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Invariance of critical exponents for renormalization groups generated by a flow vector

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An invariance theorem for exponents of scaling variables is proved for a class of renormalization groups introduced by Wegner. (Renormalization groups generated by a flow vector.) A convenient form for the renormalization group linearized both around the fixed point and around the renormalization trajectory is derived. The latter is used to derive a coordinate system for perturbations around a critical Hamiltonian from that around a fixed point. It is shown that the classification scaling redundant applies also to the vectors of this critical-point coordinate system. The effect of an infinitesimal perturbation of the flow vector on the fixed point and on the renormalization group linearized around the fixed point is investigated. If the unperturbed linearized group has no marginal scaling operator, two cases arise. (i) The perturbed group has no fixed point near that of the unperturbed group. This case may obtain if there is a marginal redundant observable. (ii) The perturbed group has a fixed point which differs from that of the unperturbed group by a redundant perturbation. In the second case the scaling eigenvalues of the perturbed linearized group will be unchanged. Eigenvalues of redundant observables may change. The effect of a perturbation in a flow vector on a renormalized trajectory is considered and shown to suggest the concept of manifolds of equivalent Hamiltonians. Some difficulties involved in this concept are discussed.

I. INTRODUCTION

The renormalization-group approach to critical phenomena and other problems in many-body physics is one of the most successful syntheses of recent theoretical physics. Its predictions for a wide variety of phenomena in a wide variety of physics systems are being confirmed almost daily.[1] New syntheses in theoretical physics often contain features which apparently do injury to previously accepted common sense. One such feature of the renormalization-group approach is the apparent arbitrariness of the choice of renormalization transformation. A perusal of the literature[2] on applications of the renormalization-group approach would quickly reveal that there are many different transformations on the Hamiltonian of a system which are called renormalization transformations. The renormalization-group approach is evidently not a single well-defined mathematical procedure but rather a paradigm which becomes concrete only by fixing elements which are arbitrary within certain rules. These elements may be fixed by historical accident, by requirements of formal elegance or mathematical convenience, or possibly so as to reduce the errors of an approximation scheme.[3]

One of the successes of the renormalization-group (RG) method is the explanation of universality,[4] the fact that unlike most properties of matter, critical exponents and scaling functions do not depend on such specifics of the system as the range and shape of intermolecular forces but only on discrete properties such as the dimensionality of space and of the order parameter. Since universal properties cannot depend on the Hamiltonian, they must be properties of the renormalization group and the question arises immediately, "are the universal properties independent of the arbitrary elements of the renormalization group?"

This question has been addressed for special cases by a number of authors. DiCastro showed that in the ϵ expansion η is the same to order ϵ^2 for the Gell-Mann–Low and the Wilson recursion relation.[5] Bell and Wilson[6] showed that the exponents of the Gaussian fixed point of a linear RG are unchanged to first order if a small nonlinear perturbation is made on the generator of the group. Shukla and Green[7] have shown for a class of groups, which they called Gaussian and which includes the Wilson recursion relation, the linear group of Bell and Wilson, as well as the incomplete integration RG of Wilson, that η is independent of the two arbitrary functions which characterize an RG of the class to order ϵ^2, and that the other relevant eigenvalue is invariant to order ϵ.[8]

The question of the invariance of component critical exponents and scaling functions has been discussed recently by Wegner[9] and Jona-Lasinio[10] from a more general point of view. Jona-Lasinio has pointed out that if the generator of a very arbitrary defined RG is subject to a similarity transformation with a nonsingular differentiable renormalization transformation (diffeomorphism),[11] the eigenvalues of the linearized RG do not change. Wegner has derived two important invariance theorems for exponents for a class of RG's which are less general than that considered by Jona-Lasinio but include the general Gaussian RG. Wegner's first theorem introduces a distinction among the eigenvectors of the linearized RG which is different from, but is as fundamental for the question of invariance of critical exponents, as the distinctions relevant, irrelevant, and marginal[4,9] are for the question of universality. The theorem states that if the RG is changed by a small perturbation which does not change the fixed-point Hamiltonian, eigenvalues corresponding to certain variables, called scaling variables, are invariant but others corresponding to variables, called redundant, may change. Only the scaling variables have physical significance. Wegner's second theorem is a restatement for the more restricted class of RG's of Jona-Lasinio's theorem about diffeomorphisms. In this theorem both the group and the fixed point change, but all exponents, those corresponding to scaling as well as those corresponding to redundant variables, remain unchanged. Our conclusion is based on Wegner's two invariance theorems.

In this paper we consider the invariance of critical exponents and other universal features from a very general point of view. What happens to the universal quantities if the renormalization transformation is changed by an infinitesimal perturbation, while still remaining a renormalization transformation? Ideally we should characterize a renormalization transformation in the most general way possible. We are not able to proceed in this way but rather deal with the class of renormalization transformations defined by Wegner[9] which are generated by divergenceless flows in phase space. Although this class is not manifestly the most general renormalization transformation, it includes the continuous renormalization transformations actually in use, and it is not unlikely that it is essentially the most general. To simplify the calculations we confine ourselves to the case in which the underlying field has one component. A small perturbation in the RG may have two consequences for the fixed point. The new RG may not have a fixed point which is infinitesimally close to that of the original RG. This will be the case if the unperturbed group has a redundant marginal operator. If it has an infinitesimally close fixed point but no marginal scaling operator, the new RG linearized around the new fixed point will differ from the old by an infinitesimal similarity transformation that changes none of the eigenvalues coupled with a transformation which changes only the eigenvalues of the redundant variables. The net effect leaves the eigenvalues which correspond to the scaling variables unchanged. The case in which the original group had a marginal scaling eigenvector is an exceptional case in which a number of anomalous situations may arise.[9] A line or manifold of fixed points with continuously varying exponents may occur, or logarithmic singularities may appear in the free energy. Although there are certainly features of such fixed points which are invariant with respect to the renormalization group, we do not consider them in this paper.

In Sec. II, we review the general defining characteristics of renormalization transformations, fixed points, and eigenoperators, emphasizing certain features which will be important in our discussion of invariance. In Sec. II A we define the concept introduced by Wegner of a differential renormalization transformation generated by a flow vector, and in Sec. II B we discuss certain commutator identities between flow vectors of two different renormalization transformations. Special attention is devoted to the case in which one of the transformations is a simple scale change while the other is a nonscale-changing transformation. It is shown in some detail that a commutator identity is also valid in this case and the resulting flow vector is nonscale changing. The title of Sec. II C, emphasizes the fact that iterations of a differential renormalization transformation generate a semigroup rather than a group. Fixed points are defined and a notation is introduced for the dependence of the flow vector on the Hamiltonian which enables us to write the renormalization group linearized around a fixed point in a convenient form. The eigenoperators of the linearized group are defined. In Sec. II D the linear subspace of redundant perturbations is defined as the space perturbations generated by a flow vector and by using the commutator identity it is shown to be an invariant subspace of the linearized renormalization group. This fact is used to classify the eigenoperators into scaling operators and eigenoperators in Sec. II E. The renormalization transformation is linearized around a renormalization trajectory which lies in the domain of attraction of the fixed point. The linearized equations are used to define a natural coordinate system around a critical point which can by any point of the domain of attraction. The vectors of this coordinate system like the eigenoperators can be classified as redundant and scaling. The fields corresponding to the redundant vectors do not enter into the thermodynamic potential. In Sec. II F the question of how general the renormalization transformations generated by a flow vector are is discussed. It is pointed out that there is a contradiction between the desirable characteristics of a flow vector which generates a renormalization transformation and one which generates a redundant perturbation.

In Sec. III we consider the case in which the flow vector of a renormalization transformation depends on a parameter. In Sec. III A we derive the equation for the perturbation in the fixed point. In

Sec. III B and III C we present the derivations of the two invariance theorems of Wegner in a form which then can be used in Sec. III E, to discuss the case of an arbitrary infinitesimal change in the flow vector. In Sec. III D we consider the properties of the solution of the equation for the perturbation in the fixed point which are used in Sec. III E. In Sec. III F we consider the effect of an infinitesimal change in the flow vector on the renormalization trajectories themselves.

In Sec. IV we make some concluding comments in the direction of extending our results about infinitesimal changes in the group to finite changes. In Sec. IV A we consider an integrated form of our invariance theorem and the factors which might inhibit the application of this integrated form for arbitrary finite changes. In Sec. IV B we consider the manifold of equivalent fixed points, and speculate in Sec. III C on the possibility of extending this notion to the notion of manifolds of equivalent Hamiltonians. In Sec. IV D we write the fundamental functional equation for the thermodynamic potential density in an invariant form using manifolds of equivalent Hamiltonians.

II. RENORMALIZATION TRANSFORMATIONS

The defining characteristics of a renormalization transformation have been stated many times[2] but it is worthwhile to repeat them here in a form which emphasizes the features which will be important to us. Simply stated, a renormalization transformation is a transformation on a space of Hamiltonians, or rather of logarithms of Boltzmann factors, which leaves the thermodynamic potential invariant. We will use the notation $\mathcal{H}([\sigma])$ for the Hamiltonian of a system, where $[\sigma]$ stands for some set of variables which specify the microscopic state of the system. The set of variables $[\sigma]$ is taken to be a continuous valued field $\sigma(x)$ defined in a Euclidean space, a continuous valued function defined on the sites of a lattice, or more often the Fourier components σ_k of such field or lattice function. Specifically excluded are cases like the Ising model, in which $[\sigma]$ represents a set of variables which take on a discrete set of values on the sites of a lattice.[12] $\mathcal{H}([\sigma])$ will denote the negative logarithm of the Boltzmann factor $P([\sigma])$ and is usually, but not always equal to $\beta H([\sigma])$, where $\beta = 1/kT$. We will also refer to $\mathcal{H}([\sigma])$ as the Hamiltonian where it will cause no confusion.

$$P([\sigma]) = e^{-\beta H([\sigma])} = e^{-\mathcal{H}([\sigma])} . \qquad (1)$$

$P([\sigma])$ is the statistical weight of the state σ. A renormalization transformation transforms $\mathcal{H}([\sigma])$ into a new Hamiltonian, $\mathcal{H}'([\sigma'])$,

$$\mathcal{H}'([\sigma']) = R\mathcal{H}([\sigma]) , \qquad (2)$$

so that the thermodynamic potential F is unchanged:

$$e^{-F'} = \int_{[\sigma']} e^{-\mathcal{H}'([\sigma'])} = \int_{[\sigma]} e^{-\mathcal{H}([\sigma])} = e^{-F} . \qquad (3)$$

(The symbol $\int_{[\sigma]}$ indicates functional integration over $[\sigma]$.) The new Hamiltonian is supposed to represent a new physical system of the same type as the original system. The Hamiltonian $\mathcal{H}([\sigma])$ may depend on parameters, such as the temperature chemical potential and the magnetic field. The new Hamiltonian will, of course, depend on the same parameters. The importance of the condition [Eq. (3)], is that the new Hamiltonian is just as suitable as the old for computing the dependence of the thermodynamic potential on physical parameters.

The systems we deal with are supposed to be extended systems defined on a d-dimensional Euclidean space or on a d-dimensional lattice. The variables $[\sigma]$ may represent among many possibilities the actual positions and moments of the particles of the system, the presence or absence of a particle on a lattice site, a single, or multicomponent field. Defined throughout d-dimensional Euclidean space or on a d-dimensional lattice, a renormalization transformation preserves the fundamental symmetries of the system. If the initial Hamiltonian is translationally symmetric, the transformed Hamiltonian is translationally symmetric. If the initial Hamiltonian is defined on a lattice, the transformed Hamiltonian is defined on the same lattice. The renormalization transformations which are most useful in the theory of critical phenomena average over, and therefore ignore, the information contained in a fraction of the degrees of freedom of the system. In this case the set of variables $[\sigma']$ cannot be identical to the set of variables $[\sigma]$, since there are fewer variables in $[\sigma']$. Such a renormalization transformation does not have a reciprocal. The condition that the new system should represent a system of the same physical type as the original can be met, however; if the new system is defined on a smaller volume than the original system, in such a way that the number of degrees of freedom per unit volume is the same for the new and old systems. The free energy of the new system in its smaller volume is of course the same as that of the old system in its larger volume. Such a renormalization transformation may have a fixed point, i.e., a Hamiltonian $\mathcal{H}^*$ such that

$$R\mathcal{H}^* = \mathcal{H}^* . \qquad (4)$$

The symbol, = (*is equal to*) in Eq. (4) cannot mean the identity of $R\mathcal{H}^*$ and $\mathcal{H}^*$ as functions defined on the same domain, since the volume in which $R\mathcal{H}^*$ operates, will in general be smaller than that in which $\mathcal{H}^*$ operates. Equality of $R\mathcal{H}^*$ and $\mathcal{H}^*$ means that they have the same local properties.

A. Differential renormalization transformations generated by a flow vector

Wegner[9] has given a general scheme for constructing renormalization groups for Hamiltonian $\mathcal{H}([\sigma_k])$ which are functions of the Fourier components σ_k of a scalar (or vector) field. The Hamiltonians $\mathcal{H}([\sigma_k])$ are supposed to be represented by the first few terms of a functional power series

$$\mathcal{H}([\sigma_k]) = u_0 V + \sum_k u_1(k)\sigma_k + \frac{1}{2}\sum_{k_1 k_2} u_2(\vec{k}_1, \vec{k}_2)\sigma_{k_1}\sigma_{k_2} + \cdots + \frac{1}{n!}\sum_{k_1,\ldots,k_n} u_n(\vec{k}_1,\ldots,\vec{k}_n)\sigma_{k_1}\cdots\sigma_{k_n} + \cdots . \tag{5}$$

V is the volume of the system. We will usually be concerned with spacially uniform Hamiltonians for which $u_n(\vec{k}_1, \vec{k}_2, \vec{k}_n) = 0$ unless $\vec{k}_1 + \vec{k}_2 + \cdots \vec{k}_n = 0$. A flow vector, ψ_k, in the space of spin components is introduced and a nonscale-changing infinitesimal renormalization transformation is constructed.

$$\delta(\exp - \mathcal{H}'([\sigma'])) = \delta l \sum_k \frac{\partial}{\partial\sigma_k}\psi_k \exp - \mathcal{H}([\sigma]), \tag{6}$$

where δl is an infinitesimal scalar. The summation on the right-hand side has the form of a divergence of a vector field in the infinite dimensional space of the σ_k. Thus Eq. (6) will generate a differential renormalization transformation and the change in the partition function, which is the integral of the left-hand side of Eq. (6) over the components σ_k will be zero if $\psi_k \exp - \mathcal{H}([\sigma])$ vanishes sufficiently rapidly for large σ_k. Expressed in terms of $\mathcal{H}$ the infinitesimal renormalization transformation becomes

$$\delta\mathcal{H} = \delta l \left(\sum_k \psi_k \frac{\partial\mathcal{H}}{\partial\sigma_k} - \sum_k \frac{\partial\psi_k}{\partial\sigma_k} \right) = \delta l G(\psi_k)\mathcal{H}. \tag{7}$$

The flow vector ψ_k may be a functional of the σ_k either directly or implicitly through a dependence on

$$\psi_k = \psi_k([\sigma], \mathcal{H}([\sigma])). \tag{8}$$

We will suppose the dependence of ψ_k on $\mathcal{H}$ is such that ψ_k does not depend on the constant term in $\mathcal{H}$.

$$\psi_k([\sigma], \mathcal{H} + wV) = \psi_k([\sigma], \mathcal{H}). \tag{9}$$

Interesting renormalization transformations which lead to insight into phase transitions involve scale changes. In Wegner's representation, an infinitesimal scale transformation is given by

$$\delta\mathcal{H} = \delta l \sum_k (\vec{k}\cdot\nabla_k\sigma_k + \tfrac{1}{2}d\sigma_k)\frac{\partial\mathcal{H}}{\partial\sigma_k} + du_0 V = \delta l G(D_k)\mathcal{H}, \tag{10}$$

where δl now has the well-defined meaning of the infinitesimal change in the length scale. $u_0 V$ is the constant term in the Hamiltonian. d is the space dimension. The notation $G(D_l)$ for the differential scaling operator in Eq. (10) with

$$D_k = \vec{k}\cdot\nabla_k\sigma_k + \tfrac{1}{2}d\sigma_k, \tag{11}$$

is supposed to suggest a parallel to $G(\psi_k)\mathcal{H}$. Although the parallelism is not complete, it is useful to consider D_k as a flow vector similar to ψ_k. An infinitesimal scale change is a renormalization transformation. In contrast to a nonscale-changing renormalization transformation, however, the density of the thermodynamic potential changes. Only the product of the density and the volume remains constant. A general scale-changing renormalization transformation is effected by combining an infinitesimal scale change [Eq. (10)] with the nonscale-changing renormalization transformation [Eq. (7)]. The combination is effected by simply adding the two changes in the Hamiltonian:

$$\delta\mathcal{H} = \delta l[G(D_k)\mathcal{H} + G(\psi_k)\mathcal{H}] = \delta l\, G(D_k + \psi_k)\mathcal{H}. \tag{12}$$

The best-known examples of a renormalization group of this type is the smooth cutoff renormalization transformation of Wilson[13] and a generalization by Shukla and Green,[7] the general Gaussian renormalization transformation. For the latter the flow vector is

$$\psi_k = \beta(k)\sigma_k - \Gamma(k)\frac{\partial\mathcal{H}}{\partial\sigma_k}, \tag{13}$$

where $\beta(k)$ and $\Gamma(k)$ are analytic functions of k^2, which approach infinity for large k. The former is the special case for which $\Gamma = \beta = k^2$. The sharp cutoff renormalization transformation of Wilson and the linear renormalization transformation of Bell and Wilson[6] can be understood as limiting cases of this form.

B. Commutator identities

Wegner[9] has demonstrated a commutator identity involving two infinitesimal transformations $G(\psi_k)\mathcal{H}$, $G(\phi_k)\mathcal{H}$. We will use this identity in two forms. If we define $G'(\psi_k)$ to be the linear part of $G(\psi_k)\mathcal{H}$,

$$G'(\psi_k)\mathcal{H} = \sum_k \psi_k \frac{\partial\Delta\mathcal{H}}{\partial\sigma_k}, \tag{14}$$

the first form of the commutator identity is

$$G'(\psi_k)G(\phi_k)\mathcal{H} - G'(\phi_k)G(\psi_k)\mathcal{H} = G(\chi_k)\mathcal{H}, \tag{15}$$

where

$$\chi_k = K_k(\psi_k, \phi_k) = \sum_k \psi_{k'}\frac{\partial \phi_k}{\partial \sigma_{k'}} - \phi_{k'}\frac{\partial \phi_k}{\partial \sigma_{k'}}. \tag{16}$$

The second form is

$$G'(\psi_k)G'(\phi_k) - G'(\phi_k)G'(\psi_k) = G'(\chi_k). \tag{17}$$

Both forms may be easily verified by direct computations using the definitions Eqs. (7) and (14). Even though the operations $G(D_k)\mathcal{H}$ and $G(\psi_k)\mathcal{H}$ are basically different, commutator identities of the forms Eqs. (15)-(17) also hold for $G(D_k)\mathcal{H}$. Since the identities involving $G(D_k)\mathcal{H}$ are fundamental for what follows, we will present their demonstration at some length. We first compute the commutator flow vector $K_k(D_k, \phi_k)$.

$$\begin{aligned} K_k(D_k, \phi_k) &= \sum_{k'} D_{k'}\frac{\partial \phi_k}{\partial \sigma_{k'}} - \sum_{k'} \phi_{k'}\frac{\partial D_k}{\partial \sigma_{k'}} \\ &= \sum_{k'} (\vec{k}' \cdot \nabla_{k'}\sigma_{k'} + \tfrac{1}{2}d\sigma_{k'})\frac{\partial \phi_k}{\partial \sigma_{k'}} - \sum_{k'} \phi_{k'}(\vec{k}\cdot\nabla_k + \tfrac{1}{2}d)\delta(\vec{k} - \vec{k}') \\ &= \sum_{k'} (\vec{k}' \cdot \nabla_{k'}\sigma_{k'} + \tfrac{1}{2}d\sigma_{k'})\frac{\partial \phi_k}{\partial \sigma_{k'}} - (\vec{k}\cdot\nabla_k + \tfrac{1}{2}d)\phi_k. \end{aligned} \tag{18}$$

We note that if we write

$$\phi_k = \phi(k)\sigma_k + \tilde{\phi}_k, \tag{19}$$

where $\tilde{\phi}_k$ contains all terms of order σ^2 and higher, we have

$$K_k(D_k, \phi_k) = (\vec{k}\cdot\nabla_k\sigma_k + \tfrac{1}{2}d\sigma_k)\phi(k) - [(\vec{k}\cdot\nabla_k + \tfrac{1}{2}d)\sigma_k]\phi(k) + K_k(D_k, \tilde{\phi}_k) = [-\vec{k}\cdot\nabla_k\phi(k)]\sigma_k + K_k(D_k, \tilde{\phi}_k). \tag{20}$$

Thus unlike D_k itself $K_k(D_k, \phi_k)$ contains no terms $\vec{k}\cdot\nabla_k\sigma_k$ and therefore generates no dilation. Factors $\vec{k}\cdot\nabla_k\sigma_k$ are, however, contained in the higher-order terms in $K_k(D_k, \tilde{\phi}_k)$.

We next compute

$$G'(D_k)G(\phi_k)\mathcal{H} - G'(\phi_k)G(D_k)\mathcal{H}, \tag{21}$$

where the linear operator $G'(D_k)$ is defined to be

$$G'(D_k)\Delta\mathcal{H} = \sum_k D_k\frac{\partial \Delta\mathcal{H}}{\partial \sigma_k} + du_0 V. \tag{22}$$

We have for the quantity [Eq. (21)], the expression,

$$\sum_k K_k(D_k, \phi_k) + dG_0(\phi_k)\mathcal{H} - \sum_{kk'} D_k\frac{\partial^2 \phi_{k'}}{\partial \sigma_k \partial \sigma_{k'}}, \tag{23}$$

where $G_0(\phi_k)\mathcal{H}$ is the constant term in $G(\phi_k)\mathcal{H}$. The term linear in σ_k in Eq. (19) is the only term which contributes to $G_0(\phi_k)\mathcal{H}$. We have

$$G_0(\phi_k)\mathcal{H} = -V\sum_k \phi(k). \tag{24}$$

Equation (23) is to be identified with the expression

$$G(K_k(D_k, \phi_k))\mathcal{H},$$

in which K_k is to be identified as an ordinary flow vector. Thus the second and third terms on the right-hand side must be identified with

$$-\sum_k \frac{\partial K_k}{\partial \sigma_k}.$$

To show this we again write ϕ_k in the form [Eq. (19)] and substitute in Eq. (20). We have

$$\begin{aligned} -\sum_k \frac{\partial K_k}{\partial \sigma_k} &= V\sum_k \vec{k}\cdot\nabla_k\phi(k) + \sum_{kk'} -D_k\frac{\partial^2\tilde{\phi}_{k'}}{\partial\sigma_k\partial\sigma_{k'}} \\ &\quad + \sum_{kk'} \vec{k}'\cdot\nabla_{k'}\delta(k'-k)\frac{\partial\tilde{\phi}}{\partial\sigma_{k'}} \\ &\quad - \sum_k \frac{\partial(\vec{k}\cdot\nabla_k\tilde{\phi}_k)}{\partial\sigma_k}. \end{aligned} \tag{25}$$

If $\phi(k)$ vanishes rapidly enough for large k, the first term on the right-hand side may be shown, by partial integration to be

$$-dV\sum_k \phi(k) = dG_0(\phi_k)\mathcal{H}.$$

The second term on the right-hand side in Eq. (25) may be identified with the corresponding term in Eq. (23), since ϕ_k and $\tilde{\phi}_k$ have the same second and higher derivatives. Carrying out the sum on k in the third term shows that it cancels against the last term. The identity

$$G'(D_k)G'(\phi_k) - G'(\phi_k)G'(D_k) = G'(K_k(D_k, \phi_k)), \tag{26}$$

may be easily demonstrated by purely algebraic manipulations.

C. Semigroup of renormalization transformations, fixed points, and eigenoperators

The indefinite iteration of the infinitesimal renormalization transformation generates a renormalization group, or, more properly since $R(l)$, in general, has no reciprocal, a semigroup of renormalization transformations:

$$\mathcal{H}(l) = R(l)\mathcal{H}, \tag{27}$$

satisfies the equation

$$\frac{\partial \mathcal{H}(l)}{\partial l} = G(D_k + \psi_k)\mathcal{H}(l). \tag{28}$$

A fixed point $\mathcal{H}^*$ of this semigroup satisfies

$$G(D_k + \psi_k)\mathcal{H}^* = 0. \tag{29}$$

In what follows it will be necessary to develop a compact notation for Eq. (28) linearized about the fixed point $\mathcal{H}^*$. To do this we introduce the notation

$$\psi_k^* = \psi_k([\sigma], \mathcal{H}^*). \tag{30}$$

We also define an operator $\mathcal{L}_k$ such that

$$\Delta\psi_k = \mathcal{L}_k \Delta\mathcal{H} = \int_{[\sigma]} \frac{\delta \psi_k}{\delta \mathcal{H}}\bigg|_{\mathcal{H}=\mathcal{H}^*} \Delta\mathcal{H}\, d[\sigma]. \tag{31}$$

The symbol $\int_{[\sigma]} \ldots d[\sigma]$ or simply $\int_{[\sigma]}$ indicates functional integration over the variables $[\sigma]$.

$\Delta\psi_k$ is the change in ψ_k caused by a small change $\Delta\mathcal{H}$ in the Hamiltonian around the fixed point. $\mathcal{L}_k$ thus takes into account the fact that ψ_k depends explicitly on $\mathcal{H}([\sigma])$. The fact that ψ_k satisfies Eq. (9) is reflected in the relationship

$$\mathcal{L}_k \Delta u_0 V = 0. \tag{32}$$

With these definitions, noting that ψ_k always appears linearly in the expression for $G(\psi_k)\mathcal{H}$ we have

$$\frac{d}{dl}\Delta\mathcal{H} = G'(D_k + \psi_k^*)\Delta\mathcal{H} + G(\mathcal{L}_k\Delta\mathcal{H})\mathcal{H}^* = L\Delta\mathcal{H}, \tag{33}$$

where

$$\Delta\mathcal{H} = \mathcal{H} - \mathcal{H}^*. \tag{34}$$

We make the usual assumption that the solutions of Eq. (33) may be expanded in a complete set of eigenoperators O_i,

$$LO_i = y_i O_i. \tag{35}$$

Then we have

$$\Delta\mathcal{H}(l) = \sum_i \mu_i e^{y_i l} O_i. \tag{36}$$

Because of Eq. (32), $O_0 = wV$, where w is a constant, is always an eigenoperator with eigenvalue d.

D. Redundant perturbations

At this point it is appropriate to define the concept of redundant perturbation introduced by Wegner.[9] We will, however, find it useful to include explicitly perturbations from nonfixed point Hamiltonians in the definition. A perturbation of a Hamiltonian of the form

$$\Delta\mathcal{H} = G(\phi_k)\mathcal{H} \tag{37}$$

is a redundant perturbation. The space of redundant perturbations of a given Hamiltonian, and in particular of a fixed point Hamiltonian, is a linear subspace, since the operation, $G(\phi_k)\mathcal{H}$, is linear in ϕ_k.

An important property of the linear operator L is that when it operates on a redundant perturbation to $\mathcal{H}^*$ it produces a redundant perturbation. In fact using the commutator identity and Eqs. (3), (29),

$$LG(\phi_k)\mathcal{H}^* = G(K_k(D_k + \psi_k^*, \phi_k) + \mathcal{L}_k G(\phi_k)\mathcal{H}^*)\mathcal{H}^*. \tag{38}$$

Thus the linear subspace of redundant perturbations is an invariant subspace of L. This means that if L is not pathological, its eigenfunctions may be classified into three disjoint sets, the redundant eigenfunctions which span the linear subspace of redundant perturbations, the constant eigenoperator O_0, and the complementary set of these two, the scaling eigenfunctions.

From the property of the G operation discussed in Sec. II C [Eqs. (6) and (7)], it is clear that for small perturbations from the fixed point Hamiltonian, the free energy of the perturbed Hamiltonian is independent of the fields μ_i corresponding to the redundant perturbations.

E. Natural coordinate system

The eigenvectors O_i [Eq. (35)] form a natural coordinate system in terms of which any perturbation from the fixed point Hamiltonian may be represented. The object of physical interest, however, is not the fixed point Hamiltonian but a critical Hamiltonian. A natural coordinate system for perturbations from a critical Hamiltonian can be constructed by considering perturbations from RG trajectories which approach the fixed point. Any Hamiltonian through which such a trajectory passes is called an element of the domain of attraction of the fixed point, and every point on the domain of attraction is a critical point. We generalize the concept of linearized RG introduced earlier [Eqs. (30), (31), and (33)]. We consider the effect of the differential operator of the RG on a perturbation from an RG trajectory $\mathcal{H}_0(l)$ which approaches $\mathcal{H}^*$ as $l \to \infty$. $l = 1$ is supposed to correspond to the critical Hamiltonian, $\mathcal{H}_c = \mathcal{H}_0(1)$.

$$\mathcal{H}(l) = \mathcal{H}_0(l) + \Delta\mathcal{H}, \tag{39}$$

where $\mathcal{H}_0(l) \to \mathcal{H}^*$ as $l \to \infty$. If we substitute Eq. (39) into Eq. (28) and retain linear terms, we obtain

$$\frac{\partial \Delta\mathcal{H}}{\partial l} = L(l)\Delta\mathcal{H}, \tag{40}$$

with

$$L(l)\Delta\mathcal{H}=G'(D_k+\psi_k([\sigma],\mathcal{H}_0(l)))\Delta\mathcal{H} + G(\mathcal{L}_k(l)\Delta\mathcal{H})\mathcal{H}_0(l) \quad (41)$$

and

$$\mathcal{L}_k(l)=\frac{\delta\psi_k}{\delta\mathcal{H}}\bigg|_{\mathcal{H}=\mathcal{H}_0(l)} . \quad (42)$$

$L(l)$ approaches L as $l\to\infty$. We define $\tilde{O}_i$ to be a perturbation from $\mathcal{H}_0(1)$ which approaches the eigenfunction O_i as $\mathcal{H}(l)\to\mathcal{H}^*$. The set of perturbations $\tilde{O}_i$ forms a coordinate system for perturbations from the critical Hamiltonians. We may call the $\tilde{O}_i$ the critical coordinate system. $\tilde{O}_0$ may be taken equal to be O_0.

It is easy to see that a redundant perturbation to $\mathcal{H}_0(1)$ will remain redundant under the operation of the RG. If

$$\Delta\mathcal{H}=G(\phi_k(l))\mathcal{H}_0(l), \quad (43)$$

with the boundary condition

$$\tilde{O}_i=G(\phi_k(1))\mathcal{H}_0(1), \quad (44)$$

we have from Eqs. (40) and (41),

$$G\left(\frac{\partial\phi_k(l)}{\partial l}\right)\mathcal{H}_0(l)+G(\phi_k(l))\frac{\partial\mathcal{H}_0(l)}{\partial l} = G'(D_k+\phi_k(\mathcal{H}_0(l)))\,G(\phi_k(l))\mathcal{H}_0(l) + G(\mathcal{L}(l)\,G(\phi_k(l))\mathcal{H}_0(l))\mathcal{H}_0(l) . \quad (45)$$

Noting that $\mathcal{H}_0(l)$ satisfies Eq. (28) and as well as the commutator identity [Eq. (15)], we have

$$G\left(\frac{\partial\phi_k(l)}{\partial l}-K_k(D_k+\psi_k(\mathcal{H}_0(l)))-\mathcal{L}_k(l)G(\phi_k(l))\mathcal{H}_0(l)\right)\times\mathcal{H}_0(l)=0 . \quad (46)$$

$\phi_k(l)$ may be taken to satisfy the equation

$$\frac{\partial\phi_k(l)}{\partial l}=K_k(D_k+\psi_k(\mathcal{H}_0(l)))+\mathcal{L}_k(l)\,G(\phi_k(l))\mathcal{H}_0(l) =0 . \quad (47)$$

Equation (47) implies that an $\tilde{O}_i$ which corresponds to a redundant O_i must itself be redundant. The thermodynamic potential of a critical Hamiltonian will not be changed by a perturbation $\tilde{O}_i$ which corresponds to a redundant perturbation O_i. The expansion coefficients of a small perturbation from a critical Hamiltonian will be linear functions of the deviations, say $\Delta\mu$, ΔT of the physical parameters from their critical values. Only the fields of the nonredundant $\tilde{O}_i$ will enter into the thermodynamic potential. These latter are perturbations from their critical values of what have been called by Wegner, the scaling fields.

F. Some questions

We close this section by raising several questions which represent hiatuses in our logic and for which our only answers are suggestions from specific renormalization groups.[14] On the one hand, if we rely on an intuition based on finite dimensional spaces, we would conclude that the groups produced by flows are very general since the primary requirement of the nonscale-changing part of a differential renormalization transformation is that the change in the Boltzmann factor should integrate to zero. In finite dimensional spaces a scalar function which goes to zero rapidly enough for large values of the independent variables, and whose integral is zero, can always be represented, indeed, in many ways as a divergence of a vector field. On the other hand, if we apply this reasoning to the linear subspace of redundant perturbations, we would reach the conclusion that any perturbation to a Hamiltonian whose average value is zero can be represented in the form $G(\phi_k)\mathcal{H}$ and therefore is a redundant perturbation. Thus, the space of small perturbations to a Hamiltonian is exhausted by the constants and the redundant perturbations, leaving no room for the scaling operators, which are the *raison d'être* of the whole procedure. It is to be presumed that this apparent internal contradiction of the theory is a consequence of our imprecise definition of the function space in which Hamiltonians and flow vectors are supposed to lie. We may hope that with suitable restrictions the flow vectors ψ_k will generate essentially all renormalization groups while the flow vectors ϕ_k will generate a linear subspace of redundant perturbations which is small enough to leave room for scaling operators.

III. FLOW VECTOR DEPENDING ON A PARAMETER

We turn now to the basic question of this paper and suppose that ψ_k depends explicitly on σ_k, implicitly on σ_k through the Hamiltonian and explicitly on a parameter α, $0\leqslant\alpha\leqslant 1$.

$$\psi_k=\psi_k([\sigma_k],\ \mathcal{H}([\sigma_k]),\ \alpha). \quad (48)$$

Equation (28) then represents a renormalization group depending on the parameter α. For a particular value say 0 of α, the renormalization group Eq. (28) is supposed to have a fixed point $\mathcal{H}_0^*$. If we make an infinitesimal change in α away from 0, it may happen that the new renormalization group has a fixed point $\mathcal{H}_\alpha^*$ which is near $\mathcal{H}_0^*$. The new renormalization group may be linearized around $\mathcal{H}_0^*$.

$$\frac{d\Delta\mathcal{H}}{dl}=L_\alpha\Delta\mathcal{H} , \quad (49)$$

where L_α is given by Eq. (33) with ψ_k given by Eq. (48) for $\mathcal{H}=\mathcal{H}_\alpha^*$. We ask the question "Will the eigenvalues of L_α be the same as those of L_0?" Two points need to be made before addressing this question in more detail. The first is that the alternative, that the new renormalization group does not have a fixed point $\mathcal{H}_\alpha^*$ near $\mathcal{H}_0^*$ is a real one, and happens in familiar cases. Thus if in the group defined by Eq. (13) the value of the function $\beta(k)$ at $k=0$ is changed slightly, there is no nearby fixed point. The second is that we cannot expect all eigenvalues of L to be unchanged, only the eigenvalues corresponding to *scaling variables*, will be invariant. The eigenvalues corresponding to *redundant variables* may change. Critical exponents, however, are related only to the y_i corresponding to scaling variables.

A. Perturbation in the fixed point

If we differentiate the equation for the fixed point Eq. (29) with respect to α and set $\alpha=0$; note that D_k is independent of α; and $G(\psi_k)\mathcal{H}$ is linear in ψ_k, we obtain

$$G'(D_k+\psi_{k0}^*)\mathcal{H}_0^{*\prime}+G(\psi_{k0}'+\mathcal{L}_{k0}\mathcal{H}_0^{*\prime})\mathcal{H}_0^*=0\ , \tag{50}$$

or

$$L_0\mathcal{H}_0^{*\prime}+G(\psi_{k0}')\mathcal{H}_0^*=0\ , \tag{51}$$

where

$$\mathcal{H}_0^{*\prime}=\frac{d}{d\alpha}\mathcal{H}_\alpha^*\big|_{\alpha=0} \tag{52}$$

and

$$\psi_{k0}'=\frac{\partial}{\partial\alpha}\psi_k\big|_{\alpha=0,\,\mathcal{H}=\mathcal{H}_0^*}\ .$$

If Eq. (51) has a solution, the new fixed point to first order in α will be

$$\mathcal{H}_\alpha^*=\mathcal{H}_0^*+\alpha\mathcal{H}_0^{*\prime}\ . \tag{53}$$

Equation (51) may be expected to have a solution if L_0 is nonsingular. If L_0 is singular, Eq. (51) will have a solution only if $G(\psi_{k0}')\mathcal{H}_0^*$ satisfies a condition which we will discuss below. In order to prove that under certain circumstances the scaling eigenvalues of L_α are the same as those of L_0, at least for small α, we make use of two theorems of Wegner.

B. Wegner's first invariance theorem (Ref. 9)

Wegner's first invariance theorem states that an infinitesimal change in an RG, which changes neither the fixed point Hamiltonian nor the fixed point value of the flow vector, produces no change in eigenvalues corresponding to scaling variables.

From the Eq. (51) for the shift in the fixed point, this will be the case if $\psi_{k0}'=0$ and L_0 is nonsingular. From Eq. (33) we see that if $\psi_{k0}'=0$, the only way L can change is through a change in $\mathcal{L}_k$. In fact

$$L_\alpha\Delta\mathcal{H}=L_0\Delta\mathcal{H}+\alpha G(\mathcal{L}_{k0}'\Delta\mathcal{H})\mathcal{H}_0^* \tag{54}$$

to first order in α, where

$$\mathcal{L}_{k0}'=\frac{\partial\mathcal{L}_k}{\partial\alpha}\bigg|_{\alpha=0}\ . \tag{55}$$

The perturbation to L_0, represented by the second term on the right-hand side in Eq. (54) is a linear operator which always produces a contribution lying in the subspace of redundant perturbations. The essence of Wegner's first theorem is that such a perturbation can produce no change in the eigenvalues of the scaling operators which do not lie in this subspace.

Let us consider the eigenvalue equation for the linear operator L and its derivative with respect to α. We have

$$(L-y_i)O_i=0\ , \tag{56}$$

where O_i is the eigenvector corresponding to the eigenvalue y_i. If O_i and y_i are continuous and differentiable functions of α for $\alpha=0$, we must have

$$(L_0-y_{i0})O_{i0}'=-(L_0'-y_{i0}')O_{i0}\ ,$$

with

$$O_{i0}'=\frac{d}{d\alpha}O_i\bigg|_{\alpha=0}\ ,\qquad y_{i0}'=\frac{d}{d\alpha}y_i\bigg|_{\alpha=0}\ . \tag{57}$$

In the particular case in which ψ_{k0}' and $\mathcal{H}_0^{*\prime}$ are both zero,

$$L_0'\Delta\mathcal{H}=G(\mathcal{L}_{k0}'\Delta\mathcal{H})\mathcal{H}_0^*\ . \tag{58}$$

We determine y_{i0}' by requiring that Eq. (57) which is an inhomogeneous linear equation for O_{i0}' has a solution. Assuming completeness of the eigenfunctions of L_0 we may expand this solution as a linear combination

$$O_{i0}'=\sum_j C_{ij}O_{j0}\ . \tag{59}$$

The left-hand side of Eq. (57) becomes

$$\sum_j C_{ij}(y_{j0}-y_{i0})O_{j0}\ . \tag{60}$$

In this series the coefficient of O_{i0} is zero. This must be true of the right-hand side. The coefficient of O_{i0} on the right-hand side is $y_{i0}'-C_{i0}'$ where C_{i0}' is the coefficient of O_{i0} in the expansion of $L_0'O_{i0}$. If O_{i0} is a scaling variable, C_{i0}' must be zero since $L_0'O_{i0}$ is a redundant perturbation to $\mathcal{H}_0^*$ which can be expressed solely as a linear combination of the redundant eigenvectors. In this case,

$y'_{i0}=0$. The exponent corresponding to a scaling variable is invariant. If O_{i0} is a redundant eigenvector, C'_{i0} is not necessarily zero, and y'_{i0} is not necessarily zero. The exponent for a redundant eigenvector need not be invariant. As has been pointed out exponents corresponding to redundant eigenvectors have no physical significance and their variability has no physical consequences.

C. Wegner's second invariance theorem (Ref. 9)

The second of the two invariance theorems proved by Wegner refers to the case in which the fixed point Hamiltonian is changed by a redundant perturbation

$$\bar{\mathcal{H}}^* = \mathcal{H}^* + \alpha G(\phi_k)\mathcal{H}^* , \tag{61}$$

while the flow vector to first order in $\Delta\mathcal{H}$ is concommitantly changed to

$$\bar{\psi}_k = \bar{\psi}_k^* + \bar{\mathfrak{L}}_k \Delta\mathcal{H} , \tag{62}$$

with

$$\bar{\psi}_k^* = \psi_k^* - K_k(D_k + \psi_k^*, \phi_k) ,$$

$$\bar{\mathfrak{L}}_k \Delta\mathcal{H} = \mathfrak{L}_k \Delta\mathcal{H} - \mathfrak{L}_k G'(\phi_k)\Delta\mathcal{H} - K_k(\mathfrak{L}_k\Delta\mathcal{H}, \phi_k) . \tag{63}$$

In this case all eigenvalues remain unchanged. [In comparing Eqs. (61)–(63) with Wegner's statement of the theorem in Ref. 9, it should be noted that what we have called $\mathfrak{L}_k\Delta\mathcal{H}$ corresponds to $\sum_i \mu_i \psi_i$ and $\mathfrak{L}(1+G'(\phi))\Delta\mathcal{H}$ corresponds to $\sum_i \mu'_i \psi'_i$.]

Wegner's second theorem is a special case of a theorem of Jona-Lasinio[10] which states that if R, R^+ are differentiable renormalization transformations and T is a differentiable renormalization transformation with reciprocal (diffeomorphism), and the similarity relation

$$TR = R^+ T \tag{64}$$

is valid, then the linear transformations $\tilde{R}$, $\tilde{R}^+$, $\tilde{T}$ obey the relation

$$\tilde{T}\tilde{R} = \tilde{R} + \tilde{T} , \tag{65}$$

where $\tilde{T}$ and $\tilde{R}$ are linearized around the fixed point, $\mathcal{H}^*$, of R and $\tilde{R}^+$ is linearized around the fixed point, $T\mathcal{H}^*$, of R^+. Equation (65) implies, of course, that $\tilde{R}$ and $\tilde{R}^+$ have the same eigenvalues. When expressed in terms of linearized differential renormalization transformations $G'(\phi)$, L, and $\bar{L}$,

$$\tilde{R}\Delta\mathcal{H} = (1+\gamma L)\Delta\mathcal{H} , \tag{66}$$

$$\tilde{R}^+\Delta\mathcal{H} = (1+\gamma \bar{L})\Delta\mathcal{H} , \tag{67}$$

$$\tilde{T}\Delta\mathcal{H} = [1+\gamma G'(\phi_k)]\Delta\mathcal{H} , \tag{68}$$

where γ is an infinitesimal constant, Jona-Lasinio's theorem[10] states

$$\bar{L} - L = G'(\phi_k)L - LG'(\phi_k) , \tag{69}$$

where $\bar{L}$ is constructed from $\bar{\mathcal{H}}^*$, $\bar{\psi}^*$, and $\bar{\mathfrak{L}}_k$ in the same way that L is constructed from $\mathcal{H}^*$, ψ_k^*, and $\mathfrak{L}_k$.

$$\begin{aligned}\bar{L}\Delta\mathcal{H} = {} & G'(D_k + \psi_k^* - K_k(D_k + \psi_k^*, \phi_k))\Delta\mathcal{H} \\ & + G(\mathfrak{L}\Delta\mathcal{H} - \mathfrak{L}G'(\phi_k)\Delta\mathcal{H} - K_k(\mathfrak{L}_k\Delta\mathcal{H}, \phi_k)) \\ & \times[\mathcal{H}^* + G(\phi_k)\mathcal{H}^*] .\end{aligned} \tag{70}$$

Keeping only terms linear in $\Delta\mathcal{H}$ and ϕ_k gives

$$\begin{aligned}(\bar{L} - L)\Delta\mathcal{H} = {} & G'(-K_k(D_k + \psi_k^*, \phi_k))\Delta\mathcal{H} \\ & - G(\mathfrak{L}_k|G'(\phi_k)\Delta\mathcal{H})\mathcal{H}^* \\ & + G'(\phi_k)G(\mathfrak{L}_k\Delta\mathcal{H})\mathcal{H}^* .\end{aligned} \tag{71}$$

A short computation, making use of the definition of L [Eq. (33)] and the commutator identity [Eq. (15)] shows that the right-hand side of Eq. (71) is indeed equal to the commutator, Eq. (69). The fact that the difference between $\bar{L}$ and L is a commutator to first order in ϕ_k means that they have the same eigenvalues to this order in agreement with Jona-Lasinio's more general theorem.

D. Solution of the equation for $\mathcal{H}_0^{*\prime}$

In order to proceed further we must learn more about $\mathcal{H}_0^{*\prime}$ the derivative of the fixed point Hamiltonian at $\alpha=0$. $\mathcal{H}_0^{*\prime}$ is a solution of Eq. (51). Since the right-hand side of this equation is a redundant perturbation and since the subspace of redundant perturbations is an invariant subspace of L_0, we may expect that $\mathcal{H}_0^{*\prime}$ is a redundant perturbation or at least contains a redundant part. Indeed if we write

$$\mathcal{H}_0^{*\prime} = \tilde{\mathcal{H}} + G(\phi_k)\mathcal{H}_0^* , \tag{72}$$

where $\tilde{\mathcal{H}}$ belongs to the invariant linear subspace of nonredundant operators, we must have using Eq. (38),

$$L_0\tilde{\mathcal{H}} = 0 \tag{73}$$

and

$$G(K(D_k + \psi_{k0}^*, \phi_k) + \mathfrak{L}_k G(\phi_k)\mathcal{H}_0^* + \psi'_{k0})\mathcal{H}_0^* = 0 . \tag{74}$$

Equation (73) states that $\tilde{\mathcal{H}}$ must be a nonredundant marginal eigenvector of L. The case in which L_0 has a marginal scaling eigenvector is well known to be anomalous and will be excluded from our considerations. We note only that in this case non-power-law singularities may appear, or there may be a manifold of fixed points near $\mathcal{H}_0^*$ with varying exponents.[9] Thus a theorem of invariance of critical exponents may be inapplicable or false. Excluding this anomalous case, $\mathcal{H}_0^{*\prime}$ is redundant.

$$\mathcal{H}_0^{*\prime} = G(\phi_k)\mathcal{H}_0^* , \tag{75}$$

where ϕ_k may be taken to be a solution of

$$K_k(D_k+\psi^*_{k0},\phi_k)+\mathfrak{L}_k G(\phi_k)\mathcal{H}^*_0+\psi'_{k0}=0\ . \tag{76}$$

Equation (76) is a linear inhomogeneous equation for the flow vector ϕ_k. Such an equation can be expected to have a solution for an arbitrary ψ_k if the homogeneous part has no null vector. Noting Eq. (38), this means that L_0 has no redundant marginal eigenvectors. In this case there is a fixed point $\mathcal{H}^*_\alpha$ near $\mathcal{H}^*_0$ for small α for an arbitrary dependence of $\psi_k([\sigma],\mathcal{H}^*,\alpha)$ on α. If on the other hand L_0 has one or more redundant marginal eigenvectors $G(\phi_k^{(1)})\mathcal{H}^*_0$, $G(\phi_k^{(2)})\mathcal{H}^*_0$, this means that the original unperturbed group has a manifold of fixed points near $\mathcal{H}^*_0$

$$\mathcal{H}^*=\mathcal{H}^*_0+G(a_1\phi_k^{(1)}+a_2\phi_k^{(2)}+\cdots)\mathcal{H}^*_0\ . \tag{77}$$

(We will see below that these will all have the same scaling eigenvalues.) The perturbed group will have a fixed point near $\mathcal{H}^*_0$ only if Eq. (76) has a solution. This will be the case only if the expansion of ψ'_{k0} in terms of eigenvectors of $K_k(D_k+\psi_k,\phi_k)+\mathfrak{L}_k G(\phi_k)\mathcal{H}^*_0$ has no terms in $\phi_k^{(1)},\phi_k^{(2)},\ldots$. The new fixed point will be

$$\mathcal{H}^*_\alpha=\mathcal{H}^*_0+G(a_1\phi_k^{(1)}+a_2\phi_k^{(2)}+\cdots)\mathcal{H}^*_0+\alpha G(\phi_k^0)\mathcal{H}^*_0 \tag{78}$$

where $a_1, a_i,\ldots$, are arbitrary infinitesimal constants, and ϕ_k^0 is a particular solution of Eq. (74). Thus, the perturbed group will have a manifold of fixed points parallel to, but slightly displaced from, the manifold of fixed points of the unperturbed group. For the more general case in which ψ'_{k0} does not satisfy the above condition, the new group will have no fixed point near $\mathcal{H}^*_0$.

E. Invariance of scaling exponents

We have shown in Sec. III D that except for an anomalous case in which an invariance theorem cannot be expected to apply, the perturbation in the fixed point, if it exists at all, will be redundant. In this section we will show that for such a perturbed fixed point the scaling eigenvalues will be unchanged.

The derivative of Eq. (33) with respect to α for $\alpha=0$ is

$$L'\Delta\mathcal{H}=G'(\psi'_k+\mathfrak{L}_k\mathcal{H}^{*\prime}_0)\Delta\mathcal{H}+G(\mathfrak{L}'_k\Delta\mathcal{H})\mathcal{H}^*_0+G'(\mathfrak{L}_k\Delta\mathcal{H})\mathcal{H}^*_0\ . \tag{79}$$

If $\mathcal{H}^{*\prime}_0$ is redundant there exists a ϕ_k satisfying Eq. (79) and we may write the expression for L' [Eq. (79)] as

$$L'\Delta\mathcal{H}=G'(-K_k(D_k+\psi^*_k,\phi_k))\Delta\mathcal{H}+G(\mathfrak{L}'_k\Delta\mathcal{H})\mathcal{H}^*_0+G'(\mathfrak{L}_k\Delta\mathcal{H})\,G(\phi_k)\mathcal{H}^*_0\ . \tag{80}$$

Adding and subtracting the two terms

$$G'(\phi_k)\,G(\mathfrak{L}_k\Delta\mathcal{H})\mathcal{H}^*\quad\text{and}\quad G(\mathfrak{L}_k G'(\phi_k)\Delta\mathcal{H})\mathcal{H}^*_0\ ,$$

we may rewrite Eq. (80) as

$$L'\Delta\mathcal{H}=G'(-K_k(D_k+\psi^*_k,\phi_k))\Delta\mathcal{H}+G'(\phi_k)\,G(\mathfrak{L}_k\Delta\mathcal{H})\mathcal{H}^*_0-G(\mathfrak{L}_k G'(\phi_k)\Delta\mathcal{H})\mathcal{H}^*_0+G(K_k(\mathfrak{L}_k\Delta\mathcal{H},\phi_k)+\mathfrak{L}_k G'(\phi_k)\Delta\mathcal{H}+\mathfrak{L}_k\Delta\mathcal{H})\mathcal{H}^*_0\ . \tag{81}$$

By Eqs. (69) and (71) the expression in square brackets is a commutator, and the second term is a redundant perturbation. By Wegner's first and second theorem, neither causes any change in the eigenvalues y_i of the scaling eigenfunctions.

F. Perturbed renormalization trajectories

In Secs. III D and III E we have considered the effect of an infinitesimal modification of the flow vector ψ_k on the fixed point $\mathcal{H}^*$ and on the linearized group operator L. In this section we consider the effect of such a change on the renormalization trajectory $\mathcal{H}_\alpha(l)$ itself. If we differentiate Eq. (28) with respect to α at $\alpha=0$, we obtain

$$\frac{\partial\mathcal{H}'_0(l)}{\partial l}=G'(D_k+\psi_k(\mathcal{H}_0(l)))\mathcal{H}'_0(l)+G(\mathfrak{L}_k(l)\mathcal{H}'_0(l))\mathcal{H}_0(l)+G(\psi'_k(l))\mathcal{H}_0(l)=L(l)\mathcal{H}'_0(l)+G(\psi'_k(l))\mathcal{H}_0(l)\ , \tag{82}$$

where $L(l)$ is the linear operator defined in Eq. (41) and

$$\psi'_k(l)=\frac{\partial}{\partial\alpha}\psi_k([\sigma],\mathcal{H}(l),\alpha)\Big|_{\alpha=0,\ \mathcal{H}(l)=\mathcal{H}_0(l)}\ . \tag{83}$$

This equation may be compared to Eq. (51), of which it may be considered to be the l-dependent version, and to Eq. (40) of which it is an inhomogeneous form. Equation (82) determines the modification in the renormalization trajectory $\mathcal{H}_0(l)$ if ψ_k is changed infinitesimally. The modified trajectory has the same initial Hamiltonian as the unmodified trajectory, i.e., $\mathcal{H}'_0(0)=0$. We assume that the homogeneous equation [Eq. (40)] has no solution for which $\mathcal{H}'_0(0)$ is zero except $\mathcal{H}'_0(l)=0$. The inhomogeneous equation will have a unique solution which will be redundant,

$$\mathcal{H}'_0(l)=G(\phi_k(l))\mathcal{H}_0(l)\ , \tag{84}$$

where $\phi_k(l)$ may be taken to satisfy the linear inhomogeneous equation

$$\frac{\partial\phi_k(l)}{\partial l}=-K_k(D_k+\psi_k(\mathcal{H}_0(l)),\phi_k(l))+\mathfrak{L}_k(l)\,G(\phi_k(l))\mathcal{H}_0(l)+\psi'_k(l)\ , \tag{85}$$

with the initial condition $\phi(0)=0$. The modified trajectory differs from $\mathcal{H}_0(l)$ by a redundant perturbation.

IV. SOME CONCLUDING COMMENTS

The theorem which has just been proven does not, of course, exhaust all questions about invariance of renormalization groups which might be asked, even for the restricted class of renormalization transformations, for which it was derived. One might wish to ask what happens when L_0 has a nonredundant null vector. This is the case in which the thermodynamic singularities may be no longer represented by power laws and exponents but rather by logarithmic singularities or in which exponents may vary continuously with parameters in the Hamiltonian. Another question is the invariance of the thermodynamic potential whose computation is the primary objective of the approach. A third question is how far can we go in making our theorem global rather than differential. We conclude our considerations with several speculative comments which attempt to answer partially some of the above questions.

A. Continuation for finite values of the parameter

In this section we consider what happens if the parameter in the flow vector $\psi_k([\sigma],\mathcal{H}([\sigma],\alpha))$ is varied continuously from zero to some finite value. For simplicity we suppose that we are in the case in which L_0 has no null vectors. We assume first that this is also the case for $0\leq\alpha\leq 1$. Then there is a unique solution of Eq. (76) for the flow vector $\phi_k(\alpha)$ for $0\leq\alpha\leq 1$. Noting the role of the flow vector as the generator of an infinitesimal substitution transformation we may write[15]

$$\frac{d\sigma_k(\alpha)}{d\alpha}=\phi_k(\alpha) \tag{86}$$

and rewrite Eq. (75) in the form

$$\frac{d\mathcal{H}^*}{d\alpha}=G(\phi_k(\alpha))\mathcal{H}^*_\alpha\ . \tag{87}$$

Equations (86) and (87) together with Eq. (76) constitute a system of equations for $\sigma_k(\alpha)$ and $\mathcal{H}^*$ which may be integrated to yield the finite substitution transformation

$$\sigma_k(\alpha)=\sigma_k(\alpha,[\sigma_k(0)])\ , \tag{88}$$

with the inverse

$$\sigma_k(0)=\sigma_k(0,[\sigma_k(\alpha)])\ , \tag{89}$$

determined from $\mathcal{H}^*_0$ through

$$\mathcal{H}^*_\alpha([\sigma_k(\alpha)])=\mathcal{H}^*_0([\sigma_k(0,[\sigma_k(\alpha)])])+\ln\frac{\partial\{\sigma_k(\alpha)\}}{\partial\{\sigma_k(0)\}}\ , \tag{90}$$

where $\partial\{\sigma_k(\alpha)\}/\partial\{\sigma_k(0)\}$ is the Jacobian of the transformation Eq. (88). There is thus a connected one-dimension manifold of fixed point Hamiltonians beginning with $\mathcal{H}^*_0$ and ending with $\mathcal{H}^*_1$. This manifold is different from the manifolds of fixed points previously discussed in that each fixed point of the manifold belongs to a different renormalization transformation. Any $\mathcal{H}^*$ of the manifold is connected to any other $\mathcal{H}^*$ by a substitution transformation of the type given by Eqs. (88)–(90). The scaling exponents are the same for each member of the manifold as is the thermodynamic potential. The nonscaling exponents of course may vary. The existence of such a manifold of fixed points illustrates the point which has often been made that there is much about the precise form of the fixed point Hamiltonian that has no physical significance. This procedure would break down if as some value of α is approached $\phi_k(\alpha)$ would tend to go outside the class of acceptable flow vectors or cease to exist at all. For instance since the nonscaling exponents may change, it may happen that one of these becomes zero for a value $\alpha=\alpha_0$. Then it is possible that Eq. (86) cannot be integrated beyond α_0 and no fixed point Hamiltonian $\mathcal{H}^*_\alpha$ for $\alpha_0\leq\alpha\leq 1$ which is related to $\mathcal{H}^*_0$ by a substitution transformation [Eqs. (88)–(90)] exists. The renormalization transformation specified by the parameter 1 either has no fixed point or if it has a fixed point, it may not have the same scaling exponents as those of $\mathcal{H}^*_0$.

B. Manifolds of equivalent fixed point Hamiltonians

The considerations of Sec. IV A suggest that it might be useful to define the concept of a manifold of equivalent fixed points as the manifold of Hamiltonians which are related to the fixed point Hamiltonian $\mathcal{H}^*_0$ by a continuous sequence of infinitesimal substitution transformations [Eqs. (88)–(90)]. Such a manifold presumably represents all fixed point Hamiltonians having the same qualitative features which are reached by some renormalization transformation. While no Hamiltonian of the manifold has a unique physical significance, the manifold as a whole does. Critical exponents and other universal physical quantities may be considered to be properties of the manifold rather than of any individual member. It is clear that the linear manifold of redundant perturbations is the tangent manifold to the manifold of fixed point Hamiltonians at the point $\mathcal{H}^*_0$, and that the tangent manifold around any other point of the manifold $\mathcal{H}^*_\alpha$ is the linear manifold of redundant perturbations with respect to $\mathcal{H}^*_\alpha$. For linear renormalization groups, Kawasaki and Gunton[16] have demonstrated an asymptotic property of the fixed point Hamiltonian for large σ which may perhaps characterize the manifold of fixed point Hamiltonians.

C. Manifolds of equivalent Hamiltonians

It is of some interest to generalize the concept of the manifold of equivalent fixed point to the concept of manifold of equivalent Hamiltonians. The manifold of Hamiltonians equivalent to a Hamiltonian $\mathcal{H}_0$ is the manifold of all Hamiltonians which can be generated from $\mathcal{H}_0$ by a sequence of infinitesimal substitution transformations. The theorem proved above that an infinitesimally perturbed renormalization group will carry a redundant perturbation into a redundant perturbation can be expressed in integrated form through the concept of manifolds of equivalent Hamiltonians. If a renormalization group $R_0(l)$ transforms a Hamiltonian $\mathcal{H}_0(0)$ into an $\mathcal{H}_0(l)$, the renormalization group $R_1(l)$ will transform a Hamiltonian $\mathcal{H}_1(0)$ equivalent to $\mathcal{H}_0(0)$ into a Hamiltonian $\mathcal{H}_1(l)$ equivalent to $\mathcal{H}_0(l)$. More succinctly stated, a renormalization transformation $R(l)$ will transform manifolds of equivalent Hamiltonians into manifolds of equivalent Hamiltonians, which depend only on l and not on the renormalization transformation. From the mode in which manifolds of equivalent Hamiltonians are constructed all equivalent Hamiltonians have the same thermodynamic potential density.

D. Thermodynamic potential density functional

Because of the questions raised at the end of Sec. III, it is not clear whether a manifold of equivalent Hamiltonians is smaller than the set of all Hamiltonians having the same free energy. The thermodynamic potential density is a functional of the manifold of equivalent Hamiltonians. (We may indicate by $\{\mathcal{H}\}$ the manifold of Hamiltonians equivalent to $\mathcal{H}$.) We may indicate the thermodynamic potential density by $f(\{\mathcal{H}\})$. We denote by $\{\mathcal{H}\}_l$ the equivalence manifold of $\mathcal{H}(l)$, $\{\mathcal{H}\}_l \equiv \{\mathcal{H}(l)\}$. The notation is intended to emphasize the fact noted above that $\{\mathcal{H}(l)\}$ does not depend on the renormalization transformation. The invariance property of the total thermodynamic potential under a renormalization transformation $R(l)$ can be expressed as a relationship for the thermodynamic potential density functional $f(\{\mathcal{H}\})$ through the well-known equation

$$f(\{\mathcal{H}\}) = \exp dl\, f(\{\mathcal{H}\}_l)\,. \tag{91}$$

In contrast to more familiar forms of this equation, however, Eq. (91) is invariant with respect to the choice of renormalization transformation since $\{\mathcal{H}\}_l$ is independent of such choice. The fact that $f(\{\mathcal{H}\})$ is a generalized homogeneous function near a fixed point is a consequence of Eq. (91), and not unexpectedly the scaling function which expresses this homogeneity is independent of the renormalization group. The invariance properties of fixed points with nonredundant marginal operators may presumably be derived from this same invariant equation.

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