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Destruction of first-order transitions by symmetry-breaking fields

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Various physical systems with Hamiltonians of cubic and lower symmetry are predicted classically to exhibit second-order phase transitions but known to yield first-order transitions within the renormalization-group approach either (a) because no appropriate "stable" fixed point exists, or (b) because the stable fixed point is not physically accessible. [MnO seems to be an example of (a).] This situation is discussed under circumstances where imposition of a further symmetry-beaking field, $g \gg 1$, restores a continuous transition. Two possible types of phase diagram are identified for $g \rightarrow 0$, either (i) without or (ii) with tricritical points. Renormalization-group trajectory calculations for examples of (b), namely a cubic Hamiltonian under a quadratic anisotropy field, are presented: tricritical points are found and a universal amplitude ratio governing their location is calculated to first order in $\epsilon = 4 - d$.

I. INTRODUCTION

Renormalization-group calculations, especially those using ϵ-expansion techniques,[1] have proved most instructive in studying the critical behavior of systems with interactions more complex than isotropic short-range couplings. In particular, the effects of interactions which break various symmetries are important in studying phase transitions in solids. A central concept in the renormalization-group approach is the association of criticality in a system with the existence of a fixed-point Hamiltonian $\mathcal{H}^*$ of an appropriate renormalization transformation $\mathcal{H}' = \mathsf{R}[\mathcal{H}]$. However, a critical point will be observed under variation of the physical fields (e.g., temperature, pressure, magnetic field, stress, etc.) *only* if the fixed point satisfies certain conditions.

In the first place, for observable criticality the fixed point must be "stable" or, more properly, "stable on the critical manifold." To appreciate this, consider an ordinary critical point in a ferromagnet: the critical point is found by varying just the temperature and the magnetic field, so that the "codimension" of the critical manifold (in the terminology introduced by Griffiths[2]) is 2. Thus, the corresponding fixed point must have only two relevant critical operators or only two independent directions of instability. Any other perturbations $\Delta\mathcal{H}$ about the fixed point should correspond to irrelevant operators not under physical control. More generally, we will say a fixed point is stable if, under all small perturbations of $\mathcal{H}^*$, *other* than those corresponding to the physical critical fields, the successively renormalized Hamiltonians $\mathcal{H}^* + \Delta\mathcal{H}'$, $\mathcal{H}^* + \Delta\mathcal{H}''$, ... "relax" back to $\mathcal{H}^*$.

In the second place, even if $\mathcal{H}^*$ is stable it must be *accessible* under action of R from those "initial" or physical Hamiltonians, $\mathcal{H} = \mathcal{H}_{(0)}$, which lie within the range of the physically controlled fields.

It is of interest, however, to ask what the phase transition behavior of a physical system is like when one or other of these conditions fails. To see the significance of this question we remark, first, that in such circumstances a *first-order phase transition* is often to be expected (in place of a continuous or λ-like transition) *even though* classical mean-field or Landau theory will normally predict a continuous transition. On the other hand, the introduction of a new symmetry-breaking physical field, say g, will frequently bring into existence a new fixed point which is both stable and accessible.[3] Thus, at least for sufficiently large g, a continuous transition with critical behavior is then to be expected. Hence the symmetry-breaking field destroys the first-order nature of the transition and the overall phase diagram differs qualitatively from the direct classical predictions. But what happens as $g \rightarrow 0$? Is the first-order transition destroyed by *any* nonzero value of g, or does it survive until g reaches some threshold value? In this paper we answer these questions explicitly within one context. However, before introducing the particular model which we attack, it may be helpful to look at concrete examples in which no stable and accessible fixed point exists, in order to see more clearly the physical meaning of such situations.

Accordingly, let us recall that to study a given physical system using the standard momentum shell integration renormalization technique, the first step is to construct an appropriate model Hamiltonian which can describe the phase transitions of interest and which embodies the correct symmetries. Typically, one constructs an effective (Landau-Ginzburg-Wilson) reduced Hamiltonian for n-component, continuous local variables

$\vec{\psi}(\vec{R}) = [\psi_\alpha(\vec{R})]$ $(\alpha = 1, 2, \ldots, n)$ which takes the form

$$\mathcal{H} = \int d\vec{R}\left[-\tfrac{1}{2}|\nabla\vec{\psi}|^2 - \tfrac{1}{2} r \sum_{\alpha=1}^{n} \psi_\alpha^2 - u_1\left(\sum_{\alpha=1}^{n}\psi_\alpha^2\right)^2 - \sum_{l=2}^{L} u_l O_l(\psi_\alpha)\right], \quad (1.1)$$

where $r \sim T - T_0$ is the basic temperature-like variable. The $O_l(\psi_\alpha)$ are terms of fourth order in the components ψ_α chosen to be invariant under the symmetry group of the disordered phase. The critical behavior in $d = 4 - \epsilon$ dimensions is then studied by locating the appropriate stable fixed point of the Hamiltonian. For Hamiltonians of the form (1.1) it has been shown by Brézin, LeGuillou, and Zinn-Justin[4] that the *isotropic fixed point* ($u_1^* > 0$, $u_2^* = \ldots = u_L^* = 0$) is stable provided the order parameter $\vec{\psi}$ has less than $n^\times(d)$ components, where in the important case of perturbations of cubic symmetry we have[4]

$$n^\times(d) = 4 - 2\epsilon + O(\epsilon^2) \quad \text{for } \epsilon = 4 - d \geq 0 ,$$

and

$$n^\times(3) \simeq 3.1 > 3 . \quad (1.2)$$

[More generally, Brézin *et al.* establish only $n^\times(d) = 4 - O(\epsilon)$.] However, for $n > n^\times(d)$ the isotropic fixed point is unstable with respect to a subset of the $u_2, \ldots, u_L$ perturbations. More recently,[5-7] it has been discovered that the effective Landau-Ginzburg-Wilson Hamiltonians appropriate to many real materials for which $n \geq 4 > n^\times(3)$ (such as MnO, UO_2, Cr, Eu, TbP, etc.) yield *no* stable fixed points. As mentioned above, this fact has been interpreted as indicating that the transitions in these materials should be of first order even though Landau theory predicts continuous transitions. However, by applying a suitable symmetry-breaking field (such as an anisotropic stress or a magnetic field), the symmetry of the original Hamiltonian is lowered and the dimensionality of the new order parameter is thus reduced, say to $m < n$. The resulting Hamiltonians may then become accessible to a stable fixed point.[3] Consider, for example, a field g which enters the Hamiltonian (1.1) via the symmetrized term[8]

$$\frac{g}{n}\left((n-m)\sum_{\alpha=1}^{m}\psi_\alpha^2 - m\sum_{\alpha=m+1}^{n}\psi_\alpha^2\right) . \quad (1.3)$$

For large enough positive g such a term clearly supresses the fluctuations of the $n - m$ components $\psi_{m+1}, \ldots, \psi_n$, and their effect on the resulting critical behavior can hence be asymptotically neglected. The transition is thus properly described by an m-component model, which might well yield a stable fixed point and a continuous transition exhibiting criticality. The situation for negative g is the same except that m and $n - m$ must be interchanged. However, when the symmetry-breaking field is small enough the remaining $n - m$ or m components can no longer be neglected since their presence is responsible for the first-order transition at $g = 0$. Two (r, g) phase diagrams seem plausible for small g although more complex possibilities may be conceived. These are shown schematically in Fig. 1. In (a) the transition is taken to be first order *only* for $g = 0$, but continuous for $g \neq 0$, i.e., the first-order transition is destroyed however small the symmetry-breaking field. Conversely in (b) the transition is assumed to remain of first order for small enough g and to become continuous only for large enough g. In this case one may anticipate *tricritical points* (possibly of anomalous character) at two finite values of g (one positive and one negative). The original first-order transition at $g = 0$ now appears as an ordinary *triple point*.

Recently, the phase diagram of MnO under a uniaxial stress has been studied in a most interesting experiment.[9] This compound is an fcc antiferromagnet of type II and exhibits a first-order transition. The transition is quite "close to critical," under zero stress. However, a tricritical point is found at a finite value of g, the uniaxial [111] stress, and beyond this the transition becomes continuous. It has been subsequently pointed out[3] that this change in the nature of the phase transition might be explained by noting that under zero stress the transition is properly described by an $(n = 8)$-component vector model which yields no stable fixed point; however, for large enough uniaxial stress $n - m = 6$ components are suppressed and the transition can be described by a simple $(m = 2)$-component vector model which *does* have

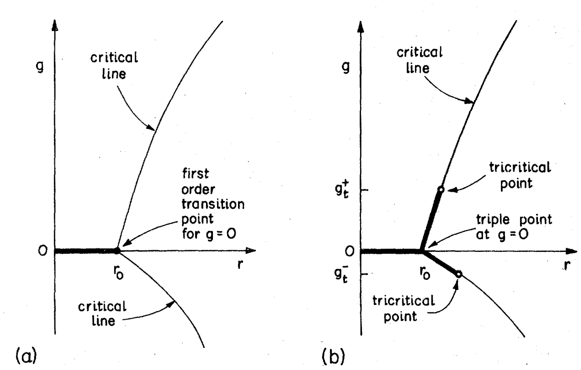

FIG. 1. Possible types of phase diagram in the (g, r), or anisotropy field-temperature plane showing how a first-order transition in zero field $(g = 0)$ might develop. In (a) a continuous or critical transition line exists for any $g \neq 0$; in (b) first-order lines arise for small g and terminate in tricritical points at nonzero g.

an associated stable fixed point. However, the character of the corresponding phase diagram [whether it is of the type (a) or (b) in Fig. 1] has not been determined theoretically.

We hope to study this particular issue in the future. However, the model Hamiltonians so far studied which yield no stable fixed points have rather complicated fourth-order anisotropic terms. Accordingly, we will first study (in this paper) a simpler example of the same underlying situation but one which arises from the inaccessibility rather than the nonexistence of a stable fixed point.[10]

Thus consider the n-component model with cubic anisotropy as described by

$$\mathcal{H}= \int d\vec{R}\Big(-\tfrac{1}{2}\,|\nabla\vec{\psi}|^2-\tfrac{1}{2}\,r\sum_{\alpha=1}^{n}\psi_\alpha^2-u\sum_{\alpha=1}^{n}\psi_\alpha^4 - v\sum_{\alpha<\beta}^{n}\psi_\alpha^2\psi_\beta^2\Big)\,. \tag{1.4}$$

The critical behavior associated with this model has already been studied by several authors.[11,12] It was found that for $n\leq n^\times(d)$ the isotropic fixed point is stable (in agreement with the general result of Brézin *et al.*[4]), but for $n>n^\times(d)$ a new cubic fixed point is stable.[11] However, by examining the flow diagram of this model on the critical manifold it is discovered that although there is a stable isotropic fixed point for $n<n^\times(d)$ there are regions in the (u,v) plane which lie outside its domain of attraction as shown in Fig. 2. Consequently, if the initial physical Hamiltonian is in such a region the stable fixed point is not accessible. In fact, the renormalization group flows take Hamiltonians in these regions to domains where the renormalized Hamiltonian, in the form (1.4), is thermodynamically unstable (i.e., where the fourth-order term $u\sum_i\psi_i^4+v\sum_{i<j}\psi_i^2\psi_j^2$ is not positive definite). In these circumstances, of course, positive higher-order terms play a crucial role and should be included in the initial Hamiltonian. This situation has been studied using parquet graph approaches[13-15] and by integrating the recursion relations.[16] These calculations establish the fact, expected on heuristic grounds, that the phase transition in a system which is not accessible to the stable fixed point is of first order. It may be mentioned again that Landau theory does not recognize this possibility; instead, the transition is predicted to be continuous for all values of (u,v) for which the Hamiltonian itself is stable. However, we will take this point up again briefly in Sec. II.

The nature of the (r,g) phase diagram can, therefore, be examined in this model by studying the effect of the symmetry breaking term (1.3) on the transition associated with the Hamiltonian (1.4). The anisotropic terms of this Hamiltonian are sufficiently simple that the calculations are tractable and can be performed analytically as we show below.

In outline, the remainder of this paper is as follows: In Sec. II we first analyze the phase diagram of the Hamiltonian (1.4) with the symmetry-breaking term (1.3), using scaling arguments. This leads to the definition of a *universal amplitude ratio* associated with the two tricritical points induced by the symmetry-breaking field [assuming Fig. 1(b) applies]. We also comment on the purely classical phase diagram within a wider context. In Sec. III, we use a perturbation expansion in the cubic anisotropy parameter v to demonstrate the existence of critical and tricritical points [see Fig. 1(b)] in the limit of large symmetry-breaking field $|g|\gtrsim 1$. The limit $|g|\ll 1$ is discussed in Sec. IV using renormalization-group techniques in $d=4-\epsilon$ dimensions, specifically the integration of the differential recursion relations. The existence of tricritical points is confirmed and the universal amplitude ratio defined in Sec. II is then calculated to leading (zeroth) order in ϵ. Rudnick's[16] solution of the recursion relations for u and v is recapitulated and generalized in the Appendix.

II. SCALING ANALYSIS

Consider the n-component cubic model with a symmetry-breaking field g described by the reduced Hamiltonian

$$\mathcal{H}=\int d\vec{R}\Big(-\tfrac{1}{2}\,|\nabla\vec{\psi}|^2-\tfrac{1}{2}\,r_1\sum_{\alpha=1}^{m}\psi_\alpha^2-\tfrac{1}{2}\,r_2\sum_{\alpha=m+1}^{n}\psi_\alpha^2 - u\sum_{\alpha=1}^{n}\psi_\alpha^4-v\sum_{\alpha<\beta=1}^{n}\psi_\alpha^2\psi_\beta^2\Big)\,, \tag{2.1}$$

with

$r_1=r-[1-(m/n)]g$ and $r_2=r+(m/n)g$.

For stability of the free energy we require $u>0$ and $u+\frac{1}{2}(n-1)v>0$. Within the ϵ expansion this model is found to have two regions in the (u,v) plane (with u and v of order ϵ) which lie *outside* the domain of attraction of the stable fixed point: these are

(a) $w>0$,

where
$$w=\begin{cases} v-6u \text{ for } n=2\,, \\ v-3u+O(\epsilon^2) \text{ for } n=3\,, \\ v-2u \text{ for } n\geq 4\,, \end{cases} \tag{2.2}$$

(b) $v<0$

(see Fig. 2). In this section we apply scaling arguments to discuss the phase diagram in the first region (a).

The free energy associated with the Hamiltonian

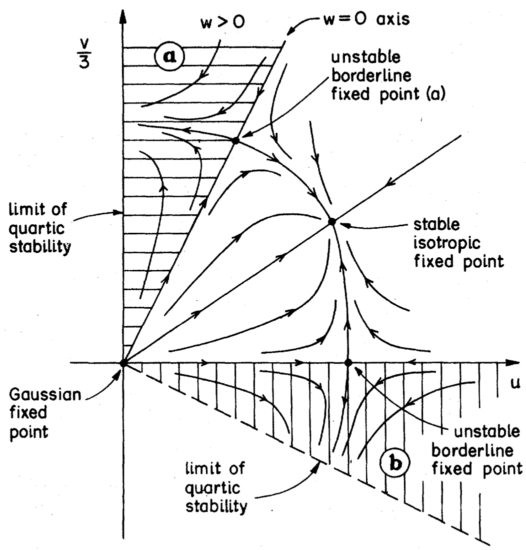

FIG. 2. Schematic renormalization-group flow diagrams in the (u,v) or quartic-coupling critical manifold for $n<n^{\times}(d)$ (and small ϵ). The stable isotropic fixed point is not accessible from the two shaded regions marked (a) $w>0$, and (b) $v<0$.

(2.1) has been studied on the line $g=0$ for $w>0$ by several authors,[13-15] who found a first-order transition. However, for $g=w=0$ the model exhibits a continuous transition associated with the cubic fixed point [for $n<n^{\times}(d)$] or with the isotropic Heisenberg fixed point [for $n\geq n^{\times}(d)$]. These fixed points are unstable with respect to both g and w perturbations. One can thus apply scaling arguments to discuss the phase diagram in the vicinity of the $g=w=0$ point which has the character of a multicritical point.

According to general scaling theory[17] the singular part of the free energy derived from the Hamiltonian (2.1) should vary as

$$f(t,g,w)\approx t^{2-\alpha}W(g/t^{\phi_g},\, w/t^{\phi_w}) \tag{2.3}$$

in the limit $t\to 0$, $g\to 0$, and $w\to 0$. As usual, $t=1-T/T_c$, where T_c is the critical temperature for $g=w=0$; the critical exponent for the specific heat is α, and ϕ_g and ϕ_w are the appropriate crossover exponents.[11,18] The critical exponents are, of course, those associated with the $g=w=0$ fixed point (see Fig. 2).

The mean-field or classical phase diagram in the (t,g,w) space or small g and w is shown in Fig. 3(a). However, according to the renormalization group calculations the transition becomes of first order for $w>0$. We will show in Secs. III and IV that the transition remains first order for sufficiently small nonzero g (with $w>0$). On the other hand, we will show that for large positive g and $m<n^{\times}(d)$ [or, by symmetry, for large $-g>0$ and $n-m<n^{\times}(d)$] the transition becomes continuous. In this case the model must exhibit a tricritical point at some finite value of g as shown in Fig. 3(b). Tricriticality is necessarily associated with singular behavior of the scaling function $W(x,y)$ at some point (x_t,y_t). This singularity will, in fact, describe a *line* of tricritical points

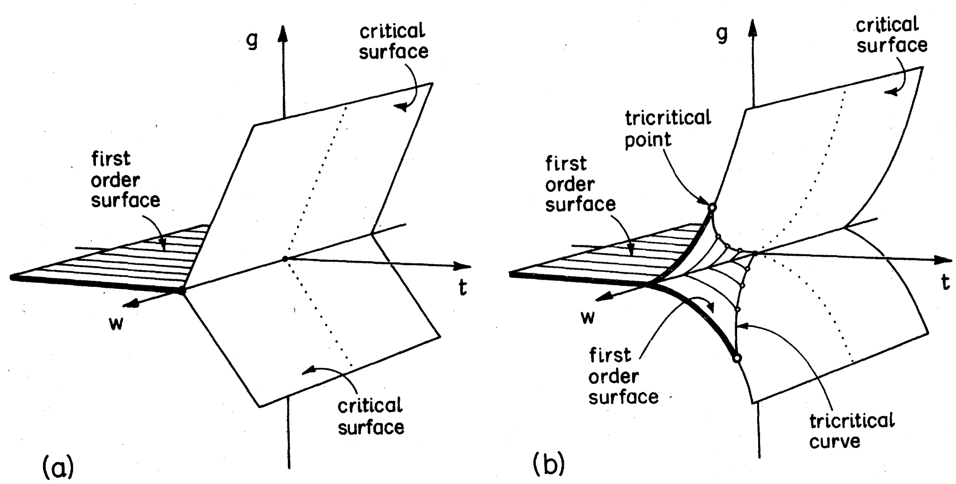

FIG. 3. Schematic phase diagrams for the anisotropic system in the space (t,g,w) as predicted (a) by classical theory and (b) by scaling theory assuming the existence of tricritical points. First-order transition surfaces have been shaded; critical surfaces are shown clear. Note that the critical surfaces in (b) are expected to meet one another, and the main first-order surface, tangentially along a line (of bicritical points). On the other hand, the first-order surfaces starting at the tricritical lines will meet one another and the remaining first-order surface, along the triple-point line, at a definite nonzero angle.

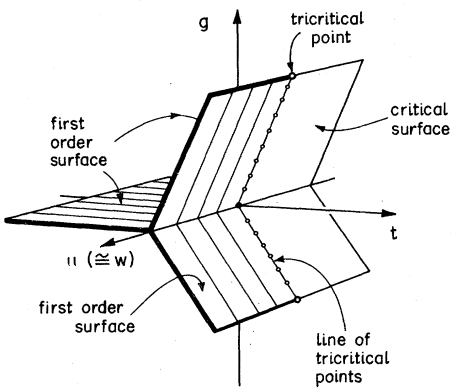

FIG. 4. Schematic phase diagram of the predictions of classical theory for an extended thermodynamic space in which u can become negative but positive sixth-order terms in the Hamiltonian maintain thermodynamic stability. Now lines of tricritical points appear (at $u=0$) in contrast to Fig. 3(a) and with greater similarity to Fig. 3(b).

in the (t, g, w) space [see Fig. 3(b)]. If *both* inequalities $m < n^{\times}(d)$ and $n-m < n^{\times}(d)$ are satisfied, two tricritical lines are present: one for $g>0$ and one for $g<0$, Fig. 3(b).

In the scaling limit, (2.3), the projection of these tricritical lines on the (g, t) plane must take the form

$$g = A_m t^{\phi_g} \quad \text{for } g>0, \tag{2.4}$$

and

$$g = -A_{n-m} t^{\phi_g} \quad \text{for } g<0 . \tag{2.5}$$

The ratio A_m/A_{n-m} should be a universal quantity, independent of the irrelevant variables (such as u etc.) and equal to x_t^+/x_t^-. This ratio is calculated generally in Sec. IV to leading (zeroth) order in $\epsilon = 4-d$. However, by symmetry the ratio is clearly equal to unity in the case $n=2$, $m=1$.

We have made the point that within the stability region of (2.1) in the (u, v) plane, classical Landau-type theory can give a qualitatively misleading phase diagram inasfar as no tricritical lines and associated first-order surfaces are predicted. However, within a broader parameter space in which a positive sixth-order term is included in (2.1) the phenomenological theory is less misleading (see Fig. 4). Then the instability condition $w>0$ (needed for tricriticality) is replaced by $u<0$. The ψ^6 terms ensures stability of the free energy, but lines of tricritical points are now evident.

III. CUBIC MODEL WITH A LARGE ANISOTROPY FIELD

In this section we discuss the (g, r) phase diagram of the cubic model with a large symmetry-breaking, anisotropy field g. For computational simplicity we consider only the case $n=2$, but the results can be generalized readily to $n \geq 2$. The analysis to be presented is directly applicable only in the domain $0 < u \lesssim v^2 \ll 1$ which is included in the region (a) defined in Sec. II: recall that this is one of the two regions in the (u, v) plane lying outside the domain of attraction of the stable, XY fixed point (see Fig. 2). However, the nature of the answers to be obtained should apply throughout region (a) since, as is clear from Fig. 2, the flows in this region drive u towards zero while v remains of order v^*. We will show that the (g, r) phase diagram (for $g>0$) exhibits a tricritical point at a finite value of g as indicated in Fig. 1(b). By symmetry a similar tricritical point arises for $g<0$. The Hamiltonian now simplifies to

$$\mathcal{H} = -\tfrac{1}{2}|\nabla\vec{\psi}|^2 - \tfrac{1}{2}r_1\psi_1^2 - \tfrac{1}{2}r_2\psi_2^2 - u(\psi_1^4+\psi_2^4) - v\psi_1^2\psi_2^2 , \tag{3.1}$$

where $r_1 = r - \frac{1}{2}g$ and $r_2 = r + \frac{1}{2}g$, $(g>0)$. We are interested in the thermodynamic behavior associated with this Hamiltonian in the critical region for the first component ψ_1 of the order parameter, that is $r_1 \simeq 0$. For $r_2 = O(1)$ which corresponds to a large anisotropy field g, the second component of the order parameter ψ_2 can be integrated out using a perturbation expansion in powers of u and v. One then obtains a reduced, single-component, or Ising-like Hamiltonian,

$$\mathcal{H} = -\tfrac{1}{2}(\nabla\psi_1)^2 - \tfrac{1}{2}\tilde{r}\psi_1^2 - \tilde{u}\psi_1^4 - \tilde{w}\psi_1^6 , \tag{3.2}$$

where to leading order in u and v we find

$$\tilde{r} = r_1 + 2vA(r_2, d) , \tag{3.3}$$

$$\tilde{u} = u - v^2 B(r_2, d) , \tag{3.4}$$

while $\tilde{w}$ is of order v^3 and will remain positive in the regions of interest. The functions $A(r_2, d)$ and $B(r_2, d)$ are integrals over the ψ_2 propagator, namely,

$$A(r_2, d) = \int_{|q|\leq 1} \frac{d^dq}{(2\pi)^d}\frac{1}{r_2+q^2} , \tag{3.5}$$

$$B(r_2, d) = \int_{|q|\leq 1} \frac{d^dq}{(2\pi)^d}\frac{1}{(r_2+q^2)^2} , \tag{3.6}$$

and d is the spatial dimensionality.

Now, in three or more dimensions, the reduced Hamiltonian (3.2) will yield a continuous transition for $\tilde{u} \gtrsim 0$, a first-order transition for $\tilde{u} \lesssim 0$ $(\tilde{w}>0)$, and a tricritical point at $\tilde{r} \simeq O(\tilde{w}) = O(v^3)$ and $\tilde{u} = O(\tilde{w}) = O(v^3)$. The tricritical point can therefore be located to leading order in v by solving the equations $\tilde{r}(r, g, u, v) = \tilde{u}(r, g, u, v) = 0$ using (3.3) and (3.4).

Consider now the integral $B(r_2, d)$: this is a decreasing function of r_2 which approaches zero as

$r_2 \to \infty$. For $d=3$ and $d=4$ the explicit values are

$$B(r_2;3)=\tfrac{1}{2}K_3[r_2^{-1/2}\cot^{-1}r_2-(1+r_2)^{-1}]\ , \quad (3.7)$$

and

$$B(r_2;4)=\tfrac{1}{2}K_4[\ln(1+r_2^{-1})-(1+r_2)^{-1}]\ , \quad (3.8)$$

where $K_d=2^{-(d-1)}\pi^{-d/2}[\Gamma(\tfrac{1}{2}d)]^{-1}$. Let us fix u and v and vary r_2. For large enough r_2 we have

$$\tilde{u}=u-v^2B(r_2,d)\simeq u>0\ , \quad (3.9)$$

and so the Hamiltonian (2.2) exhibits a continuous transition. As r_2 is decreased, however, the integrals $B(r_2,d)$ become indefinitely large (for $d \le 4$) and $\tilde{u}$ will change sign. For smaller r_2 the transition thus becomes first order. The system hence exhibits a tricritical point at $r_2=r_{2,t}$ given by

$$B(r_{2,t},d)\approx u/v^2\ . \quad (3.10)$$

However, since this expression is valid only if $r_{2,t}\gtrsim 1$ [with $B(r_{2,t},d)\lesssim O(1)$ see (3.7) and (3.8)] the existence of the tricritical point has been demonstrated only in the region $0<u\ll v^2$. This approach cannot be extended to the region $|g|\ll 1$ since the expansion (3.3) for $\tilde{r}$ is then no longer valid. This region will be discussed in the next section using a renormalization group approach.

IV. RENORMALIZATION GROUP ANALYSIS OF TRICRITICAL REGION

A. Outline

In this section we will demonstrate the occurrence of tricritical points at small anisotropy fields $|g|\ll 1$ in the region $0<w/u\ll 1$. We will utilize the renormalization group trajectory integral method[19] to calculate the amplitude ratio A_m/A_{n-m} to leading (zeroth) order in $\epsilon=4-d$ but general m and n. The results involve an integral which is evaluated explicitly only for the case of principal interest, namely $m=1$ and $n=3$. (The result for $m=2$, $n=3$ follows trivially by symmetry and likewise the amplitude ratio must be equal to unity in the case $m=1$, $n=2$ as already observed.)

Now, as explained above, the integration over the ψ_2 components is justified only when $r_2=O(1)$. However, for small g in the critical region $r_1=r-O(g)\simeq 0$, we also have $r_2=r+O(g)$ small. To handle this situation we use the trajectory integral method,[19] to relate the initial Hamiltonian with small r and g to a renormalized Hamiltonian in which r_2 is of order unity. Only then is the ψ_2 field integrated out, and the tricritical point identified. For $m=1$ the procedure is then straightforward and yields the tricritical relation

$$g_t\approx A_1\, t_t^{\phi}. \quad (4.1)$$

For $m=2$ we may follow the same procedure. However, after integrating out the $(n-m)$-component ψ_2 variables we are left with a two-component, or XY-like reduced Hamiltonian with cubic anisotropy. If this Hamiltonian lies in the domain of attraction of the isotropic $n=2$ fixed point, it will display critical (rather than tricritical) behavior. On the other hand, if it lies outside of the domain of attraction, it can display only a first-order transition. Thus the tricritical point may be located by requiring that the reduced Hamiltonian lies *on the borderline* of the domain of attraction of the stable, XY fixed point. This leads to a relation of the same form as (4.1) but with a different amplitude A_2.

In what follows, we shall explicitly derive the form (4.1) and its analogs but we calculate the amplitude ratio explicitly only for $n=3$, $m=1$, as mentioned.

We remark that tricritical points exist only for $m<n^{\times}(d)$. If $n>m>n^{\times}(d)$, the region in the (u,v) plane which lies outside the domain of attraction of the stable fixed point of the n-component model coincides with the corresponding region of the m-component model, namely $0<v<2u$. Thus, when the initial Hamiltonian corresponds to $0<v<2u$, the renormalization group recursion relations transform the n-component Hamiltonian into an unstable m-component model. The transition is then expected to be first order *even* for a large symmetry-breaking field $|g|\gg 1$ and hence no tricritical point occurs. The phase diagram for $n>m\ge n^{\times}(d)$ then resembles Fig. 1(b) but with the tricritical points removed to infinity.

B. Recursion relations and matching

Consider the Hamiltonian (2.1). Under action of the renormalization group[1] it is transformed but to leading order in $\epsilon=4-d$ the renormalized Hamiltonians $\mathcal{H}(l)$ remain in the parameter space (r_1,r_2,u,v). To this order the differential recursion relations for $r_1(l)$, $r_2(l)$, $u(l)$, $v(l)$ are found to be

$$\frac{dr_1}{dl}=2r_1+[12u+2(m-1)v](1-r_1) + 2(n-m)v(1-r_2)\ , \quad (4.2)$$

$$\frac{dr_2}{dl}=2r_2+[12u+2(n-m-1)v](1-r_2) + 2mv(1-r_1)\ , \quad (4.3)$$

$$\frac{du}{dl}=\epsilon u-36u^2-(n-1)v^2\ , \quad (4.4)$$

$$\frac{dv}{dl}=\epsilon v-24vu-2(n+2)v^2\ , \quad (4.5)$$

where we have also expanded the propagator factors as $(1+r_i)^{-1} \simeq 1 - r_i$. It has been shown[19–21] that as long as $r_i \lesssim O(1)$ the u and v relations (4.4) and (4.5) yield results correct to $O(\epsilon)$. This means that providing r_1 and r_2 do not become too large, the flows in the (u, v) plane are not affected, to $O(\epsilon)$, by the breaking of the isotropic quadratic symmetry. This point, which is essential for the treatment of the present problem, was first emphasized by Nelson and Domany,[20] and has been used previously in calculating the thermodynamic functions of a bicritical system.[21,22]

As to the r_1 and r_2 relations, (4.2) and (4.3), where terms of order r_i^2 have been neglected, it should be noted that these terms would be important if we were here interested in calculating scaling functions that are correct to $O(\epsilon)$; however, the various exponents and amplitudes remain correct to orders ϵ and ϵ^0, respectively, when the truncated Eqs. (4.2) and (4.3) are used.

The Hamiltonian (2.1) and its flows have been investigated extensively in the context of bicritical behavior.[22] In that case, the initial parameters u_0, v_0 are assumed to be in the vicinity of the stable isotropic fixed point [for $n<n^\times(d)$]. Here we study the case where u_0 and v_0 are such that the Hamiltonian with $r_1 = r_2$ will not flow (in the u, v plane) into the stable fixed point (see Fig. 2), but rather tend to flow out of the classical stability wedge. To be specific, we assume $0<u_0/v_0 <\frac{1}{6}(n-1)$ with $n<n^\times(d)$. (For $n=3$ this is correct only to leading order in ϵ.)

Our procedure will now be as follows. Equations (4.2)–(4.5) will be solved in an approximation valid for any l for which $r_2(l) \lesssim O(1)$. We then select a value of l^* by setting

$$r_2(l^*) = 1 + O(\epsilon) \ . \tag{4.6}$$

At this point the renormalized Hamiltonian $\mathcal{H}(l^*)$ is noncritical with respect to fluctuations of the ψ_2 variables. Thus, the trace over ψ_2 can be performed, keeping terms of appropriate order. After this step we obtain a reduced Hamiltonian that depends only on the ψ_1 components and has the form

$$\mathcal{H}_{\rm red} = -\int d\vec{R}\Big[\frac{1}{2}\sum_{\alpha=1}^{m}(\vec{\nabla}\psi_\alpha)^2 + \tfrac{1}{2}\,\tilde{r}\,|\vec{\psi}|^2 + \tilde{u}\sum_{\alpha=1}^{m}(\psi_\alpha)^4 + \tilde{v}\sum_{\alpha<\beta}^{m}(\psi_\alpha)^2(\psi_\beta)^2\Big] \ . \tag{4.7}$$

(Obviously, for $m=1$ we do not have a $\tilde{v}$ term.)

In $\mathcal{H}_{\rm red}$ we easily identify a temperature-like variable $\tilde{t}$. If $(\tilde{u}, \tilde{v})$ are such that $\mathcal{H}_{\rm red}$ lies in the domain of attraction of the m-component isotropic fixed point, $\tilde{t}=0$ defines a critical (reduced) Hamiltonian. If, however, $\tilde{u}$ and $\tilde{v}$ satisfy

$$\tilde{u}/\tilde{v} < \tfrac{1}{6}(m-1) \quad (n^\times \geq m \geq 1) \ , \tag{4.8}$$

then $\mathcal{H}_{\rm red}$ will yield a first-order transition as $\tilde{t}$ is varied. Clearly, the borderline case between criticality and a first-order transition, i.e., tricriticality, occurs when

$$\tilde{u}/\tilde{v} = \tfrac{1}{6}(m-1), \tag{4.9}$$

for $n^\times > m \geq 1$ (to leading order in ϵ). Thus the tricritical Hamiltonians are specified [to $O(\epsilon)$] by the conditions

$$r_2(l^*) = 1, \quad \tilde{t} = 0, \tag{4.10}$$

[see (4.6)] in conjunction with (4.9).

C. Solution of quartic-term equations

The coupled (u, v) differential Eqs. (4.4) and (4.5) have been solved in closed form by Rudnick[16] (see also Appendix A). We will use the fact that $u(l)$ and $v(l)$ remain of order u^* and v^* (or less), that is of order ϵ. In order to solve Eqs. (4.2) and (4.3), we reintroduce the original variables r and g [see (2.1)] for which the recursion relations can be written

$$\frac{dg}{dl} = \lambda_2 g - 12g\Delta u + 2g\Delta v, \tag{4.11}$$

$$\frac{dr}{dl} = \lambda_1 r + \frac{2\epsilon(n-1)}{3n} + [12\Delta u + 2(n-1)\Delta v](1-r), \tag{4.12}$$

where the eigenvalues

$$\lambda_1 = 2 - \frac{2\epsilon(n-1)}{3n}, \quad \lambda_2 = 2 - \frac{n-2}{3n}\,\epsilon, \tag{4.13}$$

correspond to the unstable, borderline fixed point [(a) in Fig. 2]

$$u^* = \frac{n-1}{36n}\,\epsilon, \quad v^* = \frac{\epsilon}{6n}, \tag{4.14}$$

and

$$\Delta u(l) = u(l) - u^*, \quad \Delta v(l) = v(l) - v^*. \tag{4.15}$$

Again, the expressions (4.11)–(4.14) are correct only to order ϵ.

Given $u(l)$ and $v(l)$ the solutions of (4.11) and (4.12) may be written

$$g(l) = e^{\lambda_2 l}\bar{g}(l), \tag{4.16}$$

$$\bar{g}(l) = g_0 \exp\Big(-\int_0^l [12\Delta u(l') - 2\Delta v(l')]dl'\Big), \tag{4.17}$$

and

$$r(l) = t(l) - 6u(l) - (n-1)v(l), \quad t(l) = e^{\lambda_1 l}\bar{t}(l), \tag{4.18}$$

$$\bar{t}(l) = t_0 \exp\Big(-\int_0^l [12\Delta u(l') + 2(n-1)\Delta v(l')]dl'\Big), \tag{4.19}$$

The reason for subtracting off the fixed point values u^* and v^* will become apparent below.

D. Location of tricritical points

Having solved the recursion relations, we turn to determine l^*. It proves convenient to impose the requirement in the form

$$t(l^*)+(m/n)g(l^*)=1. \quad (4.20)$$

Since $u(l)$ and $v(l)$ remain of $O(\epsilon)$, or less, this ensures, via (2.1), that $r_2(l^*)=1+O(\epsilon)$ in accordance with the prescription (4.6). In principle, this equation can be solved for l^* for any (t_0, g_0, u_0, v_0). However, we want to find the tricritical points. Accordingly we call on (4.9) and (4.10).

On performing the trace over the ψ_2 variables, we obtain a reduced Hamiltonian with temperature-like variable[19-21]

$$\tilde{t}=r_1(l^*)+O(\epsilon). \quad (4.21)$$

Imposing $\tilde{t}=0$ in accord with (4.10) yields, to leading order, the relation

$$t(l^*)-(1-m/n)g(l^*)=0. \quad (4.22)$$

The tricriticality conditions are completed, to leading order, by

$$f(l^*)=\frac{u(l^*)}{v(l^*)}=\frac{m-1}{6}. \quad (4.23)$$

These two equations together with (4.20) impose three conditions on five variables t_0, g_0, u_0, v_0, and l^*. Thus, for fixed u_0 *and* v_0 we can, in principle, eliminate l^*, and obtain $t_{0,t}, g_{0,t}$.

Consider first Eq. (4.20) which, by using (4.16) and (4.18), may be rewritten

$$\bar{t}e^{\lambda_1 l^*}+(m/n)\bar{g}e^{\lambda_2 l^*}=1. \quad (4.24)$$

If we now define

$$z=\bar{g}/\bar{t}^{\phi} \quad \text{with } \phi=\lambda_2/\lambda_1, \quad (4.25)$$

and put

$$e^{l^*}=\bar{t}^{-1/\lambda_1}\Psi, \quad (4.26)$$

we may substitute in (4.24) to obtain

$$\Psi^{\lambda_1}+(m/n)z\Psi^{\lambda_2}=1, \quad (4.27)$$

from which we see that Ψ is a function of z alone. The criticality condition (4.22) now becomes

$$\Psi^{\lambda_1}-(1-m/n)z\Psi^{\lambda_2}=0, \quad (4.28)$$

which on using (4.27) and (4.13) yields

$$z=n/(n-m) \quad (4.29)$$

to leading order ϵ.[23] Finally, we must invoke the condition (4.23) for tricriticality. This means that for given u_0, v_0, and m we must also have

$$l^*=l_m, \quad (4.30)$$

where l_m is the value of l for which the flow in the (u, v) plane (which, one recalls, is decoupled in leading order) goes from (u_0, v_0) to $[u(l_m), v(l_m)]$ such that the condition (4.23) is satisfied. Combining this requirement with (4.29) and (4.25) yields

$$\bar{g}(l_m)/[\bar{t}(l_m)]^{\phi}=n/(n-m)$$

with

$$g_0=g_t \quad \text{and} \quad t_0=t_t, \quad (4.31)$$

where t_t and g_t denote the tricritical point values in the initial Hamiltonian. On using the definitions (4.17) and (4.19) we see this is equivalent to the announced result

$$g_t \approx A_m t_t^{\phi} \quad (4.32)$$

together with the trajectory integral expression

$$A_m=\frac{n}{n-m}\exp\left(-2n\int_0^{l_m}\Delta v(l)dl\right). \quad (4.33)$$

Obviously, for $g<0$ we will get quite analogous expressions with A_m in (4.32) replaced by $-A_{n-m}$. Thus the amplitude ratio is given by

$$\frac{A_m}{A_{n-m}}=\frac{m}{n-m}\exp\left(-2n\int_{l_{n-m}}^{l_m}\Delta v(l)dl\right). \quad (4.34)$$

In order to calculate explicitly the integral entering this expression we have generalized Rudnick's exact solution of the (u, v) equations for the case[16] $n=2$, to general n. By utilizing the tricritical condition (4.23) to determine l_m and noting that $l=-\infty$ corresponds to the unstable fixed point (4.14) we can then obtain an expression for A_m/A_{n-m} valid for general m and $n-m$ $[<n^{\times}(d)]$. These calculations are presented in Appendix A. The result, however, involves an integral [see (A6)] that is somewhat intractable for nonintegral m and n. Nevertheless, for the case of most practical interest, namely $m=1$ for $n=3$, the integral can be performed straightforwardly and we conclude

$$\frac{A_1}{A_2}=\left(\frac{9}{134}\right)^{1/3}\simeq 0.406\,486 \quad (n=3). \quad (4.35)$$

Of course, this result is subject to corrections of order ϵ.

E. Conclusion

This completes our analysis. We have demonstrated explicitly how a symmetry-breaking term can alter the transition behavior of a Hamiltonian so that a first-order transition corresponding to an inaccessible fixed point, becomes a triple point with two associated tricritical points, beyond which the transitions become continuous. In the scaling regime the universal ratio determining the location of these tricritical points has been found to leading order in ϵ. Further calculations using the trajectory integral approach could yield the full scaling function $W(x,y)$ introduced in (2.3). However, it seems more worthwhile to study one of the more complicated examples, discussed in the Introduction, in which the original first-order transition occurs because the symmetry is such that no stable fixed point exists. We hope to discuss such a situation in the future.

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APPENDIX

To calculate the amplitude ratio from the formula (4.34) we first solve the (u,v) Eqs. (4.2) and (4.3). Following Rudnick's solution[16] for $n=2$, we introduce the substitutions

$$y=e^{\epsilon l}/\epsilon, \quad u=\epsilon y U, \quad v=\epsilon y V, \tag{A1}$$

$$f=u/v=U/V. \tag{A2}$$

These yield

$$\frac{df}{dy}=-V(2f-1)[6f-(n-1)], \tag{A3}$$

and

$$\frac{dV}{df}=\frac{V[24f+2(n+2)]}{(2f-1)[6f-(n-1)]}. \tag{A4}$$

The second equation here may be integrated easily for $n\neq 4$ to yield

$$V(f)=A(2f-1)^{-(n+8)/(n-4)}(6f-n+1)^{3n/(n-4)}, \tag{A5}$$

where A is a constant of integration. This result may then be substituted into (A3) which is then formally solved for $y(f)$ by

$$y=B\int_{f_0}^{f}\frac{(6f'-n+1)^{4(n-1)/(4-n)}}{(2f'-1)^{12/(4-n)}}\,df'+y_0, \tag{A6}$$

where $B=A^{-1}$. Inversion of this relation gives f as a function of y and hence $f(l)$. Substitution in (A5) then gives $V(l)$ and finally $u(l)$ and $v(l)$ follow from (A1) and (A2).

To use these results to evaluate the integrals entering the amplitude ratio expression we rewrite (4.34) as

$$\frac{A_m}{A_{n-m}}=\frac{m}{n-m}R_1R_2, \tag{A7}$$

with

$$R_1=\exp\left(-2n\int_{l_{n-m}}^{l_m}v(l)dl\right), \tag{A8}$$

$$R_2=\exp[2nv^*(l_m-l_{n-m})]. \tag{A9}$$

To calculate R_1 we transform to an integral on f according to

$$\int_{l_{n-m}}^{l_m}v(l)dl=\int V\frac{dy}{df}\,df$$
$$=-\int_{f_{n-m}}^{f_m}df/(2f-1)(6f-n+1), \tag{A10}$$

where for $k=m$ or $(n-m)$ we have, by definition,

$$f_k=f(l_k)=\frac{u(l_k)}{v(l_k)}=\frac{k-1}{6}, \tag{A11}$$

in which, finally, the tricriticality condition (4.23) has been used. On integrating (A10) explicitly we obtain

$$R_1(m;n)=\left(\frac{m(4-m)}{(n-m)[4-(n-m)]}\right)^{n/(4-n)}, \tag{A12}$$

which correctly inverts on replacement of m by $(n-m)$. For $m=1, n=3$ we find $R_1(1,3)=\frac{27}{64}$.

To handle the factor R_2 we rewrite (A9) as

$$R_2(m;n)=(I_m/I_{n-m})^{\epsilon/2nv^*}\approx(I_m/I_{n-m})^{1/3}, \tag{A13}$$

where, for $\epsilon>0$,

$$I_k=e^{\epsilon l_k}/\epsilon=\int_{-\infty}^{l_k}e^{\epsilon l}dl=\int_{y_0}^{y_k}dy=y(f_k), \tag{A14}$$

in which $y_0=y(l=-\infty)=0$. We may then use the solution (A6) for $y(f)$ if we recall (A11) for f_k and note that

$$f_0=\frac{u(-\infty)}{v(-\infty)}=\frac{u^*}{v^*}=\frac{n-1}{6}. \tag{A15}$$

This follows by recognizing that for small w all the (u,v) trajectories in question emerge originally (i.e., at $l=-\infty$) from a region close to the unstable borderline fixed point specified by (4.14).

This analysis thus expresses the ratio R_2 and hence, via (A12) and (A7), the amplitude ratio for general n and m in terms of an explicit integral. The integral in (A6) is not simple for general n and m but for $m=1$ and $n=3$ (or other integral values) it may be performed trivially. On substitution in (A13) we thence obtain

$$R_2(1;3)=(2^{20}/3^7\times 67)^{1/3}. \tag{A16}$$

On combination with (A12) and (A7), we find the result quoted in the text.

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