Critical properties of a two-dimensional planar model

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Recently, critical exponents in the continuum Baxter model have been calculated using an equivalence of this model to the Luttinger model of a one-dimensional spinless fermion gas. We discuss here the extension of this method to the two-dimensional planar model, using the Stoeckly-Scalapino representation of the transfer matrix. This representation is shown to be equivalent to the spin-1/2 fermion gas in one dimension, which further separates into a Luttinger model and the quantum-mechanical sine-Gordon equation. We emphasize the role that soliton bound states play in determining the critical properties. We find that a critical temperature T_c exists below which the susceptibility is infinite but without long-range order. At T_c, the correlation-function exponent η takes the values $1/\sqrt{8}$, and decreases as the temperature is lowered, consistent with low-temperature results. As in the Baxter-model calculations, we argue that these results are exact for the continuum limit of the transfer matrix, and therefore provide a solution for the asymptotic properties near T_c. We discuss the correlation length and susceptibility, and sugget that nonuniversality is responsible for the disagreement between different numerical calculations.

I. INTRODUCTION

There has long been speculation about the nature of critical fluctuations in the two-dimensional system with a continuous symmetry. In contrast to the two-dimensional Ising system, these systems are known to have no broken symmetry or long-range order.[1] Nonetheless, high-temperature series[2,3] have indicated that there appears to be a critical temperature and singular behavior in the susceptibility which can be fit with power-law singularities as in the usual second-order phase transition.

At low temperatures, the problem with rotational symmetry in a plane, the planar model, has been solved.[4] The "order parameter" correlation function decays algebraically with an exponent that approaches zero as the temperature vanishes. Consequently, the susceptibility is everywhere infinite in this temperature region, and the "canonical" picture presumes the susceptibility diverges at a characteristic temperature T_c as the temperature is lowered from infinity, remaining infinite everywhere below T_c. The behavior near this characteristic temperature and the nature of any singularities are, until now, unknown.

We have undertaken a study of this problem using the methods previously used to study the Ising and Baxter models of a second-order phase transition.[5] These methods involve construction of an equivalent one-dimensional fermion problem which gives correct order-parameter equations of motion. Calculation of correlation functions with this method circumvents the complications of evaluating a determinant; the continuum limit of the fermion problem can be solved using standard techniques of field theory.

There are several proposals for the dynamic variables in the planar model. Kosterlitz and Thouless[6] have extended the spin-wave picture of Rice and Berezinsky to include the vortex excitations which are necessary to understand all the degrees of freedom in this model. There is widespread recognition that the resulting problem is then equivalent to the backward scattering model of the one-dimensional electron gas,[7,8] provided the interaction between spin waves and vortices is neglected. A finite correlation length in this approximation would correspond to the solitons of that one-dimensional quantum problem.[9] Applying a recent exact solution of the electron-gas problem, and using the relationship to the planar model, we can solve this approximation. This work also solves the two-dimensional plasma, a problem known to be equivalent to the planar model treated within this approximation.

The exact solution of these equations,[10] when used at higher temperatures, is consistent with the behavior proposed by Zittartz,[11] possessing a temperature T_2 where the susceptibility diverges. However, we are critical of the neglect of interactions between spin waves and vortices, implicit in this approximation, and believe these should be included. It is physically plausible that vortices are virtually indistinguishable from very-short-wavelength spin waves, and when thermal excitation of these states is important the interactions

between them should be included. This identifies the exchange energy as the characteristic temperature below which these excitations can be treated independently. It is just this temperature region where a "phase transition" could occur, and which we wish to investigate.

This high-temperature region can be studied within the framework proposed by Stoeckly and Scalapino,[12] based on performing functional integrations over one space dimension. There results from this approach an equivalent spin-one chain problem, whose eigenvalues describe the low-lying states of the transfer matrix. It is this spin-one problem which is solved exactly. It possesses a temperature T_c where eigenvalue degeneracy first occurs as the temperature is lowered from infinity. Above T_c, there is a gap in the spectrum. Since this temperature is much lower than the temperature T_2 identified in the low-temperature approximation of Zittartz,[11] we conclude it is this transition at T_c which characterizes the high-temperature behavior of the planar model.

At T_c, the number of bound vortices presumably becomes infinite and their binding energy vanishes. As the temperature is lowered below T_c, a type of long-range order sets in with power-law behavior, the vortex binding energy is finite, and the number of bound vortices decreases. Our methods of solving the Stoeckley-Scalapino transfer Hamiltonian formalism study the temperature region near the first eigenvalue degeneracy, and we therefore cannot extrapolate this solution to the low-temperature region.

The solution of these models relies on the construction of the continuum-limit equations of motion for the spin operators, and the recognition that the resulting continuum problem is exactly the one-dimensional electron gas, but with different definitions of coupling constants and temperature scales. The solution[10] for the eigenvalue spectrum of this electron-gas problem is used to provide the solution for both the low-temperature model[7,8] and the Stoeckly-Scalapino model.[12] The soliton and bound-soliton gaps in the eigenvalue spectrum determine the correlation length in the corresponding planar-model problems.

The violation of universality near T_c is both interesting and complicated. There are additional marginal operators which imply that exponents depend on the model, e.g., lattice structure or spin, in addition to symmetry class. This is perhaps not surprising since exponents are known to depend on the temperature ratio T/T_c, and T_c depends on the model. Despite this, there are scaling laws which are found to hold in our solution.

The solution we have obtained for the "order parameter" correlation function $C(r)$, near or at T_c, has the structure

$$C(r) = r^{-\eta_0(T)} C_1(r) . \tag{1.1}$$

The "prefactor" exponent $\eta_1(T)$ varies smoothly near T_c. At T_c, we find the result $\eta_0(T_c) = 8^{-1/2}$.

The correlation function $C_1(r)$ is equal to a standard second-order phase transition correlation function of the form

$$C_1(r) = r^{-\eta_1} f(mr) , \tag{1.2}$$

with $m \sim |T - T_c|^{\nu}$. It exhibits long-range order for $T < T_\nu$, $C_2(r) \to \text{const}$, as $r \to \infty$, while $f(mr) \sim e^{-mr}$ for $T > T_c$. The exponents η_1 and ν are related by the scaling relation $2\nu = (1 - \eta_1)^{-1}$, which has also been shown to apply to the Baxter-model electric exponents,[5] in the continuum limit. Indeed, "unphysical" variables can be introduced to make the $C_1(r)$ formally the same as an electric correlation function for the Baxter model.

We propose the name "displaced scaling" to describe this situation, since the η exponent is displaced by η_1. However, once we have proven $C_1(r)$ is a Baxter-model correlation function, for which the usual scaling laws are known to hold, in addition to the $2\nu = (1 - \eta_1)^{-1}$ above, we have completely established the nature of the scaling properties of the correlation function in the two-dimensional x-y model. The model we solve has $\eta_1 = 0$, $\nu = \frac{1}{2}$, and we suggest that $\eta_1 = 0$ is true for more general models.

Our results thus verify the conventional qualitative picture of diverging susceptibility as T_c is approached from above. The susceptibility exponent is found to be $\gamma = \nu(2 - \eta_1 - \eta_0)$, and our results further make clear the nature of the infinite susceptibility for $T \leq T_c$, and nature of a "long-range order" in that interval. The introduction of the two sets of exponents, η_0 and η_1, suggest this model is equivalent to two Baxter models, and consequently there should be two marginal operators of the Baxter type. These operators are explicitly constructed.

It is now appropriate to recall the nature of the Stoeckly-Scalapino[12] transfer matrix for a complex two-dimensional Ginzburg-Landau field ψ. The energy associated with a given configuration of ψ is

$$F[\psi] = \int dx\, dy \left(a|\psi|^2 + b|\psi|^4 + c_x \left| \frac{d\psi}{dx} \right|^2 + c_y \left| \frac{d\psi}{dy} \right|^2 \right) . \tag{1.3}$$

Here $a = (T - T_{c0})\bar{a}$, with T_{c0} the mean-field transition temperature and $\bar{a}, b, c_x, c_y$ are positive. The partition function is then given by a sum of the Boltzman factor over all configurations of ψ which

leads to the functional integration

$$Z=\int \delta\psi\, e^{-\beta F[\psi]}. \tag{1.4}$$

Correlation functions are similarly constructed

$$\langle\psi(x)\psi^*(x')\rangle=\int \delta\psi\, e^{-\beta F[\psi]}\psi(x)\psi^*(x')Z^{-1}. \tag{1.5}$$

Now the two-dimensional space is sliced in strips parallel to the x direction of width Δy. The width Δy remains macroscopic but is smaller than the scale of variation of the order parameter in the y direction. It will cancel out of the final form. Next, the one-dimensional function integrations along each strip are carried out, leading, in the usual way, to a transfer matrix generated by the Hamiltonian

$$\mathcal{H}=\sum_i \frac{-1}{4\beta^2 C_x(\Delta y)^2}\frac{\partial^2}{\partial\psi_i^2}+a|\psi_i|^2 + b|\psi_i|^4+\frac{c_y}{(\Delta y)^2}|\psi_i-\psi_{i+1}|^2. \tag{1.6}$$

When $c_x \gg c_y$, order will be well developed along a strip before the coupling between the strips becomes significant. In this weak-coupling case, the Hamiltonian for a single strip is well represented as a rigid rotor with a length $|a|/2b$ set by the minimum of $a|\psi|^2+b|\psi|^4$. In this case $\mathcal{H}$ can be written

$$\mathcal{H}=-\sum_i \Delta\frac{\delta^2}{\delta\psi_i^2}-2\sum_i \cos(\psi_{i+1}-\psi_i), \tag{1.7}$$

with $\Delta=(b/|a|)^2/\beta^2 c_x c_y$. As the temperature is lowered, Δ decreases in a smooth way. Introducing angular momentum operators $\vec{L}_i$,

$$\mathcal{H}=\Delta\sum_i (L_i)^2-\sum_i (L_i^+L_{i+1}^-+L_i^-L_{i+1}^+). \tag{1.8}$$

We assume the lowest eigenvalues dominate the critical properties, and we restrict ourselves to the $L_i^2=0,1$ manifolds at each site. There results a spin $S=1$ problem,

$$\mathcal{H}_r=\Delta\sum_i (J_i)^2-\sum_i (J_i^+J_{i+1}^-+J_i^-J_{i+1}^+), \tag{1.9}$$

with $\vec{J}$ a spin-one operator. It is intuitively plausible that as Δ decreases from $\Delta\gg 1$, the first eigenvalue degeneracy occurs in this manifold of states because the spacing to the next set is four times greater. The equivalent assumption for the Ising case is known to be correct,[13] and the lowest two states per site correctly give the critical exponents. In Secs. II–IV, we discuss the solution to the problem posed by Eq. (1.9), restricted to the lowest three states per site. Clearly our results are directly applicable to spin-one chain problems as well as the x-y model, and this application is also discussed. In Appendix A, we discuss some further properties of the transfer Hamiltonian.

II. $S=1$ ONE-DIMENSIONAL CHAIN

In this section, we construct the continuum spin algebra and equations of motion appropriate for the spin-one problem encountered in the Introduction, Eq. (1.3). This discussion is necessary because we wish to relate this problem to the quantum sine-Gordon equation whose solution we then discuss in Sec. III.

The starting point for this discussion is the spin-$\frac{1}{2}$ continuum x-y model which has been recently discussed[5] in connection with the transfer matrix for the Baxter model. The spin operators are defined in term of the free-fermion Hamiltonian in one space dimension, the free-particle Luttinger model,[14]

$$\mathcal{H}_0=\begin{cases}\displaystyle\int \frac{dx}{i}(\psi_1^\dagger\nabla\psi_1-\psi_2^\dagger\nabla\psi_2), & \text{(2.1a)}\\[2ex] \displaystyle 2\pi L^{-1}\sum_{k>0}[\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(+k)], & \text{(2.1b)}\end{cases}$$

where ψ_1 and ψ_2 are free-fermion fields with positive and negative group velocities, respectively, and we have used the result of Mattis and Lieb[12] to write (2.1a) in boson language in (2.1b). In the latter equation,

$$[\rho_1(-k),\rho_1(k')]=\delta_{kk'}kL(2\pi)^{-1}=[\rho_2(k),\rho_2(-k')],$$

with

$$\rho_1(k)=L^{-1}\int dx\, e^{+ikx}\psi_1^\dagger(x)\psi_1(x),$$

etc., as discussed in this reference.

In terms of these operators, the continuum spin-$\frac{1}{2}$ operators have the simple form [cf. Eq. (A4) and Eq. (A5) of Ref. 5]

$$\begin{aligned}S^z(x)&=\rho_1(x)+\rho_2(x),\\ S^+(x)&=(2s)^{-1/2}[\psi_1^\dagger(x)e^{N(x)}+\psi_2^\dagger(x)e^{N'(x)}], \qquad \text{(2.2)}\\ S^-(x)&=[S^+(x)]^\dagger,\end{aligned}$$

where

$$N(x)=i\pi\int_0^{x-s/2} dy\,[\rho_1(y)+\rho_2(y)]$$

and $N'(x)=N(x+s)$, with s representing the lattice spacing which tends to zero in the continuum limit. It can be shown that it is not necessary to use the more complete spin algebra discussed in that reference, because the extra terms turn out to be irrelevant to this problem. We now recall that

the single creation or annihilation operators can be written

$$\psi_1(x) = (2\pi\alpha)^{-1/2} \exp[\phi_1(x) + ik_F x], \tag{2.3}$$

where k_F is the Fermi momentum in the continuum fermion problem, α is a cutoff parameter, and

$$\phi_1(x) = 2\pi L^{-1} \sum_k e^{-\alpha|k|/2} k^{-1} \rho_1(k) e^{-ikx}.$$

The spin operator $S^-(x)$ has the same operator dimension as

$$S^-(x) \sim \exp\tfrac{1}{2}[\phi_1(x) + \phi_2(x)], \tag{2.4}$$

since $2N(x) = \phi_1(x) + \phi_2(x)$, in the continuum limit when α and s are zero; this suggests that we can transform this operator $S^-(x)$ into a simple product of fermion operators, recalling Eq. (2.3). The appropriate canonical transformation

$$\begin{aligned} \rho_1(k) &\to \tfrac{5}{4}\rho_1(k) - \tfrac{3}{4}\rho_2(k), \\ \rho_2(k) &\to \tfrac{5}{4}\rho_2(k) - \tfrac{3}{4}\rho_1(k) \end{aligned} \tag{2.5}$$

is generated by the operator e^{iT} with

$$T = 2\pi i L^{-1} \sum_k \varphi \rho_1(k) \rho_2(k)$$

and $\sinh\varphi = -\frac{3}{4}$. The transformed spin operator then has the form

$$\mathcal{S}^-(x) \sim \exp[\phi_1(x) + \phi_2(x)] = \psi_2(x)\psi_1(x). \tag{2.6}$$

It should be realized that this argument does not determine an operator prefactor, e.g., the $\phi(0)$-type terms arising from the lower limit in Eq. (2.2). This is not necessary, however, since we are ultimately interested only in the space dependence, and for that reason we may write Eq. (2.6) as an equality.

Under the transformation given in Eq. (2.5), the free-particle Hamiltonian becomes

$$\begin{aligned} \mathcal{H}_0' = {} & \tfrac{17}{4}\pi L^{-1} \sum_{k>0} [\rho_1(k)\rho_1(-k) + \rho_2(-k)\rho_2(k)] \\ & - \tfrac{15}{4}\pi \sum_k \rho_1(k)\rho_2(k), \end{aligned} \tag{2.7}$$

and it is easily verified that the resulting spin algebra

$$\begin{aligned} \mathcal{S}^+(x) &= \psi_1^\dagger(x)\psi_2^\dagger(x), \\ \mathcal{S}^-(x) &= \psi_2(x)\psi_1(x), \\ 2\mathcal{S}^z(x) &= \rho_1(x) + \rho_2(x) \end{aligned} \tag{2.8}$$

is a canonical spin algebra. Since we have generated these operators by canonical operations, it follows that the spin-correlation functions have the correct asymptotic behavior, $\langle \mathcal{S}^-(x)\mathcal{S}^+\rangle \sim x^{-1/2}$, as is easily verified, which gives the known exponent for the spin-$\frac{1}{2}$ x-y model. The equations of motion for $\mathcal{S}^-(x)$ and $\mathcal{S}^z(x)$, needed for later purposes, are

$$\begin{aligned} & i\frac{d}{dt}\mathcal{S}^-(x) = [\mathcal{S}^-(x), \mathcal{H}_0'] = \pi \mathcal{S}^z(x)\mathcal{S}^-(x), \\ & \left(i\frac{d}{dt}\right)^2 \mathcal{S}^z = \nabla^2 \mathcal{S}^z. \end{aligned} \tag{2.9}$$

The reason for the transformation to $\psi_1^\dagger\psi_2^\dagger$ operators is that the subsequent operations necessary for the spin-one spin algebra are greatly simplified. To construct this spin-one algebra, we simply introduce an additional continuum spin field, say $\mathcal{L}(x)$, write $\vec{\mathcal{J}}(x) = \vec{\mathcal{L}}(x) + \vec{\mathcal{S}}(x)$, and deduce the equations of motion in the spin-one subspace. In terms of fermion fields, we must introduce an additional quantum number to distinguish those fields associated with $\vec{\mathcal{L}}$ from those associated with $\vec{\mathcal{S}}$. The previous result, Eq. (2.8), thus becomes relabeled as

$$\begin{aligned} \mathcal{S}^+(x) &= \psi_{1,+}^\dagger(x)\psi_{2,+}^\dagger(x), \\ \mathcal{L}^+(x) &= \psi_{1,-}^\dagger(x)\psi_{2,-}^\dagger(x), \end{aligned} \tag{2.10}$$

etc., where the other relations are obvious, following from Eq. (2.8) with the attachment of the "+" label on $\mathcal{S}$ fields, and the "−" label on the $\mathcal{L}$ fields. Clearly, the "+" and "−" labels have nothing to do with the physical spins.

Obviously the Hamiltonian, Eq. (2.7), must also receive the faith label, and the result $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$ correctly describes the problem of two noninteracting spin-$\frac{1}{2}$ fields with dynamics governed by the $x-y$ one-dimensional spin-$\frac{1}{2}$ Hamiltonian. It remains to find the proper dynamics for the spin-one x-y problem. Here we can make use of the fact that we now have constructed the total spin operator, $\vec{\mathcal{J}}(x) = \vec{\mathcal{L}}(x) + \vec{\mathcal{S}}(x)$, and know the dynamics of the $\vec{\mathcal{L}}$ and $\vec{\mathcal{S}}$ fields separately.

To deduce the correct equations of motion for the spin-one x-y model, we first observe that the spin-$\frac{1}{2}$ equations of motion lead to the result

$$[\mathcal{J}^-(x), \mathcal{H}_+ + \mathcal{H}_-] = \pi \mathcal{L}^z(x)\mathcal{L}^-(x) + \pi \mathcal{S}^z(x)\mathcal{S}^-(x). \tag{2.11}$$

This lacks the interaction terms which mix $\mathcal{L}$ and $\mathcal{S}$ in the proper manner to permit the equations of motion to be expressed entirely in the $\vec{\mathcal{J}}(x)$ operators. If this were true, then these equations would describe the correct dynamics of the spin-one x-y model provided that only states with $\mathcal{J}^2 = 1$ are involved.

We introduce an interaction term into the Hamiltonian to correct this deficiency. This term is

$$\mathcal{H}_{\text{int}} = -\tfrac{1}{2}\pi \int dx [\mathcal{S}^+(x)\mathcal{L}^-(x) + \mathcal{L}^+(x)\mathcal{S}^-(x)]. \tag{2.12}$$

The equation of motion for $\vec{\mathcal{J}}(x)$, given by $[\mathcal{J}(x), \mathcal{H}_+ + \mathcal{H}_- + \mathcal{H}_{\text{int}}]$, now becomes the same as spin-$\frac{1}{2}$ case, since $(\mathcal{J}^z, \mathcal{H}_{\text{int}}) = 0$:

$$i\frac{d}{dt}\mathcal{J}^-(x)=\pi\mathcal{J}^z(x)\mathcal{J}^-(x)\,,$$
$$\left(\frac{id}{dt}\right)^2\mathcal{J}^z(x)=\nabla^2\mathcal{J}^z(x)\,, \tag{2.13}$$

which involves only the total spin operators $\mathcal{J}$.

This equation of motion is not that of the spin-one x-y problem until we show that it involves only states with spin-one symmetry. But now, we note that the spin algebra requires $[\mathcal{J}^2(x),\vec{\mathcal{J}}(x')]=0$, which can be verified using the relation

$$\mathcal{J}^2(x)\equiv\mathcal{L}^+(x)\mathcal{S}^-(x)+\mathcal{S}^+(x)\mathcal{L}^-(x)+2\mathcal{S}^z(x)\mathcal{L}^z(x)\,.$$

There is therefore a local conservation law that states

$$\left[\mathcal{J}^2(x'),i\frac{d}{dt}\mathcal{J}^-(x)\right]=0\,,$$

and therefore the time evolution operator preserves the local $\mathcal{J}^2$ symmetry as well as the global symmetry corresponding to the conservation of

$$\sum_i J_i^2=\int dx\,\mathcal{J}^2(x)\,.$$

Further, the equations of motion for $\mathcal{J}^+$ and $\mathcal{J}^z$ also satisfy this local conservation law. Because the $\vec{\mathcal{J}}(x)$ operators have only matrix elements within the triplet subspace, a correlation function has space and time dependence determined only by this subspace. We show in the Appendix that triplet state at each site is the ground state.

We conclude that Eq. (2.13) is the spin-one, dimension-one x-y problem. This is equivalent, at least as far as the space and time dependence is concerned, to the fermion problem

$$\mathcal{H}=\mathcal{H}_+ +\mathcal{H}_- +\mathcal{H}_{\rm int}=\tfrac{17}{4}\pi L^{-1}\sum_{k>0,\,\alpha=\pm}[\rho_{1,\alpha}(k)\rho_{1,\alpha}(-k)+\rho_{2,\alpha}(-k)\rho_{2,\alpha}(k)]$$
$$-\tfrac{15}{4}\pi L^{-1}\sum_{k,\,\alpha=\pm}\rho_{1\alpha}(k)\rho_{2\alpha}(-k)-\tfrac{1}{2}\int dx[\psi^\dagger_{1+}(x)\psi^\dagger_{2+}(x)\psi_{2-}(x)\psi_{1-}(x)+\text{H.c.}]\,, \tag{2.14}$$

with the spin operators

$$\mathcal{J}^+(x)=\sum_{\alpha=\pm}\psi^\dagger_\alpha(x)\psi^\dagger_\alpha(x),\quad \mathcal{J}^-(x)=[\mathcal{J}^+(x)]^\dagger\,,$$
$$2\mathcal{J}^z(x)=\sum_{\alpha=\pm}[\rho_{1\alpha}(x)+\rho_{2\alpha}(x)]\,. \tag{2.15}$$

This fermion problem can be tranformed into several more familiar forms. In particular, the backward scattering problem of the one-dimensional electron gas is one equivalent problem, and in Sec. III we bend Eq. (2.14) into this form, and finally into the sine-Gordon equation for which solutions exist.

We will also require a fermion form for the $\Delta\sum_i (J_i^z)^2$ term in Eq. (1.9). This is readily accomplished by using $(J_i^z)^2=2L_i^zS_i^z$, and taking the continuum limit, with the result

$$\Delta\sum_i (J_i^z)^2\to 2\Delta\int dx\,\mathcal{L}^z(x)\mathcal{S}^z(x)\,. \tag{2.16}$$

Expressing these spin operators in fermion equivalents leads to an additional term bilinear in density operators which contains the temperature field Δ in the classical two-dimensional problem.

III. TRANSFORMATION AND SOLUTION OF THE SPIN-ONE CHAIN

In this section we discuss how the spin-one chain Hamiltonian Eq. (2.14) plus Eq. (2.16) can be transformed into other Hamiltonians whose properties have been extensively studied. We then use previously known results to discuss the physics of our problem. The logic involves the identification of charge-density and spin-density degrees of freedom, and the separation of these into two separate commuting Hamiltonians analogous to the charge- and spin-density separation in the backward scattering problem. The charge-density Hamiltonian has the well-known Luttinger form[14] and can be directly solved. The spin-density Hamiltonian can be transformed into three related problems from which we can deduce a number of its properties. First we show that the spin-density Hamiltonian corresponds to a continuum limit of a spin-$\frac{1}{2}$ x-y-z model which is involved in the solution of the Baxter model.[5] It can also be transformed to the sine-Gordon equation whose eigenvalue spectrum is known.[9] It also is equivalent to the backscattering problem.[10]

The Hamiltonian given by Eq. (2.14) plus Eq. (2.16) can be separated into two commuting pieces by introducing the charge- and spin-density operators $\sqrt{2}\rho_i=\rho_{i+}+\rho_{i-}$ and $\sqrt{2}\sigma_i=\rho_{i+}-\rho_{i-}$, where i stands for 1 or 2. These new variables satisfy the usual commutation relations of the density operators. In terms of these, the Hamiltonian Eq. (2.14) plus Eq. (2.16) becomes $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$, with

$$\mathcal{H}_0 = \frac{2\pi v_0}{L}\sum_{k>0}\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(k) + \frac{V_{\parallel}}{L}\sum_k \rho_1(k)\rho_2(-k) \tag{3.1}$$

and

$$\mathcal{H}_1 = \frac{2v_1}{L}\sum_{k>0}\sigma_1(k)\sigma_1(-k)+\sigma_2(-k)\sigma_2(k) + \frac{U_{\parallel}}{L}\sum_k \sigma_1(k)\sigma_2(-k)+U_{\perp}(2\pi\alpha)^{-2} \times\int dx\,(e^{\sqrt{2}(\Phi_1+\Phi_2)}+\text{H.c.}). \tag{3.2}$$

Here the velocities and coupling constants are given by

$$\begin{aligned} 2\pi v_0 &= \tfrac{17}{4}\pi+\tfrac{1}{2}\Delta, \quad V_{\parallel} = -\tfrac{15}{4}\pi+\tfrac{1}{2}\Delta, \\ 2\pi v_1 &= \tfrac{17}{4}\pi-\tfrac{1}{2}\Delta, \quad U_{\parallel} = -\tfrac{15}{4}\pi-\tfrac{1}{2}\Delta, \end{aligned} \tag{3.3}$$

and $\Phi_1=(\phi_{1+}-\phi_{1-})/\sqrt{2}$ and $\Phi_2=(\phi_{2+}-\phi_{2-})/\sqrt{2}$.

The $\mathcal{H}_0$ part has the Luttinger form while the $\mathcal{H}_1$ part can be related to the continuum limit of a spin-$\frac{1}{2}$ x-y-z Hamiltonian by making a canonical transformation to remove the $\sqrt{2}$ in the $U_{\perp}$ part. The transformation of the $\sqrt{2}$ factor is carried out by means of the usual canonical transformation

$$\begin{aligned} \sigma_1 &\to \sigma_1\cosh\varphi+\sigma_2\sinh\varphi, \\ \sigma_2 &\to \sigma_1\sinh\varphi+\sigma_2\cosh\varphi, \end{aligned} \tag{3.4}$$

with $e^{\varphi}=\sqrt{2}$. Collecting terms, the Hamiltonian of Eq. (3.2) becomes

$$\mathcal{H}_1' = \frac{2\pi v_1'}{L}\sum_{k>0}\sigma_1(k)\sigma_1(-k)+\sigma_2(k)\sigma_2(k) + \frac{U_{\parallel}'}{L}\sum_k \sigma_1(k)\sigma_2(-k)+U_{\perp}(2\pi\alpha)^{-1} \times\int dx\,(\psi_2^{\dagger}\psi_1^{\dagger}+\text{H.c.}), \tag{3.5}$$

where we have used Eq. (2.3) to replace the exponential operators by Fermi operators, and $2\pi v_1' = \frac{5}{2}\pi-\Delta$, $U_{\parallel}'=-(\frac{3}{2}\pi+\Delta)$, and $U_{\perp}=-\frac{1}{2}\pi$. This problem is now recognizable as the continuum limit of the spin-$\frac{1}{2}$ x-y-z chain problem studied in Ref. 5. If one makes the canonical transformation $\sigma_2\to-\sigma_2$ then the $U_{\perp}$ term has the backscattering form $\psi_1^{\dagger}\psi_2$.

Alternatively, one can also transform away the $U_{\parallel}$ term in Eq. (3.2) changing H_1 to the sine-Gordon Hamiltonian. First let $\sigma_2\to-\sigma_2$ so that the exponent in the $U_{\perp}$ part of Eq. (3.2) has the form $e^{\sqrt{2}(\Phi_1-\Phi_2)}$ and $U_{\parallel}'$ changes sign. Then eliminate $U_{\parallel}'$ by the usual canonical transformation with

$$\tanh 2\varphi = U_{\parallel}'(2\pi v_1')^{-1}, \tag{3.6}$$

and we have

$$\mathcal{H}_1'' = \frac{2\pi v_1''}{L}\sum_{k>0}\sigma_1(k)\sigma_1(-k)+\sigma_2(-k)\sigma_2(k) + U_{\perp}(2\pi\alpha)^{-2}\int dx\,[\exp\sqrt{2}\,e^{\varphi}(\Phi_1-\Phi_2)+\text{H.c.}], \tag{3.7}$$

with $2\pi v_1''=2\pi v_1'(1-U_{\parallel}^2)^{1/2}$. Finally setting

$$\sqrt{2}\,e^{\varphi}(\Phi_1-\Phi_2)=i\theta(x)(8\pi)^{1/2}e^{\varphi}, \tag{3.8}$$

$\mathcal{H}_1''$ takes the familiar sine-Gordon form, with scalar field $\theta(x)$.

The eigenvalue spectrum for the various equivalent Hamiltonians can be written down directly from previous work. The spectrum is characterized by a gap for creating a soliton-soliton pair Δ', with the possibility of bound solitons depending on the parameter

$$\theta=(2\pi v_1+U_{\parallel})^{1/2}(2\pi v_1-U_{\parallel})^{-1/2}.$$

The spectrum of bound solitons is given by

$$[\Delta_n'(q)]^2=(\Delta_n')^2+(v_1'q)^2, \tag{3.9}$$

where

$$\Delta_n'=\Delta'\sin\tfrac{1}{2}n\pi\theta/(1-\theta),$$

and n is any positive integer such that $n<\theta^{-1}-1$. These states are below the soliton gap at energy Δ', and occur for $\theta<\frac{1}{2}$, that is, $U_{\parallel}<-\frac{6}{5}\pi v_1$. These states cause exponential decay in the planar model. The gap Δ' is proportional to $U_{\perp}^{(1-\theta)^{-1}}$, and therefore $\Delta'\sim U_{\perp}^{1/2}$, as $\theta\to0$. At this special point, the soliton binding energy goes to zero—that is, there is always a state $n_c\sim\theta^{-1}$, with a gap vanishing proportional to θ. It is this point, which is the "critical" temperature of the model, and the parametrization in terms of the temperature leads to the conclusion that the gap vanishes as $(T_c-T)^{1/2}$, as $T\to T_c$ from below. For $T>T_c$, i.e., $U_{\parallel}<-2\pi v_1$. It is necessary to introduce a cutoff before defining a continuum limit, a procedure discussed elsewhere.[15] This procedure has been carried out, is tedious and not interesting, for it gives the symmetric result of a gap vanishing as $(T-T_c)^{1/2}$ for $T\to T_c$ from above.

In order to understand which features of the eigenvalue spectrum of these various Hamiltonians are important in determining the properties of the phase transition, we will pause to look at the order-parameter correlation function $\langle\psi(0,y)\psi^*(0,0)\rangle$. In terms of the transfer Hamiltonian formalism, this correlation function is proportional to the ground-state expectation value $\langle0|J^-(x)J^+(0)|0\rangle$. Here we have replaced y by x, and $|0\rangle$ is the ground state of the transfer Hamiltonian Eq. (2.14) plus Eq. (2.16). The original spin problem on the lattice together with spin conservation can be used to show that

$$\langle 0|J_i^- J_j^+|0\rangle = \langle 0|L_i^- L_j^+|0\rangle + \langle 0|S_i^- S_j^+|0\rangle ,$$

and hence in the continuum limit we need to consider the pair-density operators introduced in Eq. (2.10). Expressing these operators in terms of the ρ_i amd σ_i operators of Eqs. (3.1) and (3.2) we find that

$$\langle 0|J^-(x)J^+(0)|0\rangle \sim \langle e^{(1/2)[\theta_1(x)+\theta_2(x)]}\, e^{-(1/2)(\theta_1+\theta_2)}\rangle_0 \times \langle e^{(1/2)[\phi_1(x)+\phi_2(x)]} e^{-(1/2)(\phi_1+\phi_2)}\rangle_1 , \tag{3.10}$$

with the $\theta_i = (\phi_{i+} + \phi_{i-})\sqrt{2}$ and, as before, $\phi_i = (\phi_{i+} - \phi_{i-})/\sqrt{2}$. Since the ground state of $\mathcal{H}_0 + \mathcal{H}_1$ is the product of their separate ground states and since θ_i depends only upon ρ_i, while ϕ_i depends only upon σ_i, the expectation value in Eq. (3.9) has been factored. In the first term, denoted by "0", the ground state of $\mathcal{H}_0$ is used in taking the expectation value while the second expectation value denoted by "1" is taken in the ground state of $\mathcal{H}_1$. Thus the correlation function factors into two pieces $C_0(x)$ and $C_1(x)$ associated with $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

The correlation function $C_0(x)$ for the $\mathcal{H}_0$ problem has the usual power-law behavior

$$C_0(x) \sim x^{-\eta_0(T)} \tag{3.11}$$

associated with the Luttinger model. Here

$$\eta_0(T) = (2\pi v_0 + V_\parallel)^{1/2}(2\pi v_0 + V_\parallel)^{-1/2} , \tag{3.12}$$

and $2\pi v_0$ and $V_\parallel$ are related to the temperature-dependent parameter $\Delta(T)$ by Eq. (3.3). The correlation function $C_1(x)$ for the $\mathcal{H}_1$ degrees of freedom is more complicated. As we have seen $\mathcal{H}_1$ can be transformed into the continuum limit of a spin-$\frac{1}{2}$ x-y-z model. It can therefore be identified with the transfer matrix of a Baxter model. We therefore expect that $C_1(x)$ will have the form

$$C_1(x) = x^{-\eta_1} f(mx) . \tag{3.13}$$

As the temperature is lowered from some high value, Δ decreases until the parameters $2\pi v_1'$ and $-U_\parallel'$ of Eq. (3.5) become equal. This occurs for $\Delta = \frac{1}{2}\pi$ and corresponds to the ferromagnetic critical point of the spin-$\frac{1}{2}$ chain. At this value η and m vanish making $C_1(x)$ a constant. At this point, the value of η_0 obtained from Eq. (3.11) is $1/\sqrt{8}$, the result quoted in the Introduction. For $T < T_c$ the correlation function exhibits long-range order and the space dependence is determined by $C_0(x)$.

The calculation of the susceptibility exponent γ, from the modified Fisher's relation of Ref. 5, gives $\gamma = 2\nu(1 - \eta_0 - \eta_1)$, gives $1 - 8^{-1/2} \cong 0.63$.

IV. CONCLUSION

The critical region in our formulation of the two-dimensional problem corresponds for the fermion problem to a coupling parameter region where both solitons and bound solitons occur. At the critical temperature T_c, the number of bound soliton states diverges while their excitation energy vanishes. The order-parameter correlation function $C(r)$ near or at T_c has the structure.

$$C(r) = r^{-\eta_0(T)} C_1(r) . \tag{4.1}$$

The prefactor exponent $\eta_0(T)$ varies smoothly near T_c and at T_c we find that $\eta_0(T_c) = 8^{-1/2}$.

The correlation function $C_1(r)$ is equal to a standard second-order phase transition correlation function of the form

$$C_1(r) = r^{-\eta_1} f(mr) , \tag{4.2}$$

with $m \sim |T - T_c|^\nu$. It exhibits long-range order for $T < T_c$, $C_1(r) \to$ const, as $r \to \infty$, while $f(mr) \sim e^{-mr}$ for $T > T_c$. The exponents η_1 and ν are related by the scaling relation $2\nu = (1 - \eta_1)^{-1}$, which has also been shown to apply to the Baxter-model electric exponents in the continuum limit. Indeed, "unphysical" variables can be introduced to make $C_1(r)$ formally the same as an electric correlation function for the Baxter model. Since the full η exponent of $C(r)$ is equal to $\eta_0 + \eta_1$ and $C(r)$ satisfies the usual Baxter scaling relations, we propose the name "displaced scaling" to describe the behavior of $C(r)$.

These results agree with the conventional qualitative picture of a diverging susceptibility as T_c is approached from above. Here we find a susceptibility exponent $\gamma = \nu(2 - \eta_1 - \eta_0)$ and for the model we have solved $\eta_1 = 0$ and $\nu = \frac{1}{2}$.

Kosterlitz and Thouless have recently constructed a theory of the critical-point behavior by focusing on the role of vortex excitations. They find that the partition function for the vortex degrees of freedom can be viewed as that of a two-dimensional Coulomb gas:

$$Z = \sum_{n=0} \frac{z^{2n}}{(n!)^2} \int \prod_i^{2n} \frac{dr_i^2}{\alpha^2} \times \exp\left(-\beta \sum_{i \neq j} e_i e_j \ln \left| \frac{|r_i - r_j|^2}{\alpha^2} \right| \right) . \tag{4.3}$$

For an x-y model with a near-neighbor exchange coupling J, the fugacity

$$z = \exp[-(\pi J/T)(\gamma + \tfrac{3}{2}\ln 2)]$$

and $\beta e^2 = \pi J/T$. The parameter α represents a short-distance lattice cutoff.

It has been shown that this partition function Eq. (4.3) is formally equivalent to the *ground-state* expectation value of $\exp(-\beta H_1)$, where H_1 is the

backward scattering Hamiltonian Eq. (3.2). In the language of the fermion problem

$$\theta = (1 + U_{\parallel}/2\pi)^{1/2}/(1 - U_{\parallel}/2\pi)^{1/2},$$

$z = U_{\perp}/2\pi$, and the renormalized Fermi velocity is chosen to equal unity. Thus $T \to 0$ in the statistical mechanics problem corresponds to $U_{\parallel}/2\pi \to 1$. As T is increased from zero, $U_{\parallel}/2\pi$ decreases from 1 and the first "critical point" of the fermion problem is encountered at $T_c = \frac{1}{2}\pi J$, where $\theta = 1$ and $U_{\parallel}/2\pi = 0$. The region near $U_{\parallel} = 0$ is subtle and is characterized by the onset of a gap which initially increases for $T > \frac{1}{2}\pi J$, if $U_{\perp} = U_{\parallel}$. From the solution of the fermion problem one expects that this gap increases as $(T - T_c)^{1/2} \exp -(T - T_c)^{-1}$. This gives rise to "exponential scaling," which implies the correlation length diverges with this exponential behavior.[6]

A difficulty occurs if we continue to raise the temperature in this model. At the temperature corresponding to $\theta_1 = \frac{1}{2}$, $T = 2T_c$, bound solitons appear, and as the temperature becomes even larger, progressively more and more such states appear at temperatures $T_n = nT_c$ until some type of condensation occurs as $T \to \infty$. It seems implausible to us that any further transitions occur at high temperatures, and therefore we believe the model cannot be trusted in this region. Since the only temperature scale is T_c, all expansions are in powers of T/T_c, and we believe it safe to apply this model only at low temperatures $T \ll T_c$, where the vortex density is small.

Zittartz has also recently proposed a theory of the two-dimensional classical x-y model. He has emphasized the role of the magnetic field B and views the nonlinearities in the expansion of the $J\cos(\phi_i - \phi_j)$ exchange interaction as leading to a renormalization of J with the basic nature of the field dependence of the partition function determined by the quadratic part

$$J\cos(\phi_i - \phi_j) \sim J[1 - \tfrac{1}{2}(\phi_i - \phi_j)^2]$$

of the exchange. He is lead to a partition function of the form given in Eq. (4.3) but with $z = B/2T$, and $\beta e^2 \equiv \frac{1}{2}\eta(T) = T/4\pi cJA$. Here Zittlartz takes the low-lying spin-wave excitations to have energy cAk^2 as the wave vector $k \to 0$. A is assumed to depend upon T/B due to renormalization effects. Using the same back scattering identification we see that $T \to 0$ corresponds to $\eta = 0$ and $U_{\parallel}/2\pi = -1$. As T increases η increases until for a particular value $\eta = 2$ corresponding to $U_{\parallel}/2\pi v_F = -\frac{3}{5}$. At this point χ diverges. In this approach as T increases further, η continues to grow until it reaches the value 4 at $\frac{1}{2}U_{\parallel} = 0$ at which point the $K = 1 + 1/\delta \to \infty$. This occurs for $U_{\parallel}/2\pi = 0$.

Zittartz approach requires that no transition occurs until η increases to two. Our results give a value of η, at the transition temperature, of $(8)^{-1/2}$. If we view η as a measure of the temperature scale, since all models predict η to be a monotonically increasing function of temperature, clearly the gap which appears at this point controls the higher temperature behavior.

Thus the behavior of these models in the critical region is also determined by the properties of the one-dimensional backward scattering fermion problem. However, as we have seen, the models have different definitions of the coupling parameters and the critical regions correspond to different regions of the $U_{\perp} - U_{\parallel}$ plane.

What physics underlies the differences between these various models? We believe that there are various possibilities. The approach of Kosterlitz and Thouless neglects spin waves and spin-wave-vortex interactions which become important in just the temperature region near T_c. Zittartz uses a term by term cumulant form to argue that the basic structure of the critical region is determined by the quadratic terms in the exchange coupling with the nonlinearities simply renormalizing the spin-wave coefficients A. However, a similar treatment of the sine-Gordon equation does not lead to the correct energy-level spectrum. Our treatment views the nonlinear aspects as the most important part of the problem. This is put in from the beginning by working with the lowest three states at each site, which is appropriate in the high-temperature region of interest. We also have kept all the interactions between excitations. Clearly, however, our approach is not without difficulties. We cannot obtain the low-temperature limit, in which $\eta \to 0$, with our three-state approximation. This requires an infinite number of states as discussed in Appendix A. Finally, there is the nagging question of the role of the spurious $S = 0$ states which our spin-one fermion representation introduces. We discuss this in Appendix B and show that a gap exists between the ground state and states with any singlet spin parts, and believe that this gap is sufficient to suppress any singlet contamination.

Another question obviously raised by our results concerns the breakdown of universality, and the degree to which properties of this model can be extended to others. As discussed by Betts *et al.*,[3] there is evidence from high-temperature series expansions that exponents depend on the spin value. The model solved here is presumably not equivalent to a particular spin value on a lattice.

However, it may be possible to use our results to infer the critical exponents of the x-y model in the two-dimensional lattice. This can be done from Eq. (3.5), and a matching condition on cou-

pling constants for the lattice theory. The matching condition simply involves constructing that lattice theory whose continuum limit is Eq. (3.5), and has been discussed in connection with the "extra scaling law" in the Baxter model.[5] It must be kept in mind, as noted above, that the nature of the critical-point behavior for the two-dimensional planar model lacks the universality associated with critical points in higher dimensions. Thus, indices may depend not only upon the lattice to continuum limiting process but also on the spatial isotropy of the model.[16] The model considered here has $c_x \gg c_y$ rather than the usual $c_x = c_y$ coupling of the isotropic x-y model.

Some preliminary work attempting to build in the finite lattice spacing indicates that the type of transition at T_c does change. Here, we found that the region near T_c has the same characteristics as the fully isotopic (Heisenberg) x-y-z model. On the lattice, the transition is described by the Heisenberg-Ising model, and both exponents η_1 and η_2 are changed from the continuum-limit values quoted here. The tendency is to increase η_2, decrease η_1, and increase γ towards the values typical of the high-temperature series.

As mentioned in the introduction, our results can also be applied to spin-one chain problems. For example they give a definite prediction about the exponent for the transverse spin-correlation function in the spin-one x-y chain. This corresponds precisely to Eq. (1.9) with $\Delta = 0$. Calculation of the $\langle \mathcal{J}^+(x)\mathcal{J}^-\rangle$ function can be read from Eq. (3.10) giving $\langle \mathcal{J}^+(x)\mathcal{J}^-\rangle \sim x^{-1/4}$. This value might be expected, since the corresponding spin-$\frac{1}{2}$ experiment is $\frac{1}{2}$ and the large-spin calculations suggest a $1/S$ dependence.[17] We also expect that the methods introduced in this paper may be of use in other field-theoretic problems involving systems with multiple internal degrees of freedom.

APPENDIX A: TRANSFER HAMILTONIAN

The transfer Hamiltonian given by Eq. (1.6) is appropriate for a $|\psi|^4$ problem in which $c_y \ll c_x$. In this case the transition temperature T_c lies well below its mean-field value, and the potential $a|\psi|^2 + b|\psi|^4$ has a well developed double minimum for T near T_c. In this case, radial excitations of the single-site problem are not important, and the angular excitations described by Eq. (1.6) determine the ground-state properties.

Now, as discussed in Sec. I, the moment of inertia I of the rigid rotor is given by

$$I = 1/2\Delta = \tfrac{1}{2}(|a|/b)^2 c_x c_y/(kT)^2 . \tag{A1}$$

The excitation spectrum of a single rotor is $l^2/2I$, with $l = 0, \pm 1, \pm 2, \ldots$. For a nonvanishing T_c, the two degenerate excited states with $l = \pm 1$ lie $(2I)^{-1}$ above the ground state, while the next twofold degenerate state with $l = \pm 2$ lies four times higher in energy. We therefore believe that the essential structure of the continuous symmetry problem near T_c can be retained by keeping only the $l = 0$ and $l = \pm 1$ states.

However, as T decreases towards zero, the moment of inertia diverges, and the excited states collapse towards zero energy. In the limit $T = 0$, the ground state consists of completely alligned rotors. To achieve this alignment requires an infinite number of angular momentum states. For T greater than 0, but small compared to T_c, one knows that the order-parameter correlation function decays algebraically with an exponent that approaches zero as the temperature vanishes. Let us see how this result follows from the ground-state properties of the rigid-rotor transfer Hamiltonian Eq. (1.6).

Near zero temperature, one can locally expand $\cos(\varphi_{i+1} - \varphi_i)$ in powers of the small angular deviation of the neighboring rotors $\varphi_{i+1} - \varphi_i$ to obtain

$$\mathcal{H} \simeq \sum_i \left(-\Delta \frac{\partial^2}{\partial \varphi_i^2} + (\varphi_{i+1} - \varphi_i)^2 \right) . \tag{A2}$$

Fourier transforming in the usual way with

$$\varphi_n = \frac{1}{\sqrt{L}} \sum_k \frac{1}{(2\omega_k)^{1/2}} \left(a_k e^{ikn} + a_k^\dagger e^{-ikn} \right) , \tag{A3}$$

leads to

$$\mathcal{H} \simeq \sum_k \omega_k (a_k^\dagger a_k + \tfrac{1}{2}) , \tag{A4}$$

with $\omega_k = k/\Delta^{1/2}$. The order-parameter correlation function is given by the ground-state expectation value

$$C(n-m) = \langle 0 | e^{i\varphi_n} e^{-i\varphi_m} | 0 \rangle = e^{-\langle 0 | (\varphi_n - \varphi_m)^2 | 0 \rangle / 2} , \tag{A5}$$

with

$$\langle 0 | (\varphi_n - \varphi_m)^2 | 0 \rangle = \frac{1}{L} \sum_k \frac{1}{2\omega_k} (e^{ik(n-m)} - 1)$$
$$\simeq (\sqrt{\Delta}/2\pi) \ln |n-m| . \tag{A6}$$

Setting $r = n - m$ and $\sqrt{\Delta}/2 = kT/\epsilon$, with $\epsilon = 2(c_x c_y)^{1/2}|a|/b$, we have the expected result

$$C(r) = (1/r)^{kT/\epsilon} . \tag{A7}$$

In this way of looking at the problem the zero-point fluctuations of the quantum-mechanical rotors about the nearly aligned state give rise to the characteristic low-temperature power-law behavior. These fluctuations can only be described when the entire set of rigid-rotor levels are available.

The transfer Hamiltonian Eq. (3.8) was originally constructed as a means of analyzing the phase transition of a two-dimensional classical complex Ginzburg-Landau field. It can also be related to the classical two-dimensional x-y model. Consider the energy functional for a x-y model on a two-dimensional lattice

$$E\{\varphi_{ij}\} = -J_{\parallel} \sum_{(i,j)} \cos(\varphi_{i+1,j} - \varphi_{i,j}) - J_{\perp} \sum_{(i,j)} \cos(\varphi_{i,j+1} - \varphi_{i,j}) . \tag{A8}$$

The partition function is given by

$$Z = \pi \int_0^{2\pi} d\varphi_{ij}\, e^{-\beta E[\varphi_{ij}]} . \tag{A9}$$

If $J_{\parallel} \ll J_{\perp}$, it is useful to integrate out the behavior along the ith row noting that

$$\int_0^{2\pi} d\varphi'\, e^{\beta J_{\parallel} \cos(\varphi - \varphi')} \frac{e^{in\varphi'}}{(2\pi)^{1/2}} = I_n(\beta J_{\parallel}) \frac{e^{in\varphi}}{(2\pi)^{1/2}} . \tag{A10}$$

Here I_n is the modified Bessel function. If only the $n=0$ and $n=\pm 1$ terms are kept, the transfer Hamiltonian for this problem is

$$\sum \epsilon (J_i^x)^2 = -\pi J_{\perp} \sum_j (L_{j+1}^+ L_j^- + L_{j+1}^- L_j^+) , \tag{A11}$$

with $\beta\epsilon = \ln[I_1(\beta J_{\parallel})/I_0(\beta J_{\parallel})]$. This has the same structure as Eq. (3.8).

APPENDIX B: TRIPLET GROUND STATE

The discussion here centers on the treatment of the spin equations of motion discussed in Sec. II. We seek to prove, as assumed there, that the triplet state *at each site* is the lowest-energy configuration. Then we show that a gap exists to the first excited state, with one singlet excited, and that there is no degeneracy factor associated with the excited state which would destroy the ground state.

The proof requires establishing a symmetry property of the Hamiltonian considered in Sec. II. There it was argued that the spin-one problem could be replaced by the spin problem:

$$\mathcal{H}_J = -\sum_i (J_i^+ J_{i+1}^- + J^- J_{i+1}^+) + \Delta (J_i^x)^2 , \tag{B1}$$

with $\vec{J} = \vec{L} + \vec{S}$, $\vec{L}$, and $\vec{S}$ are spin-$\frac{1}{2}$ operators. This problem is only a spin-one problem if every site is in the triplet state. Since we then solve equations of motion for expectation values, at zero temperature, our calculations can only be correct if the ground state is triplet at each site. The symmetry property needed to prove this is

$$[J_i^2, \mathcal{H}_J] = 0 \quad \text{for all } i \tag{B2}$$

Obviously, $\vec{J}_i$ has no matrix elements which change J_i^2 at any site. It is intuitively plausible that the triplet states are lower in energy since they can take advantage of the hopping kinetic energy from site to site, while the singlets have no $\vec{J}$ matrix elements and cannot participate.

This property means that the energy levels of the system can be easily classified. Consider a system with N_1 consecutive triplet states starting at site 1, then one singlet, followed by N_2 triplets, to form a chain $N_1 + N_2 + 1$ sites long. The singlet site breaks the chain, for there are no matrix elements of $\mathcal{H}_J$ across it. Thus, if we have zero boundary conditions, the energy of this configuration is $E(N_1) + E(N_2)$, where $E(N_1)$ is the energy of a triplet chain N_1 sites long. Adding more singlet sites segments the chain further, for the energy of each singlet site is zero. We now show

$$E_G(N_1 + N_2 + 1) < E_G(N_1) + E_G(N_2) ,$$

which states that the segmented system, with each segment in its ground state, is higher in energy than the all-triplet ground state.

Suppose every site of a segment N links long is in the triplet site. The Hamiltonian Eq. (B1) then has the same energy eigenvalues $E(N)$ as the spin-one problem $\epsilon(N)$:

$$\epsilon(N) = E(N) . \tag{B3}$$

We expect $E(N) = \Phi_G N + \Phi_S$, where Φ_G is the ground-state energy per spin, and Φ_S is a boundary energy associated with the endpoints of the segment.

An estimate of $E(N)$ is given by using the trial function $\psi = \prod_i \psi_x(i)$, where $\psi_x(i)$ is the eigenstate of J_x at site (i). Evaluating $\langle \psi | H | \psi \rangle$ using Eq. (1.8) gives the result

$$E(N) \leq (N-1) - \tfrac{1}{2} N\Delta , \tag{B4}$$

and, at least for small Δ, the triplet chain is favored over the singlet chain, since the latter has zero energy. For the case $\Delta = 0$, the continuum limit of this problem,[15] $N \to \infty$, with fixed length L, results in an infinite energy difference per length between the triplet and singlet chains, since $E(N) = -N = -L/s$, with s the lattice constant.

The continuum limit further requires a renormalization of the parameter Δ on the left-hand side of Eq. (2.16). We treat the coupling constant Δ in the symmetry-breaking term on the right-hand side of Eq. (2.16) as finite, which implies the parameter for the lattice theory must approach zero as $N \to \infty$. For the moment, attach a subscript L or C to indicate the lattice and continuum coupling constant respectively. Dimensional analysis then requires $\Delta_L \sim s\Delta_C$, which establishes the re-

normalization. For this reason, the $\Delta = 0$ case and the infinite-energy difference between triplet and singlet chains is appropriate for the continuum-limit model here. For the lattice problem, a finite Δ is required to involve a crossing between singlet and triplet states. In the continuum limit, that energy is infinitely large, because of the infinite ground-state energy density which must be removed to correct the triplet chain to the singlet. The parallel of this result to problems of quark confinement is obviously strong.

The form $E(N) = -|\Phi_G|N + \Phi_S$ is sufficient to establish the desired inequality, provides $2\Phi_S > -|\Phi_G|$. But the boundary energy contribution is clearly positive, for we can view the zero boundary condition implied by a singlet state, as an infinitely repulsive barrier.

We must now argue that the gap of $\approx \Phi_G$ implied by the above is not overwhelmed by entropy factors as in the one-dimensional Ising model. Here, however, the situation differs because *every* singlet costs an energy $\approx |\Phi_G|$. The proper analogy is to the one-dimensional Ising model in an external field at zero temperature, for which every spin has the Zeeman energy to overcome the entropy. There is a real gap in the eigenvalue spectrum, and the ground state is fully aligned. That alignment corresponds to the triplet state at every site, with a finite gap of magnitude greater than 2 to excite the first singlet.

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