on the behavior of linear renormalization transformations in general. We compare our findings concerning the parameter dependence of the transformation to those of Bell and Wilson[19] for the Gaussian model.

II. EQUIVALENCE BETWEEN A SYSTEM OF n-COMPONENT CUBIC SPINS AND THE EXCLUDED VOLUME PROBLEM

We consider a d-dimensional lattice of N sites $i=1,2,\ldots,N$, which is periodic in each direction. The coordinate of site i will be $\vec{r}_i$. At each site there is an n-component spin $\vec{\sigma}_i=(\sigma_{i1},\sigma_{i2},\ldots,\sigma_{in})$ of fixed length n, which has access only to the $2n$ discrete states:

$$\vec{\sigma}_i=(0,\ldots,0,\pm n^{1/2},0,\ldots,0)\,. \tag{2.1}$$

For $n=1$ we recover spin-$\frac{1}{2}$ Ising spins. It is easy to verify that the cubically symmetric spins defined by (2.1) have the following properties.

$$\sum_{\alpha=1}^{n}\sigma_{i\alpha}^2=n\,,\quad \sigma_{i\alpha}\sigma_{i\beta}=\delta_{\alpha\beta}\sigma_{i\alpha}^2\,,\quad \sigma_{i\alpha}^p=n\sigma_{i\alpha}^{p-2}\quad (p\geq 3)$$

$$(2n)^{-1}\sum_{\vec{\sigma}_i}\sigma_{i\alpha}^2=(2n)^{-1}\sum_{\vec{\sigma}_i}1=1\,, \tag{2.2a)-(2.2e)}$$

$$\lim_{n\to 0}(2n)^{-1}\sum_{\vec{\sigma}_i}\sigma_{i\alpha}^{2p}=\delta_{p0}+\delta_{p1}\,.$$

These identities will be used frequently in the sequel.

We consider the Hamiltonian

$$\mathcal{H}_{\rm NN}=K\sum_{\langle i,j\rangle}\sum_{\alpha}\sigma_{i\alpha}\sigma_{j\alpha}\,, \tag{2.3}$$

where the first summation runs through all pairs of nearest neighbors (NN) on the lattice. The partition function will be defined as

$$Z_{\rm NN}=(2n)^{-N}\sum_{\{\vec{\sigma}_i\}}\exp\mathcal{H}_{\rm NN}\,. \tag{2.4}$$

The normalization factor $(2n)^{-N}$ amounts to adding a term $N\ln 2n$ to the free energy, which is just the entropy of N free n-component spins and becomes singular as $n\to 0$. We shall see below that the remainder of the free energy is well-behaved for $n\to 0$. Due to this normalization the infinite temperature limit $K\to 0$ of the free energy is zero for all n. This of course does not affect any of those quantities (such as correlation functions) that are calculated as derivatives of the free energy. Neither does it affect any of the singularities that the free energy might have as a function of K.

The relation between the Hamiltonian (2.3) and the self-avoiding walk has been discussed by several authors[10-15] for the case where the $\vec{\sigma}$'s are continuous, spherically symmetric variables. The proof for the present case is very similar; it must be given, however, since there is no *a priori* certainty that for $n\to 0$ the properties of (2.3) are independent of the spin symmetry.

To calculate $Z_{\rm NN}$ we expand

$$Z_{\rm NN}=(2n)^{-N}\sum_{\{\vec{\sigma}_i\}}\left(1+\mathcal{H}+\frac{1}{2!}\mathcal{H}^2+\ldots\right). \tag{2.5}$$

The Lth term in the expansion is a sum of contributions of general type

$$Z(G_L)=(2n)^{-N}$$

$$\times\sum_{\{\vec{\sigma}_i\}}K^L\sum_{\alpha_1,\ldots,\alpha_L}(\sigma_{i_1\alpha_1}\sigma_{j_1\alpha_1})\ldots(\sigma_{i_L\alpha_L}\sigma_{j_L\alpha_L})\,. \tag{2.6}$$

Here G_L stands for the set $\{\langle i_1,j_1\rangle,\ldots,\langle i_L,j_L\rangle\}$, or alternatively for the graph which one obtains by drawing all bonds $(i_1,j_1),\ldots,(i_L,j_L)$ on the lattice. Evidently $Z(G_L)$ vanishes unless all vertices of G_L are even. Due to (2.2b), the summations on $\alpha_1,\ldots,\alpha_L$ in (2.6) can be replaced by a single summation for each connected subgraph of G_L. Consider first a connected subgraph g. A vertex at site i where $2p_i+2$ bonds meet is represented in the summand of (2.6) by a factor $\sigma_{i\alpha}^{2p_i+2}$. Using (2.2c) we then have

$$Z(g)=(2n)^{-N}\sum_{\{\vec{\sigma}_i\}}K^{L(g)}n^{p(g)}\sum_{\alpha}\sigma_{i_1\alpha}^2\sigma_{i_2\alpha}^2\ldots\sigma_{i_{M(g)}\alpha}^2\,, \tag{2.7}$$

where $L(g)$ is the number of bonds in g, $M(g)$ is the number of distinct vertices in g, and

$$p(g)=\sum_{m=1}^{M(g)} p_{i_m}. \tag{2.8}$$

The $\vec{\sigma}$ average in (2.7) is easily carried out with the aid of (2.2b)–(2.2d) and we find

$$Z(g)=K^{L(g)}n^{p(g)+1}. \tag{2.9}$$

The contribution of a graph G_L with $C(G_L)$ disconnected subgraphs is obtained as the product of the subgraph contributions. Hence,

$$Z(G_L)=K^L n^{\sum_g p(g)+C(G_L)}. \tag{2.10}$$

Thus the order in n of any graph follows directly from its number of disconnected subgraphs and the multiplicities of its vertices. The trivial graph without any bond gives a contribution of one. Graphs of order n are necessarily connected and can only have vertices with two bonds meeting, i.e., such graphs form nonintersecting closed loops on the lattice. Graphs of order n^2 may either consist of two disconnected nonintersecting closed loops or of a single loop that once intersects itself.

Since each graph of L bonds may be formed in $L!$ ways, we have from (2.5), (2.6), and (2.10)

$$\begin{aligned} Z_{NN}&=\sum_{L=0}^{\infty}K^L\sum_{G_L}Z(G_L)\\ &=Z_0+nZ_1+\tfrac{1}{2}n^2Z_2+\cdots, \end{aligned} \tag{2.11}$$

where Z_k is the sum of all contributions of order n^k. For $k=0$ we only have the trivial contribution, and hence

$$Z_0=1. \tag{2.12}$$

We thus find for the free energy *per spin component* F_{NN} the expression

$$\begin{aligned} -F_{NN}&\equiv\frac{1}{n}\ln Z_{NN}\\ &=Z_1+\tfrac{1}{2}n(Z_2-Z_1^2)+\cdots. \end{aligned} \tag{2.13}$$

Here Z_1 contains a contribution K^L for each nonintersecting closed loop of L bonds, and hence

$$-N^{-1}\lim_{n\to 0}F_{NN}=\sum_{L=0}^{\infty}c_L K^L, \tag{2.14}$$

where c_L is the number of self-avoiding closed loops that can be drawn on the lattice per lattice site.[20] Since the logarithm eliminates all disconnected graphs, the higher-order terms in (2.13) can be expressed as contributions from connected graphs with one self-intersection, two self-intersections, etc. The inverse of the value K_c at which (2.14) diverges is the critical temperature of the spin system, or, in the terminology of self-avoiding walks, the connective constant of the lattice.

If in (2.4) one adds a magnetic field $H\sum_i\sigma_{i\gamma}$ to the Hamiltonian, then expansion in powers of H generates all correlation functions. For the case of spherically symmetric spins des Cloizeaux has shown[11] that the $2p$ point correlation function corresponds to the Laplace transform of the number of configurations of p nonintersecting chains that pairwise connect these points. Thus for $n\to 0$ the partition function in a magnetic field corresponds to the grand-canonical partition function of a polymer solution. The physical implications of this have been discussed in Ref. 11. When starting from the cubic Hamiltonian and expanding as above, one can derive des Cloizeaux's equivalence in a completely analogous way. We shall not do this, but merely consider two special cases for illustration and future use.

The average of an arbitrary spin function $A(\{\vec{\sigma}_i\})$ with respect to the Hamiltonian $\mathcal{H}_{NN}$ will be defined by

$$\langle A\rangle=Z_{NN}^{-1}(2n)^{-N}\sum_{\{\vec{\sigma}_i\}}A(\{\vec{\sigma}_i\})\exp\mathcal{H}_{NN}. \tag{2.15}$$

Let us consider the pair correlation function $G(\vec{r}_k-\vec{r}_l)\equiv\langle\sigma_{k\gamma}\sigma_{l\gamma}\rangle$. Using the definition (2.15) and expanding the exponential as above, one can again express $G(\vec{r}_{kl})$ as a sum of graph contributions. All nonvanishing contributions stem from graphs with odd vertices at sites k and l and even vertices everywhere else. In the limit $n\to 0$ one finds

$$\lim_{n\to 0}G(\vec{r})=\sum_{L=0}^{\infty}g_L(\vec{r})K^L, \tag{2.16}$$

where $g_L(\vec{r})$ is the number of nonintersecting walks of L steps that can be drawn on the lattice between sites a distance $\vec{r}$ apart.

By summing (2.16) on all sites $\vec{r}$ one obtains for the susceptibility

$$\lim_{n\to 0}\chi=\sum_{L=0}^{\infty}g_L K^L, \tag{2.17}$$

where g_L is the number of self-avoiding walks of L steps starting from the origin. If χ diverges as $(K_c-K)^{-\gamma}$, then one finds by an inverse Laplace transformation

$$g_L\sim K_c^L L^{\gamma-1}\quad(L\to\infty). \tag{2.18}$$

This interprets the exponent γ in terms of self-avoiding walk properties. In a similar way one can show

$$\langle R^2\rangle\equiv\sum_{\vec{r}}\sum_{L=0}^{\infty}\vec{r}^2 g_L(\vec{r})K^L\sim L^{2\nu}\quad(L\to\infty), \tag{2.19}$$

$$G(\vec{r}) \sim r^{-d+2-\eta} \quad (K = K_c, r \to \infty). \tag{2.20}$$

We secondly consider the energy-energy correlation

$$G_E(\vec{r}_{kl}) \equiv \frac{1}{n}\left(\langle \sigma_{k\alpha}^2 \sigma_{l\alpha}^2 \rangle - \langle \sigma_{k\alpha}^2 \rangle \langle \sigma_{l\alpha}^2 \rangle\right). \tag{2.21}$$

One readily shows that

$$\lim_{n\to 0} G_E(\vec{r}) = \sum_L c_L(\vec{r}) K^L, \tag{2.22}$$

where $c_L(\vec{r})$ is the number of closed nonintersecting loops of L bonds passing both through the origin and through $\vec{r}$. When $G_E(\vec{r})$ is summed on $\vec{r}$ each loop of L steps contributes $L(L-1)$ times, and hence

$$\lim_{n\to 0} \sum_{\vec{r}} G_E(\vec{r}) = \sum_L c_L L(L-1) K^L$$

$$= -N^{-1} \lim_{n\to 0} K^2 \frac{d^2 F_{NN}}{dK^2}. \tag{2.23}$$

Using that $F_{NN} \sim (K_c - K)^{2-\alpha}$ for $K \approx K_c$, we finally have, for $n \to 0$,

$$G_E(\vec{r}) \sim r^{-d+\alpha/\nu} \quad (K = K_c, r \to \infty). \tag{2.24}$$

Relations (2.20) and (2.24) will play a role in later sections.

For completeness we briefly mention the relation of our spin system to the Potts model. An n-component Potts model[18] is a system of variables α_i that may be in n different states $\alpha_i = 1, 2, \ldots, n$. The interaction Hamiltonian is

$$\mathcal{H}_P = K \sum_{\langle i,j \rangle} \delta_{\alpha_i \alpha_j}. \tag{2.25}$$

In the spin system, let α_i be the nonzero component of spin $\vec{\sigma}_i$, and let ϵ_i denote the sign of this component. Then (2.3) may be rewritten as

$$\mathcal{H}_{NN} = K \sum_{\langle i,j \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j = nK \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j \delta_{\alpha_i \alpha_j}. \tag{2.26}$$

Thus we have a modified Potts model in which the variable at site i is characterized not only by its state α_i, but also by its sign ϵ_i. Crossover properties between the Potts model (2.25) and the cubic spin Hamiltonian (2.26) have recently been discussed by Aharony.[21] Real space renormalization has been applied to the Potts model by several authors,[22-24] who were especially interested in the case $n = 1$, known[25,26] to describe a model for percolation.

III. CLASSIFICATION OF SPIN OPERATORS

In the following sections we shall generate a variety of spin operators by successive application of renormalization transformations to the Hamiltonian $\mathcal{H}_{NN}$ of Sec. II. It will prove useful to classify the spin operators beforehand according to some of their general characteristics.

The initial Hamiltonian $\mathcal{H}_{NN}$ is invariant under permutation of the spin components α and under sign reversal of each of the components individually. If the renormalization transformation conserves these invariances, then the most general operators that we can expect to generate are of type

$$A(i_1, \ldots, i_{2m}; j_1, \ldots, j_l) = \sum_\alpha \sigma_{i_1\alpha} \ldots \sigma_{i_{2m}\alpha} \sigma_{j_1\alpha}^2 \ldots \sigma_{j_l\alpha}^2, \tag{3.1}$$

(where all site indices are different), and products of such operators. If initially there is a magnetic field $H\sum_i \sigma_{i\gamma}$ along one of the spin components, then one also generates products of $\sigma_{i\gamma}$ and operators of type (3.1). We shall not consider any odd operators here, however. The operators (3.1) are symmetric under permutation of the i indices as well as under permutation of the j indices. We represent such an operator by a lattice graph consisting of open circles on the sites $i_1, i_2, \ldots$, and black circles on the sites $j_1, j_2, \ldots$, as shown in Fig. 1(a). The sites have been connected by bonds in an arbitrary way in order to indicate that the corresponding variables in (3.1) occur under a single summation sign. The product of two operators of type (3.1) that have site indices in common reduces again to this type (possibly multiplied by factors of n) when one applies relations (2.2b) and (2.2c). The product of two operators not having any site indices in common cannot be further reduced; such an operator is represented by a disconnected graph, as in Fig. 1(b).

An operator of type (3.1) which is odd in $2m$ spin variables will be called a *$2m$-leg operator*. A 0-leg operator will also be called a *closed-loop operator*. All operators in $\mathcal{H}_{NN}$ are two-leg operators.

The interpretation of (3.1) in terms of a generalized excluded volume problem is obvious: this operator corresponds to a fragment of a self-avoiding walk in which the sites $j_1, j_2, \ldots$ (black

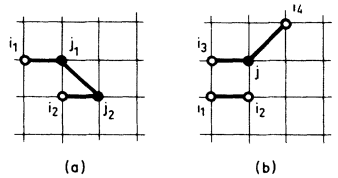

FIG. 1. (a) Graph corresponding to $A(i_1, i_2; j_1, j_2)$; (b) graph corresponding to $A(i_1, i_2)A(i_3, i_4; j)$.

circles in the graph) have been visited and are excluded from further visits, whereas the "leg" sites $i_1, i_2, \ldots$ (open circles) may still be linked to the legs of other walk fragments. Multiplication of operators corresponds to the superposition of walk fragments, with relation (2.2c) giving a factor n whenever a doubly visited site occurs. The limit $n \to 0$ then eliminates such products.

We shall now define what we mean by the order in n of an operator. Let $\mathcal{H}$ stand for a Hamiltonian which is an arbitrary sum of products of operators of type (3.1). As in Sec. II, one easily shows that the corresponding partition function $Z[\mathcal{H}]$ differs from one only by terms of order n, each of which is the contribution of a closed-loop operator. We shall call an arbitrary operator $\mathcal{O}$ *of order* n^p if, when we add $\mathcal{O}$ to a Hamiltonian $\mathcal{H}$, the new contribution to the partition function is at most of order n^p. More formally, $\mathcal{O}$ is of order n^p if

$$\max_{\mathcal{H}} (Z[\mathcal{H}+\mathcal{O}] - Z[\mathcal{H}]) \sim n^p \quad (n \to 0), \tag{3.2}$$

where $\mathcal{H}$ is in the class defined above. As $n \to 0$, an operator of order n gives a finite contribution to the free energy per component $F \equiv -n^{-1} \ln Z$. An operator of order n^p with $p \geq 2 (p \leq 0)$ gives a vanishing (diverging) contribution to F.

To find the order in n of a given operator $\mathcal{O}$, one just has to investigate in which ways products of $\mathcal{O}$ with operators of type (3.1) can yield closed-loop operators. In the evaluation of such products factors of n may be introduced by relation (2.2c); the lowest possible number of factors n determines the order of $\mathcal{O}$. One thus finds that all operators (3.1) themselves are of order n. For the order of a product of operators one easily establishes the following properties, which we shall number for future reference.

(P1) *The product of two operators of order* n^{p_1} *and* n^{p_2} *is of order* $n^{p_1+p_2-1}$ *or higher.* This means in particular that the product of two operators of order n is of order n or higher.

(P2) *The product of a closed-loop operator of order n and an operator of order* n^p *is of order* n^{p+1} *or higher.* In particular, the product of two nonoverlapping closed-loop operators of order n is of order n^2.

In Sec. II we saw that the partition function, once normalized as in (2.4), gives rise to a free energy that is well-behaved for $n \to 0$. In general this will be so as long as the Hamiltonian $\mathcal{H}$ is a sum of operators of order n or higher. Such Hamiltonians may be called self-avoiding walk Hamiltonians. A Hamiltonian containing operators of order n^p with $p \geq 0$ can no longer be given, along the lines of Sec. II, a natural interpretation in terms of self-avoiding walks. In Sec. IV we shall show how one can construct renormalization transformations that leave the subspace of self-avoiding walk Hamiltonians invariant. Repeated application of such a transformation to the initial self-avoiding walk Hamiltonian (2.3) leads to a fixed point problem in that subspace. It is this problem that we shall study.

IV. RENORMALIZATION TRANSFORMATION

In this section we shall construct a renormalization transformation for a system of n-component spins, and show how one can take the limit $n \to 0$ of such a transformation. The renormalization procedure begins by choosing on the lattice cells of c spins (c a low number ≥ 2) in such a way that the lattice of the cells differs from the original lattice only by a scale factor $c^{1/d}$. We shall label the cells by an index a and indicate the spins in a cell as $\vec{\sigma}_1^a, \vec{\sigma}_2^a, \ldots, \vec{\sigma}_c^a$. We define for each cell a cell spin $\vec{\mu}_a \equiv (\mu_{a1}, \mu_{a2}, \ldots, \mu_{an})$ which, just as the spins $\vec{\sigma}_i$, can be in any of the $2n$ states

$$\vec{\mu}_a = (0, \ldots, 0, \pm n^{1/2}, 0, \ldots, 0) . \tag{4.1}$$

Consequently the identities (2.2a)–(2.2e) also apply to the spins $\vec{\mu}_a$.

The renormalization transformation is represented by a matrix P which is a function of the site spins $\vec{\sigma}_i$ and the cell spins $\vec{\mu}_a$. We shall take P to have the usual factorized form[4,27]

$$P(\{\vec{\mu}_a\}, \{\vec{\sigma}_i\}) = \prod_a p(\vec{\mu}_a, \{\vec{\sigma}_i^a\}) . \tag{4.2}$$

We shall refer to $p(\vec{\mu}_a, \{\vec{\sigma}_i^a\})$ as the weight factor for a cell spin $\vec{\mu}_a$ associated with a set of site spins $\{\vec{\sigma}_1^a, \vec{\sigma}_2^a, \ldots, \vec{\sigma}_c^a\}$. We demand that p be normalized such that

$$(2n)^{-1} \sum_{\vec{\mu}_a} p(\vec{\mu}_a, \{\vec{\sigma}_i^a\}) = 1 . \tag{4.3}$$

With the aid of (4.2) and (4.3) the partition function for an arbitrary Hamiltonian $\mathcal{H}$ can be written as

$$\begin{aligned} Z &\equiv (2n)^{-N} \sum_{\{\vec{\sigma}_i\}} \exp \mathcal{H}(\{\vec{\sigma}_i\}) \\ &= (2n)^{-N/c} \sum_{\{\vec{\mu}_a\}} \exp[nG + \mathcal{H}'(\{\vec{\mu}_a\})] , \end{aligned} \tag{4.4}$$

with

$$e^{nG + \mathcal{H}'(\{\vec{\mu}_a\})} = (2n)^{-N} \sum_{\{\vec{\sigma}_i\}} P(\{\vec{\mu}_a\}, \{\vec{\sigma}_i\}) e^{\mathcal{H}(\{\vec{\sigma}_i\})} . \tag{4.5}$$

Here $\mathcal{H}'(\{\vec{\mu}_a\})$ is the renormalized Hamiltonian and G is a constant chosen such that $\mathcal{H}'(\{\vec{\mu}_a\})$ contains no constant (spin-independent) term.

As a generalization of (4.5) we define for any operator $A(\{\vec{\sigma}_i\})$ its $\{\vec{\mu}_a\}$-dependent average by

$$R[A(\{\vec{\sigma}_i\})] = (2n)^{-N} \sum_{\{\vec{\sigma}_i\}} P(\{\vec{\mu}_a\}, \{\vec{\sigma}_i\}) A(\{\vec{\sigma}_i\}) . \quad (4.6)$$

In Sec. V we shall give explicit examples of such averages.

We shall now discuss how to choose the weight factor $p(\vec{\mu}, \{\vec{\sigma}_i\})$. Of course p has to express first of all the tendency of the $\vec{\mu}$ spins to align with the $\vec{\sigma}$ spins. In addition, however, we have to pay particular attention to ensure that the choice of p transforms a Hamiltonian $\mathcal{H}$ of order n again into a Hamiltonian $\mathcal{H}'$ of order n plus a free energy contribution nG with G of order n^0.

A special choice for $p(\vec{\mu}, \{\vec{\sigma}_i\})$ in which we shall be interested is

$$p(\vec{\mu}, \{\vec{\sigma}_i\}) = 1 - q_2 + \frac{q^2}{nc} \sum_\alpha \mu_\alpha^2 (\sigma_{1\alpha}^2 + \ldots + \sigma_{c\alpha}^2)$$

$$+ \frac{q_1}{c} \sum_\alpha \mu_\alpha (\sigma_{1\alpha} + \ldots + \sigma_{c\alpha}) , \quad (4.7)$$

which is easily found to satisfy the normalization condition (4.3). We shall denote by $R_{q_1 q_2}$ the transformation defined by the weight factor (4.7). A brief calculation shows that $R_{q_1 q_2}$ has a property characteristic of so-called linear transformations. Let $\vec{r}_{bd}$ be the distance between two cell spins $\vec{\mu}_b$ and $\vec{\mu}_d$, and let $\langle \ldots \rangle'$ denote the thermal average with respect to the renormalized Hamiltonian $\mathcal{H}'(\{\vec{\mu}_a\})$. Reexpressing the pair correlation function $G'(\vec{r}_{bd}) \equiv \langle \mu_{b\gamma} \mu_{d\gamma} \rangle'$ in terms of the original site spins one finds, for r large,

$$G'(\vec{r}) \simeq q_1^2 G(c^{1/d} \vec{r}) . \quad (4.8)$$

A well-known argument[3,4] shows that if at the critical temperature the correlation function $G(r)$ falls off as $\sim r^{-d+2-\eta}$, then the existence of a fixed point implies the relation

$$q_1 = q_1^* \equiv c^{(d-2+\eta)/2d} . \quad (4.9)$$

Hence $R_{q_1 q_2}$ can have a fixed point only for one special value q_1^* of the parameter q_1. One similarly shows for the energy-energy correlation function defined in (2.21)

$$G_E'(\vec{r}) \simeq q_2^2 G_E(c^{1/d} \vec{r}) \quad (r \to \infty) , \quad (4.10)$$

and hence by (2.24) the transformation cannot have a fixed point unless

$$q_2 = q_2^* \equiv c^{1-1/\nu d} , \quad (4.11)$$

where we used the scaling relation $\alpha = 2 - \nu d$. Recent work by Bell and Wilson[28] suggests that such properties are limited to linear transformations; nonlinear transformations may well have fixed points—within certain limits—for any set of values of their parameters.

Finally one can show that if $R_{q_1^* q_2^*}$ has a fixed point Hamiltonian $\mathcal{H}^*$, then it has a whole two-parameter family of fixed point Hamiltonians $\mathcal{H}_{st}^*$ given by

$$\exp(nG_{st} + \mathcal{H}_{st}^*) = \mathcal{L}_{st} [\exp \mathcal{H}^*] , \quad (4.12)$$

where G_{st} is again a spin-independent constant and where $\mathcal{L}_{st}$ is the transformation defined by

$$p(\vec{\mu}_i, \vec{\sigma}_i) = 1 - s + \frac{s}{n} \sum_\alpha \mu_{i\alpha}^2 \sigma_{i\alpha}^2 + t \sum_\alpha \mu_{i\alpha} \sigma_{i\alpha} . \quad (4.13)$$

The proof follows by showing that $\mathcal{L}_{st}$ commutes with $R_{q_1 q_2}$.

V. RENORMALIZATION PROCEDURE FOR $n \to 0$

In this section we shall derive the explicit renormalization rules that connect the renormalized interactions to the original ones for the case where the number of spin components n tends to zero. We shall find that in this limit certain simplifications occur with respect to the case of general n. These are based on the fact that several of the interactions newly generated by the renormalization transformation turn out to be of order n^2 or higher, and can be shown not to couple back to the interactions of order n.

Further simplification arises from the fact that in the limit $n \to 0$ the subclass of closed-loop interactions is "driven" by the remaining independently transforming interactions. To find the fixed point properties of the transformation for $n \to 0$ it suffices to study this reduced system of independently transforming interaction constants. We shall now proceed to prove the announced results.

Consider the Hamiltonian

$$\mathcal{H} = \sum_{A_1} K_{A_1} A_1 + \sum_{A_2} K_{A_2} A_2 + \ldots , \quad (5.1)$$

where the operators $A_1, A_2, \ldots$, are of order $n, n^2, \ldots$, respectively, according to the classification of Sec. III. The renormalized Hamiltonian $\mathcal{H}'$ corresponding to $\mathcal{H}$ is given by

$$e^{nG + \mathcal{H}'} = R[e^{\mathcal{H}}]$$

$$= R\left[1 + \mathcal{H} + \frac{1}{2!} \mathcal{H}^2 + \ldots\right]$$

$$= R\left[1 + \sum_{A_1} G_{A_1} A_1 + \sum_{A_2} G_{A_2} A_2 + \ldots\right] . \quad (5.2)$$

Property (P1) of Sec. III implies that the constants G_{A_1} in the expansion are uniquely determined by the set $\{K_{A_1}\}$. The Taylor expansion

(5.2) is of practical use for small finite lattices. We remark that for finite lattices, in the limit $n \to 0$, it terminates after a finite number L of terms, viz. when L is so large that none of the graphs generated by the product $\mathcal{K}^L$ can be contained in the lattice without doubly visiting a lattice site.

We are interested in the action of the renormalization transformation R on an arbitrary operator $\mathcal{O}$,

$$\mathcal{O} = \sum_{\alpha_1,\ldots,\alpha_k=1}^{n} \prod_{i=1}^{n} \sigma_{i,\alpha(i)}^{p_i}, \tag{5.3}$$

with $\alpha(i) \in \{\alpha_1, \ldots, \alpha_k\}$ and $p_i \in \{0, 1, 2\}$. From (5.3) and (4.6) we have

$$R[\mathcal{O}] = \sum_{\alpha_1,\ldots,\alpha_k} (2n)^{-N} \sum_{\{\vec{\sigma}_i\}} \prod_a p(\vec{\mu}_a, \{\vec{\sigma}_i^a\}) \prod_i \sigma_{i,\alpha(i)}^{p_i}. \tag{5.4}$$

The average on $\{\vec{\sigma}_i\}$ can be written as a product on all cells a of averages of the type

$$C_{p_1 p_2 \ldots p_c} \equiv (2n)^{-c} \sum_{\{\vec{\sigma}_i\}} p(\vec{\mu}, \{\vec{\sigma}_i\}) \sigma_{1\gamma_1}^{p_1} \ldots \sigma_{c\gamma_c}^{p_c}. \tag{5.5}$$

Note that the exponents $p_1, \ldots, p_c$ take only the values 0, 1, or 2. Hence $R[\mathcal{O}]$ is known if all averages of type (5.5) are known. These averages constitute the essential part of the renormalization rules. Once an explicit expression for $p(\vec{\mu}, \{\vec{\sigma}_i\})$ is given, they are easily evaluated with the aid of the elementary properties (2.2) and analogous relations for the spins $\vec{\mu}_a$.

For future reference we list the results obtained with $R = R_{q_1 q_2}$ for a cell of three spins $\vec{\sigma}_1$, $\vec{\sigma}_2$, and $\vec{\sigma}_3$. Using the weight factor (4.7) in (5.5) we find

$$\begin{aligned}
&C_{000} = 1; \quad C_{200} = 1 + \tfrac{1}{3} q_2 (\mu_{\gamma_1}^2 - 1); \\
&C_{220} = 1 + \tfrac{1}{3} q_2 (\mu_{\gamma_1}^2 + \mu_{\gamma_2}^2 - 2); \\
&C_{222} = 1 + \tfrac{1}{3} q_2 (\mu_{\gamma_1}^2 + \mu_{\gamma_2}^2 + \mu_{\gamma_3}^2 - 3); \\
&C_{100} = C_{120} = C_{122} = \tfrac{1}{3} q_1 \mu_{\gamma_1}; \\
&C_{110} = C_{112} = C_{111} = 0.
\end{aligned} \tag{5.6}$$

All other averages of type (5.5) with $c = 3$ are related to those in (5.6) by permutations of site and component indices.

With the cell averages given it is easy to calculate the renormalized counterpart of any of the operators discussed in Sec. III. For the case $R = R_{q_1 q_2}$ one easily verifies the following.

(P3) *The transformation $R_{q_1 q_2}$ maps operators of order n^p onto operators of order n^p.*

(P4) *The transformation $R_{q_1 q_2}$ maps a closed-loop operator onto a closed-loop operator, and a $2m$-leg operator (with $m \geq 1$) either onto a $2m$-leg operator or onto 0.*

Although in the sequel of our discussion R will again be general, we shall nevertheless assume that it has both properties (P3) and (P4). The explicit expressions for the cell averages can be substituted into (5.4), which in turn can be substituted into (5.2). The result is an expression of the form

$$\begin{aligned}
e^{nG+\mathcal{K}'} &= R\Big[1 + \sum_{A_1} G_{A_1} A_1 + \sum_{A_2} G_{A_2} A_2 + \ldots\Big] \\
&= 1 + \sum_{A_1'} G_{A_1'}' A_1' + \sum_{A_2'} G_{A_2'}' A_2' + \ldots \\
&= \exp\Big(\sum_{A_1'} K_{A_1'}' A_1' + \sum_{A_2'} K_{A_2'}' A_2' + \ldots\Big).
\end{aligned} \tag{5.7}$$

In (5.7) the unprimed operators are functions of the site spins $\vec{\sigma}_i$ and the primed ones of the cell spins $\vec{\mu}_a$; the operators $A_1, A_2, \ldots$ and their primed counterparts are of order $n, n^2, \ldots$, respectively. We note that the summations on $A_1', A_2', \ldots$ contain in general the contribution of constant terms. The above property (P3) of the renormalization transformation implies that $\{G_{A_1'}'\}$ is uniquely determined by $\{G_{A_1}\}$. Finally, again by the multiplication property (P1) of the operators $A_1', A_2', \ldots$ we have that $\{K_{A_1'}'\}$ is uniquely determined by $\{G_{A_1'}'\}$, hence by $\{G_{A_1}\}$, and hence by $\{K_{A_1}\}$. Upon identifying the cells as the sites of a new lattice this yields the transformation

$$K_{A_1}' = R_{A_1}(\{K_{B_1}\}). \tag{5.8}$$

This is the renormalization transformation for the self-avoiding walk problem. It may be looked upon as the first term in a systematic expansion of the renormalization transformation in powers of n.

The transformation of Eq. (5.8) is further analyzed by one remark. Let the operators A_1 be divided into $2m$-leg operators A_{11} with $m \geq 1$, and closed-loop operators A_{10}. The multiplication property (P2) of $2m$-leg operators and closed-loop operators, and the above properties (P3) and (P4) of the renormalization transformation imply that the original couplings $\{K_{A_{10}}\}$ cannot contribute to the renormalized $\{K_{A_{11}}'\}$. Hence we can subdivide the coupling constants into independently transforming couplings,

$$K_{A_{11}}' = R_{A_{11}}(\{K_{B_{11}}\}), \tag{5.9a}$$

and "driven" couplings,

$$K_{A_{10}}' = R_{A_{10}}(\{K_{B_{10}}, K_{B_{11}}\}). \tag{5.9b}$$

In order to find the fixed point and the eigenvalues of the transformation it suffices to study (5.9a). We remark that in the case of Ising spin transformations (5.9b) reduces to an expression for the constant term G.

The derivation of the Eqs. (5.9a) and (5.9b) completes this section. For special cases this derivation may be followed step by step (such as in a cluster or finite lattice calculation) or with slight modifications (such as in a cumulant calculation). Section VI will be concerned with an example.

VI. APPLICATION. CUMULANT EXPANSION FOR THE TRIANGULAR LATTICE

In this section we describe a cumulant expansion of the renormalization transformation. Our calculation will illustrate the general ideas of the preceding sections and yield approximate values for the critical exponent ν and the connective constant K_c^{-1} of the self-avoiding walk. Since we believe that our technique may serve further work along these lines on this problem, we shall pay special attention to certain calculational details associated with the limit $n \to 0$.

Following the original work of Niemeijer and Van Leeuwen[1,4] we divide the initial Hamiltonian $\mathcal{H}(\{\vec{\sigma}_i\})$ up into a part $\mathcal{H}_0(\{\vec{\sigma}_i\})$ containing all intracell interactions, and a remainder $V(\{\vec{\sigma}_i\})$. The renormalized Hamiltonian can then be expressed as

$$e^{nG+\mathcal{H}'} = R[e^{\mathcal{H}_0+V}] = R[e^{\mathcal{H}_0}]\langle e^V\rangle_0 , \tag{6.1}$$

with the definition

$$\langle X\rangle_0 = R[Xe^{\mathcal{H}_0}]/R[e^{\mathcal{H}_0}] . \tag{6.2}$$

If V is small with respect to $\mathcal{H}_0$, a cumulant expansion for (6.1) makes sense and we get

$$e^{nG+\mathcal{H}'} = R[e^{\mathcal{H}_0}]e^{\langle V\rangle_0+\frac{1}{2}(\langle V^2\rangle_0-\langle V\rangle_0^2)+\cdots} . \tag{6.3}$$

From the preceding sections we know that if $\mathcal{H}_0$ and X are of order n, then $R[e^{\mathcal{H}_0}] = 1 + \mathcal{O}(n)$, and $R[Xe^{\mathcal{H}_0}]$ is of order n. By the argument of Sec. V we know that in this limit it suffices to keep terms to order n in the exponential in (6.3), and hence we obtain

$$e^{nG+\mathcal{H}'} = R[e^{\mathcal{H}_0}]\exp(R[Ve^{\mathcal{H}_0}] + \tfrac{1}{2}\{R[V^2e^{\mathcal{H}_0}] - (R[Ve^{\mathcal{H}_0}])^2\} + \cdots + \mathcal{O}(n^2)) . \tag{6.4}$$

The cumulant expansion thus amounts to the calculation of the expressions $R[V^l e^{\mathcal{H}_0}]$ for $l = 0, 1, 2, \ldots$. In practice this is most easily done by writing $\mathcal{H}_0$ as a sum of cell Hamiltonians,

$$\mathcal{H}_0(\{\vec{\sigma}_i\}) = \sum_a h(\{\vec{\sigma}_i^a\}) , \tag{6.5}$$

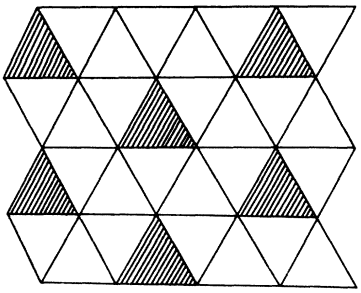

FIG. 2. Triangular lattice with cells (shaded) of three spins each.

where $\{\vec{\sigma}_i^a\}$ denotes as before the c spins in cell a. The exponential is then expanded within each cell to give

$$R[V^l(\{\vec{\sigma}_i\})e^{\mathcal{H}_0(\{\vec{\sigma}_i\})}] = R[V^l(\{\vec{\sigma}_i\})\prod_a (1 + h(\{\vec{\sigma}_i^a\}) + \ldots)] , \tag{6.6}$$

after which relations (5.4) and (5.5) can be applied and we obtain a sum of products of cell averages.

We shall now work this scheme out for a triangular lattice. We choose cells of three spins each (so that $c = 3$), as in Fig. 2, and use the linear two-parameter renormalization transformation $R = R_{q_1 q_2}$ defined by (4.7). The initial Hamiltonian will be the nearest-neighbor one of Sec. II,

$$\mathcal{H} = \mathcal{H}_{\mathrm{NN}} = K\sum_{\langle i,j\rangle}\sum_\alpha \sigma_{i\alpha}\sigma_{j\alpha} . \tag{6.7}$$

The corresponding intracell Hamiltonian for a cell with spins $\vec{\sigma}_1$, $\vec{\sigma}_2$, and $\vec{\sigma}_3$ is

$$h(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3) = K\sum_\alpha (\sigma_{1\alpha}\sigma_{2\alpha} + \sigma_{2\alpha}\sigma_{3\alpha} + \sigma_{3\alpha}\sigma_{1\alpha}) . \tag{6.8}$$

A Taylor expansion yields

$$\begin{aligned} e^{h(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)} = 1 &+ K\sum_\alpha (\sigma_{1\alpha}\sigma_{2\alpha} + \sigma_{2\alpha}\sigma_{3\alpha} + \sigma_{3\alpha}\sigma_{1\alpha}) \\ &+ K^2\sum_\alpha (\sigma_{1\alpha}\sigma_{2\alpha}^2\sigma_{3\alpha} + \sigma_{2\alpha}\sigma_{3\alpha}^2\sigma_{1\alpha} + \sigma_{3\alpha}\sigma_{1\alpha}^2\sigma_{2\alpha}) \\ &+ \tfrac{1}{2}K^2\sum_\alpha (\sigma_{1\alpha}^2\sigma_{2\alpha}^2 + \sigma_{2\alpha}^2\sigma_{3\alpha}^2 + \sigma_{3\alpha}^2\sigma_{1\alpha}^2) \\ &+ K^3\sum_\alpha \sigma_{1\alpha}^2\sigma_{2\alpha}^2\sigma_{3\alpha}^2 + \mathcal{O}(n^2) . \end{aligned} \tag{6.9}$$

We remark that all operators following the constant 1 are of order n. We now substitute (6.9) into (6.6) and take $l = 0$. Using the fact that both R and $e^{\mathcal{H}_0}$ factorize over the $\frac{1}{3}N$ cells, applying (5.4) and (5.5), and substituting (5.6), we obtain

$$R[e^{\mathcal{H}_0}] = (R[e^h])^{N/3} = 1 + \tfrac{1}{3}N(\tfrac{3}{2}K^2 + K^3)n + O(n^2). \quad (6.10)$$

The expression $\frac{3}{2}K^2 + K^3$ is readily interpreted as the free energy per cell of the intracell Hamiltonian $\mathcal{H}_0$. It contains contributions only from the last two terms in (6.9), which correspond to the closed loops that one can draw within a cell.[20]

The first-order cumulant expansion requires the calculation of $R[Ve^{\mathcal{H}_0}]$ to order n. The interaction V is a sum on terms V_{ab}, where V_{ab} contains the two two-leg interactions connecting the neighboring cells a and b. The remaining $\frac{1}{3}N - 2$ cells renormalize independently, and, V being of order n, their effect to lowest order is by (6.10) only a factor of 1. We obtain

$$R[Ve^{\mathcal{H}_0(\{\vec{\sigma}_i\})}] = \sum_{\langle a,b\rangle} R[V_{ab}e^{h(\{\vec{\sigma}_i^a\})}e^{h(\{\vec{\sigma}_i^b\})}] + O(n^2). \quad (6.11)$$

We now expand both exponentials in (6.11) with the aid of (6.9). Since V_{ab} is of order n, property (P2) allows us to neglect here the closed-loop terms in (6.9). We multiply V_{ab} by the terms in the two expansions and obtain a sum of terms to which we apply (5.4)–(5.6). The result is

$$R[Ve^{\mathcal{H}_0(\{\vec{\sigma}_i\})}] = K' \sum_{\langle a,b\rangle} \sum_{\alpha} \mu_{a\alpha}\mu_{b\alpha} + O(n^2), \quad (6.12)$$

with

$$K' = \tfrac{2}{9}q_1^2 K(1 + 2K + 2K^2)^2. \quad (6.13)$$

One recognizes without difficulty that there is a contribution $\frac{2}{9}q_1^2K^L$ to (6.13) for each self-avoiding walk of L steps beginning in cell a and ending in cell b (see Fig. 3). Substitution of (6.10) and (6.12) into (6.4) shows that the renormalized Hamiltonian is given by

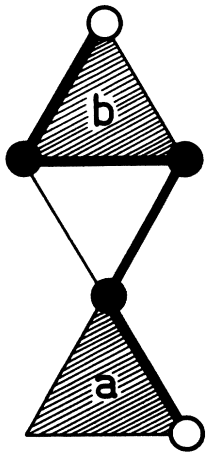

FIG. 3. A self-avoiding walk graph contributing to the interaction K' between the cell spins $\vec{\mu}_a$ and $\vec{\mu}_b$.

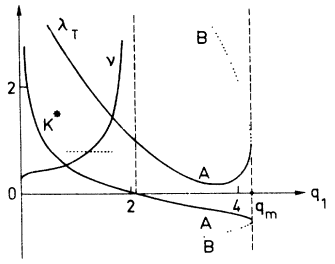

FIG. 4. The fixed point interaction strength K^* and the thermal eigenvalue λ_T as functions of q_1 in first cumulant approximation. The exponent ν has been plotted only in the region where it is positive. A dashed horizontal line indicates the Flory value $\nu = 0.75$, which is assumed for $q_1 = 1.19$.

$$e^{nG + \mathcal{H}'(\{\vec{\mu}_a\})} = \exp\Big(\tfrac{1}{3}N(\tfrac{3}{2}K^2 + K^3)n + K' \sum_{\langle a,b\rangle} \sum_{\alpha} \mu_{a\alpha}\mu_{b\alpha} + \mathcal{H}'_{\text{rem}}\Big) \quad (6.14)$$

where the remainder $\mathcal{H}'_{\text{rem}}$ denotes terms of order n^2 plus terms from the second and higher cumulants. Hence for $n \to 0$ the first cumulant approximation to the transformation $R_{q_1q_2}$ is given by (6.13). We remark that this result is independent of the parameter q_2.

Expression (6.13) illustrates a general feature of the renormalization transformation in the $n \to 0$ limit, namely that the new coupling constants become polynomials in the original ones. A fixed point K^* of the transformation is defined by $(K^*)' = K^*$. The thermal eigenvalue $\lambda_T = (dK'/dK)^*$ determines as usual[1] the critical exponent ν via $\nu = \frac{1}{2}\ln 3/\ln\lambda_T$. It is easy to obtain from (6.13) explicit analytic expressions for K^*, λ_T, and ν. We do not present these here, but show in Fig. 4 a plot of K^* and ν as a function of the parameter q_1. It appears that there is a value $q_m = 4.24$ such that for $0 < q_1 < q_m$ there are two branches of fixed points, A and B. These coalesce at $q_1 = q_m$, and for $q_1 > q_m$ no fixed point exists. This behavior of the approximate recursion (6.13) is to be compared to the prediction from general theory (Sec. IV), namely that the transformation $R_{q_1q_2}$, when carried out exactly, can have a fixed point for only one special (but unknown) value $q_1 = q_1^*$. Since we have no *a priori* way of choosing q_1^*, the first cumulant approximation does not give us unique values of ν and of K^*. We note nevertheless that the single value $q_1 = 1.19$ yields the traditional "Flory"[13] exponent $\nu = 0.75$ and $K_c = K^* = 0.30$, *both* in reasonable agreement with the best estimates from other methods (see Table I). We shall return to the problem of fixing q_1 (and q_2) after describing the results of the second-order cumulant approximation.

TABLE I. Comparison of values for the critical exponent ν at $d=2$ and the connective constant K_c^{-1} of the triangular lattice obtained by various methods.

	ν	K_c
Flory value [a]	0.75	
Self-consistent field theory [b]	0.75	
Enumeration, Monte-Carlo sampling, and series expansion	0.75 [c,d,e]	0.241 [d]
ϵ expansion [f] 1st order	0.625	
2nd order	0.735	
3rd order	0.632	
This work, second cumulant	0.740	0.229

[a] Reference 6.
[b] Reference 34.
[c] References 35 and 36.
[d] References 35 and 37.
[e] See Ref. 6 for a discussion of the accuracy of this figure.
[f] References 16 and 38.

The second-order cumulant requires the additional calculation of $R[V^2 \exp\mathcal{H}_0]$. The evaluation of this quantity involves all triples of cells connected by the interaction V. As before, in each stage of the calculation one is allowed to neglect terms of order n^2 and higher. In this order one finds that in addition to the nearest-neighbor interaction (coupling K) six new interaction types are formed, corresponding to the couplings L, M, K_1, L_1, M_1, and K_0 (see Fig. 5). Of these seven couplings, K_0 corresponds to a closed-loop interaction. Consequently (see Sec. V) it is a driven constant, i.e., the renormalized couplings of the other interactions do not depend upon K_0. When properly treating[1] the effect of the newly formed interactions in first order, we arrive after some calculations at the following expressions for the free energy contribution G and for the seven renormalized coupling constants,

$$
\begin{aligned}
G &= \tfrac{1}{3}N\{K^2[A+C+A(B+D)]+2K_0A(1+B) \\
&\quad +K_1(A+BC)\}, \\
K_0' &= K^2B(B+D)-2K^2Q^4+2K_0B^2+K_1BD, \\
K' &= 4K^2Q^2(A+C)+2KQ^2+3LQ^2+2MQ^2+2K_1(QE+Q^2A) \\
&\quad +2L_1(2QE+Q^2A)+2M_1QE, \\
L' &= K^2Q^2(A+7C)+MQ^2+M_1Q^2A, \\
M' &= 4K^2Q^2C, \\
K_1' &= 2K^2Q^2(B+D)-4K^2Q^4+K_1Q^2B+L_1Q^2B, \\
L_1' &= \tfrac{1}{2}K^2Q^2(B+D)-4K^2Q^4+\tfrac{1}{2}M_1Q^2B, \\
M_1' &= 4K^2Q^2D-4K^2Q^4,
\end{aligned}
\tag{6.15}
$$

in which we have abbreviated

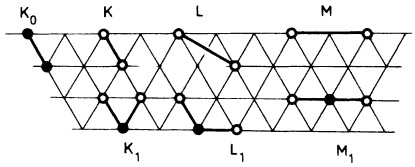

FIG. 5. The interactions and corresponding coupling constants involved in the second cumulant approximation.

$$
\begin{aligned}
A &= 1-\tfrac{1}{3}q_2, \qquad D = \tfrac{1}{3}q_2(2K+3K^2) \\
B &= \tfrac{1}{3}q_2, \qquad E = \tfrac{1}{3}q_1(1+K) \\
C &= (1-\tfrac{2}{3}q_2)K+(1-q_2)K^2, \quad Q = \tfrac{1}{3}q_1(1+2K+2K^2).
\end{aligned}
\tag{6.16}
$$

The fixed points and fixed point properties of the equations (6.15) have been found numerically as a function of the two parameters q_1 and q_2. In the region of the q_1q_2 plane which is shaded in Fig. 6, two fixed points exist. The two sheets of fixed points, A and B, join at the border line of this area, beyond which no fixed point exists. Figure 7 shows the fixed point values $\vec{K}^* \equiv (K^*, L^*, M^*, K_1^*, L_1^*, M_1^*)$ as a function of q_1, for a characteristic value of q_2. The fixed points of type A are all marked by a strongly dominant nearest-neighbor coupling K^*. We identify sheet A as the physical one. At each fixed point the linearized transformation can be represented by a 6×6 matrix. Since several matrix elements vanish identically, there are always two zero eigenvalues. Two other eigenvalues turn out to be rather small ($\lesssim 0.1$). The two remaining ones, which we shall call λ_T and λ_1, have been plotted in Fig. 8 as a function of q_1 for several values of q_2. For each q_2 there is

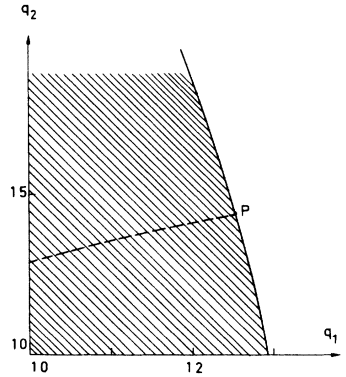

FIG. 6. The approximate recursion relation (6.15) has nontrivial fixed points only in the shaded area of the q_1q_2 plane. The dashed line corresponds to fixed points for which the exponent ν satisfies the consistency relation (4.11). The border line corresponds to fixed points possessing a marginal eigenvalue. The point of intersection P gives the "best" values for q_1 and q_2.

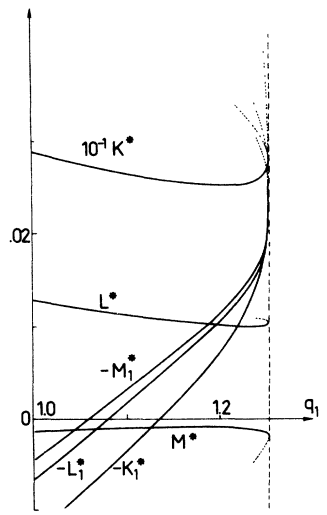

FIG. 7. Fixed point values of the coupling constants in second cumulant approximation as functions of q_1 and for $q_2 = 1.4$. The dotted parts of the curves belong to the fixed points in sheet B. The fixed points A and B coalesce at $q_1 = q_m(1.4) = 1.257$. For other values of q_2 one obtains qualitatively similar graphs.

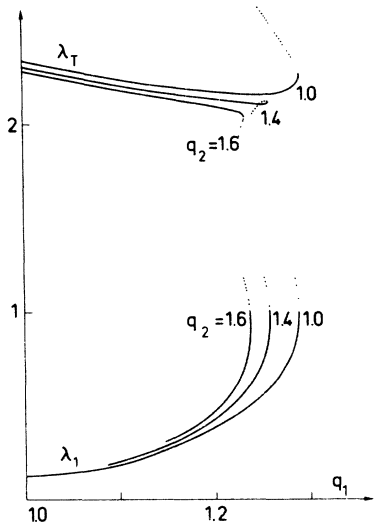

FIG. 8. The two largest fixed point eigenvalues, λ_T and λ_1, in second cumulant approximation, as functions of q_1 and for three different values of q_2. The dotted parts of the curves belong to the fixed points in sheet B. The eigenvalue λ_1 equals unity where the fixed points A and B coalesce. For $q_2 < 1.415$, branch A of λ_T comes in with slope $+\infty$ and branch B with slope $-\infty$ as $q_1 \to q_m(q_2)$. For $q_2 > 1.415$ these slopes are interchanged.

a $q_m(q_2)$ such that at $q_1 = q_m(q_2)$ the two branches of fixed points A and B coalesce and where, as expected, the second largest eigenvalue λ_1 equals unity.

In order to obtain unique fixed point properties we are now obliged to fix the parameters q_1 and q_2 at their best values. This is a delicate problem which is presently receiving a good deal of attention.[19,27-33] We shall profit especially from findings of Bell and Wilson.[19] It appears that the structure of our fixed points as a function of q_1 (viz. the existence of two branches below a certain maximum value of this parameter) is very similar to what these authors found in a study of the three-dimensional Gaussian model on a finite lattice. This is even more remarkable since the authors of Ref. 19 had to go to a $10 \times 10 \times 10$ size lattice with ten different types of interactions in order to obtain such a structure; in our second cumulant expansion with seven interaction types no term involves more than nine spins. Bell and Wilson thoroughly discuss a number of criteria for fixing a linear parameter analogous to our q_1 in their transformation. They argue that the appearance of a marginal eigenvalue $\lambda_1 = 1$ (a feature known to exist in the exact transformation) constitutes the best criterion available. Applying this same criterion here means choosing the "best" point (q_1, q_2) on the border line of the shaded area in Fig. 6. A second condition is needed to fix a unique point on this line. This condition is furnished by Eq. (4.11), which links the critical exponent $\nu = \frac{1}{2}\ln 3/\ln\lambda_T$ to the value of q_2. The dashed line in Fig. 6 indicates the points satisfying this consistency relation. Its intersection with the border line occurs at the point P given by $q_1 = 1.254$ and $q_2 = 1.428$. For these values of the parameters the fixed point $\vec{K}^*$ is given by

$$K^* = 0.278\,, \quad K_1^* = -0.025\,,$$
$$L^* = 0.008\,, \quad L_1^* = -0.022\,, \tag{6.17}$$
$$M^* = -0.003\,, \quad M_1^* = -0.022\,,$$

and the two largest eigenvalues are

$$\lambda_T = 2.100\,, \quad \lambda_1 = 1\,. \tag{6.18}$$

The corresponding values of the critical exponent ν and the critical nearest-neighbor interaction K_c (obtained by linearization of the critical surface[1]) are

$$\nu = 0.740\,, \quad K_c = 0.229\,. \tag{6.19}$$

In Table I we compare these values to the best estimates obtained by other methods. Although the classical methods all indicate a value 0.75 for ν, neither Flory's arguments nor those of self-consistent field theory are rigorous. Furthermore our value for ν is within the range of uncertainty of the results from numerical methods. It may therefore be considered as very satisfactory and as more precise than the ϵ expansion results. On

the other hand, our value for K_c is probably too low by a few percent. This fact is analogous to findings by Niemeijer and Van Leeuwen,[1] by Hsu, Niemeijer, and Gunton,[39] and by Sudbø and Hemmer,[40] who in a variety of second-order cumulant calculations for the Ising model ($n=1$) all obtain too low a value for the inverse critical temperature. Although it has been suggested[4,39,41] that the cumulant expansion may well be merely asymptotic, the work of Sudbø and Hemmer[40] gives hope that a third-order cumulant calculation will still improve both ν and K_c.

It is of interest to compare our results to two other criteria for fixing q_1 and q_2 that were suggested in Ref. 19. Firstly, consider a point (q_1,q_2) on the border line and its associated fixed point $\vec{K}^*_{q_1q_2}$. The right-hand eigenvector $\vec{\psi}^1_{q_1q_2}$ in the marginal direction (i.e., with eigenvalue $\lambda_1=1$) indicates the vestige of a line of fixed points. Such a "critical line" is known to be a feature of the exact transformation. We shall parametrize it by a parameter s:

$$\vec{K}^*_{q_1q_2}(s)=\vec{K}^*_{q_1q_2}+s\vec{\psi}^1_{q_1q_2}+\mathcal{O}(s^2)\,. \tag{6.20}$$

In $\vec{K}^*_{q_1q_2}$ this line is tangent to the surface of critical points formed by the two sheets A and B. Let us now consider the variation of the thermal eigenvalue λ_T along this line;

$$\frac{d\lambda_T}{ds}=\sum_{i=1,2}\frac{\partial\lambda_T}{\partial q_i}\frac{dq_i}{ds}\,. \tag{6.21}$$

We shall assume that λ_T and q_i are analytic in s at $s=0$. A physical criterion for fixing the parameters is that (6.21) vanish. Since $(dq_i/ds)_{s=0}=0$, this automatically happens if $\partial\lambda_T/\partial q_i$ is finite at $s=0$. Figure 8 shows, however, that the sheets A and B of the function $\lambda_T(q_1,q_2)$ join at the border line everywhere with infinite slopes ($+\infty$ and $-\infty$, within computer precision), *except* at one point where they interchange the signs of their slopes. More detailed numerical investigation shows that this happens in the point P' given by $q_1=1.255$, $q_2=1.415$, i.e., extremely close to P. Hence λ_T is stationary along the critical line associated with P'. Along the critical lines associated with points (q_1,q_2) in the neighborhood of P' (such as P), λ_T is still nearly stationary. This is illustrated in Fig. 8 by the behavior of λ_T for $q_2=1.4$ near $q_1=q_m(1.4)$. We remark that if we were to choose P' instead of P as the best point in the q_1q_2 plane, our result for ν would be $\nu=0.738$, and none of the fixed point coupling constants would differ by more than 0.001 from the values (6.17) based on the choice of P.

A second, rather qualitative criterion for fixing q_1 and q_2 is that the fixed point interactions be well localized. The reason is that in such a case the harm done by truncating the transformation in space is small. This condition is certainly satisfied by the fixed point Hamiltonian (6.17): K^* is at least 10 times larger than each of the other couplings.

Finally, it has been attempted[30] to determine the linear parameter from Eq. (4.9) and the relation of the exponent η to the magnetic eigenvalue λ_H,

$$\eta=d+2-2d\ln\lambda_H/\ln c\,. \tag{6.22}$$

Bell and Wilson reject this criterion, arguing that it should function rather poorly. We have not carried out a second-order calculation of λ_H, but did find that to first order in V this criterion gives unphysical results.

We conclude this section by a few comments on possible modifications of the calculation that we have presented. Firstly, while keeping the same renormalization transformation, one could perform a cluster or a finite lattice approximation, using a fixed number of cells. We performed a three-cell cluster[1] calculation, which involves only two interaction constants (corresponding to K and K_1 above). The results, though qualitatively similar, are not as good as the second cumulant approximation, which is in general agreement with findings for the Ising model.[1] If one wants to go to more cells, one faces the practical problem that the number of interaction types to be considered increases extremely rapidly—much faster than in the corresponding approximations for the Ising model. A four-cell[1] calculation would involve as many as eleven different interactions. This fact is due to the vector character of the spins, which is conserved in the $n\to 0$ limit.

Secondly, one could use other transformations. In a linear transformation with the properties (4.8) and (4.10) it is mandatory to have at least two free parameters related to the *a priori* unknown exponents η and ν. In Niemeijer and Van Leeuwen's nonlinear "sign rule" transformation no need for any parameters appears to exist. However, for n-component spins the weight factor of a nonlinear transformation could contain a large variety of terms, such as $\sum_\alpha\mu_\alpha\sigma_{1\alpha}\sigma_{2\alpha}\sigma_{3\alpha}$, $\sum_{\alpha\beta}\mu_\alpha\sigma_{1\alpha}\sigma_{2\beta}\sigma_{3\beta}$, $\sum_\alpha\mu_\alpha^2\sigma_{1\alpha}^2\sigma_{2\alpha}\sigma_{3\alpha}$, ... There is no natural generalization of the sign rule to vector variables which prescribes with what coefficients each of the possible terms should appear in the weight factor. Hence nonlinear transformations suffer, in our opinion, from too much indeterminacy. We have not attempted any serious calculation with such transformations.

VII. CONCLUSION

We have demonstrated how the real space renormalization of Niemeijer and Van Leeuwen can be used as an independent method for investigating the self-avoiding walk on a lattice. The self-avoiding walk is shown to belong to a universality class of more general excluded volume problems. An explicit technique has been given to perform the common approximations (cumulant, finite lattice, cluster) for this problem. As an application, a linear two-parameter renormalization transformation is carried out to second-order cumulant expansion for the triangular lattice. We discuss criteria for fixing the values of these parameters and arrive at selecting a fixed point Hamiltonian such that: (i) in its neighborhood the approximate transformation has a marginal eigenvalue; (ii) the critical exponent ν satisfies the self-consistency relation (4.11); (iii) ν is nearly stationary in the marginal direction; (iv) the fixed point interactions are highly localized in space. The resulting values for the critical exponent ν and the inverse connective constant K_c were found to be 0.740 and 0.229, respectively, which compares well with data from other sources. Finally, we have provided an interesting example of the behavior of a linear renormalization transformation when it is locally approximated. Our findings confirm and extend conclusions by Bell and Wilson based on the Gaussian model.

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