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Critical exponents for the conductivity of random resistor lattices

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This paper presents three results concerning the critical exponents which characterize the conduction threshold of a resistor lattice. (a) There are no rigorous inequalities similar to those for the phase-transition critical exponents. (b) There is a dual transformation in two dimensions which relates the critical exponents: in particular $s = t$, $u = 1/2$ for the two-dimensional bond problem. (c) The exponents for the two- and three-dimensional bond and site problems are estimated by numerically solving for the voltage distributions of large finite disordered lattices. The results are in agreement with the "scaling" exponent relationship.

I. CONDUCTION THRESHOLD AS A CRITICAL POINT

Let a finite resistor lattice of side L be constructed of elements chosen randomly to have conductance b (with probability p) or a (with probability q), where $a<b$ and $p+q=1$. In the limit that L is very large, the specific conductivity of such a system comes to have the character of a thermodynamic state function: that is, the sample-to-sample variation becomes vanishingly small, and we may refer to the specific conductivity of a given specimen as if it depended only on a, b, and p.

The case $a=0$, b finite (a resistor lattice with some of the resistors removed) has received considerable attention,[1-3] and one aspect of this case carefully noted: there is a threshold concentration p_c such that for $p<p_c$ the lattice is so fragmented that it cannot conduct, and for $p>p_c$ there are connected conducting paths of infinite extent. Furthermore, it has been noticed that in the latter regime the conductivity obeys a power-law relation[4-6] $\sigma \sim b(p-p_c)^t$ for $p>p_c$. The author has recently proposed[7] that $a/b=0$, $p=p_c$ is a critical point of the specific conductivity analogous to those studied in magnets and phase transitions, and that the analogy may be pursued to the construction of a homogeneous function representation. The representation is modeled on the enormously successful scaling theory[8,9] of the temperature dependence and field dependence of the magnetization of magnets near the Curie point. In the translation from that system to the resistor lattice the variables *temperature* and *field strength* are replaced, respectively, by *composition* $\epsilon=p-p_c$ and the *ratio of the conductivities* (a/b) of the two components. The homogeneous function representation of the conductivity takes the form $\langle\sigma\rangle=\mu S(\epsilon\lambda^{-1}, a\mu^{-1}\lambda^{-s}, \mu b^{-1}\lambda^{-t})$, where S is defined wherever its three arguments are less than unity and singular only where more than one vanishes. The dependence on the parameter μ states the invariance under simultaneous change of scale $a\rightarrow\mu a$, $b\rightarrow\mu b$, $\langle\sigma\rangle\rightarrow\mu\langle\sigma\rangle$. The parameter λ takes on arbitrary values; giving S this dependence on λ is equivalent to saying that it depends only on arguments like $(a/\mu)\epsilon^{-s}$, $(b/\mu)\epsilon^{t}$, and $(a/\mu)^t(b/\mu)^s$. By hypothesis this representation is accurate for a wide range of λ and μ in the vicinity of the critical point.

The analogy is useful in construction of the homogeneous function representation, but cannot be taken too literally: one of the interesting aspects of this critical phenomenon is the many respects in which it differs from phase transitions. The problem is not readily defined in terms of a Hamiltonian. The bulk conductivity can be given a local definition in terms of the average power dissipated under the conditions of unit average gradient of the potential; it is necessarily positive but has no convexity property, with the consequence that no rigorous exponent inequalities can be proved (as will be demonstrated below). It is only useful "order parameter" that can be defined, with the consequence that the Legendre transformation to conjugate variables does not play an interesting role.

If the critical point is approached in different ways, different critical exponents will be observed. Thus if a is finite, $b=\infty$, and $p<p_c$ (a few superconducting links inserted into a normal resistor lattice), the conductivity diverges as the threshold is approached from below as $\sigma\sim a(p_c-p)^{-s}$. We may also consider the case $p=p_c$ with both a and b finite, and then[10] $\sigma\sim a^u b^{1-u}$. These aspects of the conductivity of a random lattice are displayed in Fig. 1. The homogeneous function representation relates these exponents; specifically

$$u=t/(s+t). \qquad (1)$$

Some other examples and the extension to the case of complex a and b (as would occur in ac circuits with capacitance) have been given elsewhere,[7] and the relationship (1) has been successfully checked

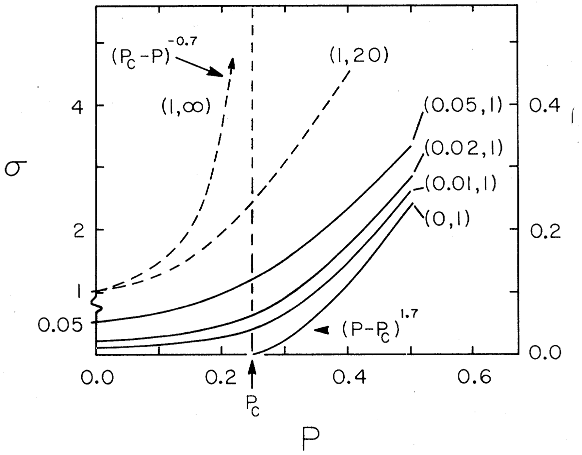

FIG. 1. Specific conductivity of a 3D bond lattice as a function of p (the fraction of b bonds), for various values of (a, b). The solid curves should be referred to the right-hand scale; the dashed curves are on the left. The two lines ($a=0.05$, $b=1$) and ($a=1$, $b=20$) are the same curve shifted in scale.

against the Bethe lattice model proposed by Stinchcombe.[11,12]

Another simple example can be provided: effective-medium theory shows a conduction threshold (e.g., see Kirkpatrick[3]), and exponents can be calculated in the context of this theory with the results $s=1$, $t=1$, $u=\frac{1}{2}$, which satisfies Eq. (1).

The Bethe "lattice" and effective-medium theory provide tractable examples of systems having a conduction threshold. However, they are not expected to be a good representation of real systems. Thus it is of considerable interest to check the theory against the more realistic two-dimensional (2D) and three-dimensional (3D) lattices, as will be done in Sec. III.

We should first note two rigorous results: (i) for the 2D bond problem there is an exact relationship $s=t$, $u=\frac{1}{2}$ (Sec. II). Thus the exponent relation is known to be exactly satisfied for this geometry. (ii) In the magnetic critical point it was possible to prove rigorous inequalities, for which the scaling equalities were the limiting case. In the resistor-lattice problem there is a rigorous nonresult, that no such inequalities can be proved. As a counter example we consider a pair of lattices having the same critical point but differing exponents, which in each case satisfy the exponent relation. Let those be coupled in parallel, so that $\sigma^* = \sigma_1 + \sigma_2$. The exponents which will be observed are $s^* = \max(s_1, s_2)$, $t^* = \min(t_1, t_2)$, $u^* = \min(u_1, u_2)$; and one readily shows that $u^* \geq t^*(s^* + t^*)^{-1}$. Now couple them in series and the reverse inequality results. Thus any combination of s^*, t^*, u^* can be achieved, and no rigorous inequality exists.

Section III presents these exponents in two and three dimensions resulting from the numerical solution of large random lattices. The technical details of the calculations are discussed in Sec. IV.

II. DUALITY RELATIONSHIP FOR TWO DIMENSIONS

In this section it will be demonstrated that there is a relationship between the conductivity of a 2D lattice with conductances chosen at random and the conductivity of a related distribution of conductances on the dual lattice. The continuum version of this relationship was demonstrated by Keller[13] and has been discussed by Mendelson.[14] The theorem and proof will only be sketched here since it differs only slightly from the foregoing discussions.

For simplicity consider a square quadratic lattice, and let the potential at the vertices be $V(i,j)$. Define $\Delta_i V(i,j) = V(i+1,j) - V(i,j)$, and define Δ_j similarly. Let the conductance of the bond in the i direction from site i to site $i+1$ be $\sigma_i(i,j)$. If $V(i,j)$ is the equilibrium potential appropriate to a distribution of conductance $\{\sigma\}$, we can write two conditions involving ΔV, which correspond to the continuum conditions $\operatorname{curl} \operatorname{grad} V = 0$ and $\operatorname{div} J = 0$:

$$\Delta_j V(i,j) - \Delta_i V(i-1,j+1) - \Delta_j V(i-1,j) + \Delta_i V(i-1,j) = 0, \quad (2)$$

$$\sigma_i(i,j)\Delta_i V(i,j) + \sigma_j(i,j)\Delta_j V(i,j) = \sigma_i(i-1,j)\Delta_i V(i-1,j) + \sigma_j(i,j-1)\Delta_j V(i,j-1). \quad (3)$$

We now wish to introduce a function $W(i,j)$ defined on the dual lattice.[15] In the present case the dual lattice is also square quadratic, and we can label the dual lattice sites so that dual site (i,j) is directly associated with direct site (i,j), as indicated in Fig. 2. Then choose $W(0,0)$ arbitrarily, and define W elsewhere by

$$\Delta_i W(i,j) = \sigma_j(i,j)\Delta_j V(i,j)$$

and

$$\Delta_j W(i,j) = -\sigma_i(i-1,j+1)\Delta_i V(i-1,j+1);$$

the W difference between regions separated by a lattice bond is the current flowing along that bond. In order that this definition be consistent, it is necessary that the ΔW around any closed loop sums to zero. This is equivalent to the condition that the net current into any vertex of the direct lattice is zero, and indeed substitution of the equations defining ΔW into the $\sum \Delta W = 0$ condition gives Eq. (3). Thus W exists. Now rewriting Eq. (2) in terms of W gives

$$\eta_i(i,j)\Delta_i W(i,j) + \eta_j(i,j)\Delta_j W(i,j) = \eta_i(i-1,j)\Delta_i W(i-1,j) + \eta_j(i,j-1)\Delta_j W(i,j-1), \quad (4)$$

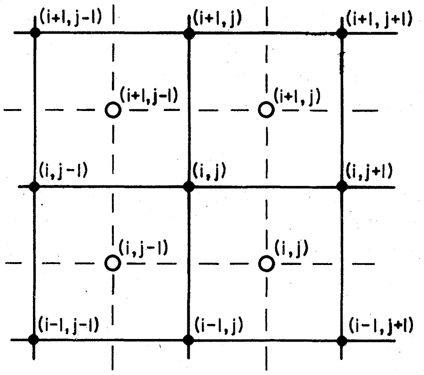

FIG. 2. Square lattice and its dual. The bonds of the lattice (solid lines) divide the plane into squares, each one of which is labeled by a dual lattice site (○) at its center. The bonds of the dual lattice (dashed lines) join the centers of squares which have a side in common. The dual of any planar lattice with noncrossing bonds can be constructed in this same way.

where $\eta(i,j)$ are functions defined on the dual lattice by $\eta_i(i,j)\sigma_j(i,j)=1$, $\eta_j(i-1,j)\sigma_i(i,j-1)=1$; the conductivity of each dual bond is the reciprocal of the conductivity of the direct bond it crosses, and W is the solution to Kirchhoff's equations for this dual problem.

The remainder of the discussion follows Mendelson very closely. Under the conditions that the specific conductivity is isotropic and is in fact a state function, we may proceed to show that the conductivity $\sigma(q,a;p,b)$ of a concentration q of conductances a and concentration p of conductances b is related to that for a system obtained by "interchanging the phases" by

$$\sigma(p,a;q,b)/ab=\sigma(p,b^{-1};qa^{-1})=[\sigma(q,a;p,b)]^{-1}. \tag{5}$$

The same relationship holds in the case of any planar graph, except that generally a lattice is different from its dual, so that Eq. (5) relates the conductivities of different lattices. We must further realize that it is only the *bond* problem of the square lattice that is self-dual. The site problem (where the four bonds connected to a randomly chosen site are simultaneously altered) is dual to a different problem ("squares," where the four bonds forming a square are simultaneously altered).

The duality relates a system above the condition threshold to another one below it, and forces a relationship between the relevant critical exponents. Thus we see that for the bond problem, $s=t$, and $s(\text{sites})=t(\text{squares})$. Furthermore, in the bond problem the lattice and its dual become indistinguishable at p_c, so that $\sigma(\frac{1}{2},a;\frac{1}{2},b)=(ab)^{1/2}$, which implies $u=\frac{1}{2}$.

III. NUMERICAL ESTIMATES OF THE EXPONENTS

The specific conductivity of several samples of large 2D (square) and 3D (simple cubic) lattices have been obtained numerically, and the exponents s, t, and u have been estimated (Table I). Needless to say, the calculations were not performed on infinite lattices. Finite lattice size manifests itself in the calculations in that there is now some sample-to-sample variation in the specific conductivity; similarly, on any one run (in which the resistors of a lattice are altered in some chosen sequence) σ is no longer a smooth function of p, since altering a given resistor may have a very large effect (e.g., by breaking the last conducting path) or no effect at all (if it already carries no current). In the studies reported here it was observed that for the 3D bond problem the rms deviation from the mean was roughly independent of ϵ and decreased as $L^{-5/2}$; for the 2D problem the rms error seemed to show a dependence on ϵ (increasing as the critical point is approached). This variability places a limit on the accuracy to which the critical exponents can be determined by studies at any fixed finite L, and prohibits meaningful study of the most immediate vicinity of the critical point. It may be possible to extend the useful range somewhat by averaging several runs based on different random sequences or by least-squares fitting to a given run, but it should be recognized that the average conductivity for finite L (thus defined) is not the same as its infinite-L limit, and indeed has no critical point whatsoever. In the present study the only data that were used were those for which the estimated error on each point (due to the finite lattice effects) was less than 5%. The quoted errors on the exponents represent estimates of the residual variability; they do not include any allowance for the likelihood that calculations closer to the critical point (on larger lattices) might find somewhat different exponents.

Let us postpone the technical details of what was

TABLE I. Critical exponents for resistor lattices.

	2D bond	2D sites	3D bond	3D sites
s	1.1[a]	1.10 ± 0.15	0.70 ± 0.05	0.5 ± 0.1
max$L$		100	20	24
t	1.10 ± 0.05	1.25 ± 0.05	1.70 ± 0.05	1.75 ± 0.1
max$L$	100	100	25	30
u	$\frac{1}{2}$[b]	0.55 ± 0.07	0.72 ± 0.02	0.83 ± 0.02
max$L$		100	25	25
p_c	$\frac{1}{2}$[b]	0.59[c]	0.247[c]	0.307[c]

[a] s was not determined, but is known to exactly equal t.
[b] Exact result.
[c] After Shante and Kirkpatrick (Ref. 2).

calculated and by what means to Sec. IV and turn to a discussion of Table I. For each dimensionality, two problems were considered, namely, the "bond" problem in which the resistors are chosen (essentially) independently; and the "site" problem in which, starting from an all-b lattice, all the resistors which meet at randomly chosen sites are altered simultaneously to a.

A. 2D bond problem

The exponent t has not been reported before. The other two exponents quoted in Table I were not calculated but rather inferred from the exact relationships $s=t$, $u=\frac{1}{2}$.

B. 2D site problem

The quoted value of t is in good agreement with the exponent which Watson and Leath[6] determined by fitting their entire curve ($t=1.28\pm 0.05$), and in reasonable agreement with the value they prefer ($t=1.38\pm 0.12$), which was based on a least-squares fit which extended somewhat into the variable region. The value quoted for s and u are quite uncertain, but Eq. (1) seems to be satisfied.

C. 3D bond problem

This problem was given the most careful study of the group. The exponent t is larger then the estimate $t=1.6\pm 0.1$ due to Kirkpatrick[4] and Webman *et al.*[16] but not clearly in conflict with it. For the purposes of checking the exponent relation the discrepancy is unimportant: Eq. (1) proves to be fairly insensitive to the value of t. For the given values of s and u, the agreement is quite good.

D. 3D site problem

The conductivity in the "s" geometry showed surprisingly large sample-to-sample variability, rendering it impossible to check the exponent relation to any accuracy. The value quoted for t is consistent with $t=1.5\pm 0.2$ found by Kirkpatrick.[3]

In summary, the exponent relation (1) is supported by a rigorous result (2D bonds) and a numerical result (3D bonds). The numerical results on the site problem are inadequate to make any judgment concerning the usefulness of the homogeneous function representation in those cases.

There is one further relationship among these exponents which should be noted. Stinchcombe and Watson[17] have argued on the basis of a renormalization-group approach that the exponents for the bond and site problems should be the same. Their argument is by no means rigorous, but the hypothesis is appealing because it places local statistics along with local geometry in the class of irrelevant variables, so that all percolation conduction problems in a given dimensionality belong to the same universality class. The exponents given in Table I are for the most part consistent with the universality hypothesis, with the exception of u for three dimensions.

IV. TECHNICAL DETAILS

The basic problem is to find the specific conductivity of a $L\times L$ square or $L\times L\times L$ cubic lattice of resistors, where the limiting value of L is indicated in Table I.

The boundaries of a lattice can give rise to significant perturbations in finite geometry. In hopes of minimizing such effects, the lattices considered here differed from those discussed heretofore in that they were continued periodically at the boundaries (so that the 2D lattice was actually the surface of a torus), and rather than establishing a potential difference by means of boundary electrodes, a constant field E was imposed along one of the coordinate axes (so that there was an emf of EL driving current around the torus).

For a similar reason, bonds on the d-dimensional lattice were altered in d-plets, one along each coordinate axis (but chosen independently and randomly otherwise). This prevented statistical fluctuations rendering the lattice anisotropic. The constraint is unimportant because it becomes unnecessary in the thermodynamic limit.

The primary tool in solving the lattice to find the potential distribution was the relaxation method discussed by Kirkpatrick,[3,4] with overrelaxation as advocated by Webman *et al.*[16] However, it was noted that the current in the direction of the field (as measured in a layer perpendicular to that direction) varied somewhat from layer to layer, even for voltage distributions which seemed to pass the convergence test at each site. This effect seemed to be due to a coherence in the errors. The algorithm was greatly improved by occasionally shifting all the voltages in a way designed to make the layer currents the same. This modification depends on global current conservation, rather than current conservation at a particular vertex, and was particularly effective at eliminating errors with slow spatial variation. Thus this modification actually helped speed convergence.

In the case $a=0$, $b=1$, it proved useful to delete from the lattice all sites which had only a single b link—these necessarily carry no current, but their presence slows down the iteration procedure. In the case $a=1$, $b=\infty$, it was *essential* to group each cluster of sites joined by b links into a single entity, whose voltage level was determined by the condition that the net current entering at its perimeter vanished (due to the presence of the imposed E field the interior of such a cluster is not an equipotential, but the relative potential differences are immediately determined by the condition $b=\infty$, and only the voltage level remains to be determined).

The distribution of voltages $\{V(m,n)\}$ which satisfies Kirchhoff's laws is also the distribution which minimizes the power dissipated in the network for the given distribution of conductances and boundary conditions.[18] The dissipated power for the correct $\{V(m,n)\}$ (which is then the minimum over all voltage distributions) is directly related to the conductivity of the lattice by $P=\sigma E^2 L^3$. These two facts explain that the conductivity should be calculated from the power dissipated, so that for an approximate solution the error in the calculated conductivity is quadratic in the errors in the $\{V(m,n)\}$. This allows accurate values for the conductivity to be extracted from poorly converged voltage distributions. The conductivity calculated this way was generally slightly smaller than the ratio of the current to the field (for the same distribution of voltages); for the distributions that passed the convergence tests, the two definitions differed by only 1%.

The critical exponents were determined by plotting the results of several "runs" (in which the critical point is approached in some way, such as successively deleting resistors from the lattice) on logarithmic paper, and drawing in straight lines (representing power laws) which seemed to characterize the data well. In the s and t geometries the sample-to-sample variability expected for finite lattices appeared in two ways: the points from any one run did not fall on a single line, and points from different runs disagreed. The error limits quoted in Table I represent the extent to which the data seemed to determine such a line. As noted above, it was not regarded useful to use data too far into the variable region. In practice, this limited ϵ to be greater than 0.1 in the t geometries, and $\epsilon > 0.05$ in the s geometries.

The u geometry is a separate case: here one can choose a particular disordered lattice with the critical concentration of good conductors, and vary the conductivity of the poor conductors. This of course gives a smooth curve when σ is plotted as a function of a, and the finite-sample variability only shows up when another sample is chosen. In the 2D sites problem, this variability was rather severe for $a \leq 0.05$. In the 3D cases, however, reproducibility was good for all $a \geq 0.01$ (beyond which convergence of the algorithm became excessively slow), and all data were well fit by a single power law. It is for this reason that the error limits on u are so small in Table I. As noted above, however, the two 3D geometries probably should have the same exponent u, which somewhat impeaches the error limits given.

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