

Thermodynamics of liquid ^4He

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(Received 1 November 1976)

Using the equation-of-motion method we derive, based on the pair approximation [Hartree-Fock (HF), BCS] of the hard-sphere model for superfluid He, the Bogoliubov equation for the superfluid phase. At the temperature $T = T_\lambda$ the off-diagonal pairing factor (BCS) has to vanish, and the excitation energy must become that of the HF solution. This condition yields a transition temperature $T_\lambda = 2.3^\circ\text{K}$. The superfluid density ρ_s as a function of temperature is also investigated. Furthermore assuming the net translational kinetic energy of the liquid to be $E_T = (1/2)\Omega(\rho_s V_s^2 + \rho_N V_N^2)$ we investigate the response of our system to the corresponding perturbation H_I . We calculate the persistent current components $\langle J^N(0) \rangle$ and $\langle J^S(0) \rangle$ which exhibits the right behavior for $T \rightarrow 0$ and $T \rightarrow T_\lambda$.

I. INTRODUCTION

Liquid ^4He undergoes a transition at $T_\lambda = 2.17^\circ\text{K}$.¹ The specific-heat curve exhibits a singularity at T_λ which appears to be logarithmic in nature. Another remarkable property of liquid ^4He is its zero viscosity (under certain conditions) or superfluidity for temperature below T_λ .

A phenomenological theory using the two-fluid concept was introduced in 1940 by Tisza² and in 1941 by Landau.³ This theory assumes that the liquid He consists of a superfluid component having zero entropy and viscosity and a normal component consisting of a gas of excitations in the liquid. Landau proposed that the existence of superfluidity in He II is related to the properties of the excitation spectrum of the liquid, which he suggested is a phonon spectrum for small momenta and a phononlike (roton) spectrum for larger momenta (see Fig. 1). In the roton region the spectrum is given by

$$\epsilon(p) = \Delta + (1/2\mu)(p - p_0)^2, \quad (1)$$

where Δ and p_0 stand for the roton energy and momentum, respectively.

Landau showed that at zero temperature, helium flowing through a capillary will experience viscosity if the flow velocity V of the liquid satisfies

$$V > [\epsilon(p)/p]_{\min} = V_c. \quad (2)$$

Elementary excitations will occur dissipating energy and momentum. With $V < V_c$ there is no viscosity at zero temperature.

At finite temperature, excitations of phonons and rotors must be taken into account. At temperatures above $T \sim 1^\circ\text{K}$ the superfluid will be depleted primarily by rotors, and the roton-roton scattering

frequency will provide the main contribution to the viscosity.⁴ According to Landau's theory, the superfluid will be depleted completely at a temperature $T_c \approx 2.6^\circ\text{K}$, at which the superfluid density vanishes, in comparison with the experimental value of $T_\lambda = 2.17^\circ\text{K}$.

Various attempts have been made to derive the above-mentioned properties like excitation spectrum,⁵⁻¹¹ viscosity,¹²⁻¹⁴ and λ point¹⁵ under special assumptions such as, for example, weakly interacting Bose gas, suitable *ad hoc* roton interaction Hamiltonians, or semiempirical considerations.

A purely microscopic theory based on hard-sphere interaction for a *dense* Bose gas has been put forward by two of the authors.^{16,17} The numerical calculation based on this model reproduces some features of the experimentally measured excitation spectrum, best fit in this case with a core diameter $a = 2.17 \text{ \AA}$. The hard-sphere approach is also very successful for exploring the interactions between excitations, such as roton-roton binding energy and roton collision frequency. Although this model seems to give rise to a coupling strength for the binding energy that is 5 times too large, a more accurate analysis reveals that this large coupling strength is compatible with other physical considerations and does not contradict the experiment.^{18,19} The coupling strength depends only very weakly on a , decreasing less than 5% for an increase of a from 2.1 to 2.5 \AA .

On the other hand, for a core diameter of $a = 2.17 \text{ \AA}$ we obtain the right upper bound for the collision frequency. The collision frequency is linearly proportional to the core diameter a .¹⁸

The aims of this paper are to calculate, based on the hard-sphere model, the transition temperature T_λ and to study the response of the system to

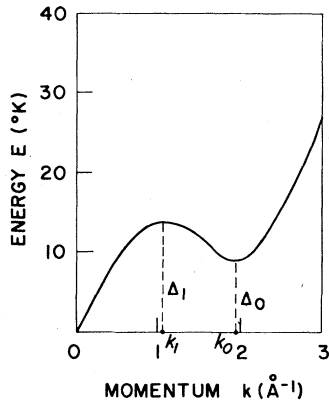


FIG. 1. Excitation energy spectrum as suggested by Landau.

external perturbations. At this point it seems to be adequate to present a brief description of this model (for further detail we refer the reader to Ref. 17 and references therein).

The analysis of a quantum-mechanical system of many particles with hard-sphere interaction is also important in other physical models such as, for example, liquid ^3He or nuclear matter. Such systems exhibit very strong and short-range repulsive forces which closely resemble those of hard spheres. Relatively weak attractive forces may be handled by ordinary perturbation theory. These are some of the reasons why the hard-sphere problem has received considerable attention for more than two decades. In order to deal with this problem, it is desirable to replace the N -particle hard-sphere boundary conditions by an equivalent Hamiltonian (the "pseudo-Hamiltonian") so that we can treat this problem in the usual second-quantization manner. However, most such treatments either are valid only for the S -wave component or they are not actually nonpenetrable. Although there exist two approaches which give the exact solution for the two-particle problem, they are hard to apply in the N -particle case. One of them requires a subsidiary condition,²⁰ and the other one leads to a non-Hermitian pseudopotential.²¹ These difficulties have perhaps limited all former pseudopotential treatments to only *dilute* systems. In this paper we utilize an exact method developed^{17,22,23} earlier. This method is equivalent to that of Ref. 21 for the two-particle problem, but the generalization to the N -particle problem leads to a Hermitian pseudopotential. In our treatment the hard-sphere boundary conditions are taken into account by introducing nonlocal field operators with generalized symmetric commutation relations. An iteration procedure devised earlier by one of us²³ allows for a successive expansion of these nonlocal hard-sphere field opera-

tors in terms of the usual Boson point-particle operators. The exact N -body hard-sphere pseudo-Hamiltonian then contains all order of interaction potentials up to genuine N -body potential interaction, which we reduce to an approximate form, maintaining the pair particle boundary conditions only. This approximate pseudo-Hamiltonian, which we call the *two-body potential Hamiltonian*, would be exact for two particles, but is only an approximation for more than two particles in the case in which the approximation becomes strictly accurate only for dilute systems.

Because of the analytical tractability of the two-body potential Hamiltonian, and the correct low-density limit, we adopt it as a *model Hamiltonian* for our problem. This reduced Hamiltonian is Hermitian, a property not retained in previous work employing similar approximate pseudo-Hamiltonians.^{5,21} The ultimate test has to be the comparison of the theoretical predictions one can make based on this model with the experimental data obtained. Before doing so, a further approximation has to be employed. We extract from the two-body potential Hamiltonian a set of terms which we call the *pair Hamiltonian*. It might be quite a delicate problem to decide to which step of approximation occurring discrepancies between experiment and theory belong to.

The range of applicability of the pair approximation is demonstrated by the following facts: Numerical calculations based on this pair Hamiltonian at liquid-helium density reproduced some features of the experimentally measured excitation spectrum. The pair Hamiltonian leads to a true upper bound of the ground-state energy of the entire two-body potential Hamiltonian. The pair Hamiltonian gives rise to a long-range ordering (off-diagonal long-range order) which is typical for the occurrence of a new thermodynamic phase. It also leads to a condensation in momentum space around the roton branch of the excitation spectrum, as must be expected in the case of a hard-sphere interaction.¹⁷ And, as shown in this paper, rather accurate statements can be made about phase-transition temperature and temperature dependence of the normal component of He II. Since the exact form of the excitation spectrum as a function of temperature is a prerequisite to achieving a quantitative evaluation of T_λ , it is important to point out¹⁸ that, based on our model, a simple Hartree-Fock calculation along the lines similar to the one presented by Ruvalds¹⁵ leads to a decrease of the roton energy with increasing temperature, in contrast to the results obtained by Parry and Ter Haar.²⁴

On the other hand, it was found¹⁷ that the pair correlation function was in poor agreement with

the expected pair correlation function obtained by Bijl-Dingle-Jastrow.²⁵

The question arises: Is it the pair approximation or the two-body potential approximation that introduces this deficiency? For two reasons we are convinced that the hard-core properties for the many-particle system are still sufficiently retained in the two-body potential approximation. Calculating the correction to the pair correlation function due to all nonpair terms gives qualitatively the right correction.¹⁷ Furthermore, computing the roton scattering frequency in liquid helium, which is dominated by short-range interaction between the rotons,¹³ one observes that the interaction Hamiltonian resulting from our model represents essentially our nonpair terms in the two-body potential Hamiltonian. The good results obtained earlier by two of us¹⁸ reinforces our conjecture that the hard-core properties of our system are still retained in the two-body potential approximation at realistic liquid-helium densities and that therefore higher-order body potential inter-

action terms can be neglected.

Bearing in mind these introductory remarks, we use this model Hamiltonian again in the pair-approximated form to calculate in Sec. II thermodynamical quantities for a bulk system. In Sec. III the transition temperature T_λ for a N -particle hard-sphere system is estimated. The results for T_λ , including attractive square-well interactions, are presented in Sec. IV.

In Sec. V we study the temperature dependence of the normal-fluid component and compare it with the experiment. In Sec. VI the response of our system to an external velocity flow is investigated and expressions for the resulting superfluid and normal-fluid current density are derived.

II. SELF-CONSISTENT-FIELD METHOD

We develop now the self-consistent-field equation for our model Hamiltonian. The hard-sphere Hamiltonian in the two-body potential approximation, using the convention $\hbar = 2m = 1$, is given by^{17,23}

$$\begin{aligned}
 H = & - \int d^3x \psi_0^\dagger(\vec{x}, t) \nabla_x^2 \psi_0(\vec{x}, t) + \lim_{\epsilon \rightarrow 0^+} - \int d^3x d^3x' [\nabla_x \psi_0^\dagger(\vec{x}, t) \psi_0^\dagger(\vec{x}', t)] [\nabla_x \psi_0(\vec{x}', t) \psi_0(\vec{x}, t)] \\
 & + \lim_{\epsilon \rightarrow 0^+} \int d^3x d^3x' \delta(r-a) \left(\frac{\partial}{\partial r} \psi_0^\dagger(\vec{x}, t) \psi_0^\dagger(\vec{x}', t) \right)_{r=a+\epsilon} \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \\
 & + \lim_{\epsilon \rightarrow 0^+} \int d^3x d^3x' \frac{1}{a} \delta(r-a) \psi_0^\dagger(\vec{x}, t) \psi_0^\dagger(\vec{x}', t) \left(\frac{\partial}{\partial r} r \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \right)_{r=a+\epsilon}, \quad (3)
 \end{aligned}$$

where $\psi_0(\vec{x}, t)$ and $\psi_0^\dagger(\vec{x}, t)$ are the Heisenberg-free field operators with time commutations given by

$$\begin{aligned}
 [\psi_0(\vec{x}, t), \psi_0^\dagger(\vec{x}', t)] &= \delta(\vec{x} - \vec{x}'), \quad (4) \\
 [\psi_0(\vec{x}, t), \psi_0(\vec{x}', t)] &= [\psi_0^\dagger(\vec{x}, t), \psi_0^\dagger(\vec{x}', t)] = 0.
 \end{aligned}$$

In order to study the thermodynamic properties of the above model Hamiltonian we make use of the self-consistent linearized equation of motion for the particle fields.²⁶ This method is essentially a generalization of the Hartree-Fock method by Bogoliubov²⁷ and includes the non-particle-conserving off-diagonal long-range-ordering terms

in the linearized equation of motion. Hence, such a method can only be applied to a grand canonical ensemble of particles. This condition is generally satisfied in the case of liquid He where the system is embedded in a thermobath and balanced by its own vapor pressure, which acts as a particles reservoir.

The equation of motion for $\psi_0(\vec{x}, t)$ is given by

$$i \frac{\partial}{\partial t} \psi_0(\vec{x}, t) = -[H, \psi_0(\vec{x}, t)], \quad (5)$$

which yields

$$\begin{aligned}
 i \frac{\partial}{\partial t} \psi_0(\vec{x}, t) &= -\nabla_x^2 \psi_0(\vec{x}, t) + \lim_{\epsilon \rightarrow 0^+} 2 \int d^3x' \delta(r-a) \left(\frac{\partial}{\partial r} \psi_0^\dagger(\vec{x}', t) \right)_{r=a+\epsilon} \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \\
 &+ \lim_{\epsilon \rightarrow 0^+} \frac{2}{a} \int d^3x' \delta(r-a) [\psi_0^\dagger(\vec{x}', t)] \left(\frac{\partial}{\partial r} r \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \right)_{r=a+\epsilon} \\
 &+ \int_{r < a} [\nabla' \psi_0^\dagger(\vec{x}', t)] [\nabla' \psi_0(\vec{x}', t) \psi_0(\vec{x}, t)] d^3x'. \quad (6)
 \end{aligned}$$

The above equation of motion is nonlinear, since it couples the solution of the single-particle field to those of two- and three-particle fields. In order to obtain an approximated solution, we linearize this equation of motion by replacing terms consisting of more than one particle operators by their thermo-

dynamic averages.

$$\begin{aligned}
i \frac{\partial}{\partial t} \psi_0(\vec{x}, t) = & -\nabla^2 \psi_0(\vec{x}, t) + \lim_{\epsilon \rightarrow 0^+} 2 \int d^3x' \delta(r-a) \left(\frac{\partial}{\partial r} \psi_0^\dagger(\vec{x}', t) \langle \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \right)_{r=a+\epsilon} \\
& + \lim_{\epsilon \rightarrow 0^+} \frac{2}{a} \int d^3x' \delta(r-a) \left(\frac{\partial}{\partial r} r \langle \psi_0^\dagger(\vec{x}', t) \psi_0(\vec{x}', t) \rangle \psi_0(\vec{x}, t) \right)_{r=a+\epsilon} \\
& + \lim_{\epsilon \rightarrow 0^+} \frac{2}{a} \int d^3x' \delta(r-a) \left(\frac{\partial}{\partial r} r \langle \psi_0^\dagger(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \psi_0(\vec{x}', t) \right)_{r=a+\epsilon} \\
& + \int d^3x' [\nabla' \psi_0^\dagger(\vec{x}', t)] [\nabla' \langle \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \rangle] + \int d^3x' \langle [\nabla' \psi_0^\dagger(\vec{x}', t)] \cdot [\nabla' \psi_0(\vec{x}', t)] \rangle \psi_0(\vec{x}, t) \\
& + \int d^3x' \nabla' \langle \psi_0^\dagger(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \cdot \nabla' \psi_0(\vec{x}', t), \tag{7}
\end{aligned}$$

where $\langle \psi(\vec{x}, t) \psi(\vec{x}, t) \rangle$ represents the grand canonical ensemble average. We note that in the above field equation only the second and fifth terms do not conserve particle numbers. These are the terms responsible for off-diagonal long-range ordering, while the rest of the terms are basically terms corresponding to the Hartree-Fock single-particle dressed energy. We have shown in detail in Ref. 17 that at liquid-helium density it is mathematically sensible to choose $a^3 n_0$ as an expansion parameter, since the density of zero-momentum condensate in actual liquid helium has been estimated by various authors^{28,29} to be no more than 8%. Based on this procedure and the numerical values of the single-particle condensate for this model Hamiltonian as calculated in Ref. 17 for realistic liquid-helium densities, which give a negligible amount of zero-particle condensate, we introduce a unitary transformation similar to the Bogoliubov transformation for the case of superconductivity, where in the thermodynamic limit we neglect the zero-momentum component.

$$\begin{aligned}
\psi_0(\vec{x}, t) = & \sum_{\vec{p}}' [U_{\vec{p}}^+(\vec{x}, t) e^{-iE_{\vec{p}}t} \alpha_{\vec{p}}^+ \\
& - V_{\vec{p}}^+(\vec{x}, t) e^{iE_{\vec{p}}t} \alpha_{-\vec{p}}^+], \\
\psi_0^\dagger(\vec{x}, t) = & \sum_{\vec{p}}' [U_{\vec{p}}^*(\vec{x}, t) e^{+iE_{\vec{p}}t} \alpha_{\vec{p}}^* \\
& - V_{-\vec{p}}^*(\vec{x}, t) e^{-iE_{\vec{p}}t} \alpha_{-\vec{p}}^*], \tag{8}
\end{aligned}$$

where $E_{\vec{p}}$ is the excitation spectrum associated with the quasiparticle operator $\alpha_{\vec{p}}^+$. The prime over the summation sign indicates that the zero-

momentum component is omitted.

To preserve the statistics, the transformation functions $U_{\vec{p}}^+$ and $V_{\vec{p}}^+$ must satisfy the following conditions

$$\sum_{\vec{p}}' [U_{\vec{p}}^+(\vec{x}, t) U_{\vec{p}}^*(\vec{x}', t) - V_{\vec{p}}^+(\vec{x}, t) V_{\vec{p}}^*(\vec{x}', t)] = \delta^3(\vec{x} - \vec{x}'), \tag{9}$$

$$\sum_{\vec{p}}' [U_{\vec{p}}^+(\vec{x}, t) V_{\vec{p}}^*(\vec{x}', t) - U_{\vec{p}}^*(\vec{x}', t) V_{\vec{p}}^+(\vec{x}, t)] = 0.$$

The inverse transformation is given by

$$\alpha_{\vec{p}}^+ = e^{iE_{\vec{p}}t} \int [U_{\vec{p}}^*(\vec{x}, t) \psi_0(\vec{x}, t) + V_{\vec{p}}^*(\vec{x}, t) \psi_0^*(\vec{x}, t)] d^3x, \tag{10}$$

such that

$$[\alpha_{\vec{p}}^+, \alpha_{\vec{p}'}^*] = \delta_{\vec{p}\vec{p}'}, \tag{11}$$

from which we can obtain two more conditions for the transformation functions

$$\begin{aligned}
\int [U_{\vec{p}}^*(\vec{x}, t) U_{\vec{k}}^+(\vec{x}, t) - V_{\vec{p}}^*(\vec{x}, t) V_{\vec{k}}^+(\vec{x}, t)] d^3x = \delta_{\vec{p}\vec{k}}, \\
\int [U_{\vec{p}}^*(\vec{x}, t) V_{\vec{k}}^*(\vec{x}, t) - V_{\vec{p}}^*(\vec{x}, t) U_{\vec{k}}^*(\vec{x}, t)] d^3x = 0. \tag{12}
\end{aligned}$$

The thermoaverage over the quasiparticle amplitudes are defined by

$$\langle \alpha_{\vec{p}}^* \alpha_{\vec{p}}^+ \rangle = 1 / (e^{E_{\vec{p}}/\hbar k_B T} - 1). \tag{13}$$

Substituting the transformation Eq. (8) into the equation of motion Eq. (7), we obtain

$$i \frac{\partial}{\partial t} [U_{\vec{p}}^+(\vec{x}, t) e^{-iE_{\vec{p}}t}] = (\hbar - \mu) U_{\vec{p}}^+(\vec{x}, t) e^{-iE_{\vec{p}}t} - S_1(\vec{x}) V_{\vec{p}}^+(\vec{x}, t) e^{-iE_{\vec{p}}t}, \tag{14}$$

where

$$\begin{aligned}
(h - \mu)\psi_0(\vec{x}, t) &= -(\nabla^2 + \mu)\psi_0(\vec{x}, t) \\
&+ \lim_{\epsilon \rightarrow 0^+} \frac{2}{a} \int d^3x' \delta(r - a) \left(\frac{\partial}{\partial r} r \langle \psi_0^\dagger(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \psi_0(\vec{x}, t) \right)_{r=a+\epsilon} \\
&+ \lim_{\epsilon \rightarrow 0^+} \frac{2}{a} \int d^3x' \delta(r - a) \left(\frac{\partial}{\partial r} r \langle \psi_0^\dagger(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \psi_0(\vec{x}', t) \right)_{r=a+\epsilon} \\
&+ \int_{r < a} [\nabla' \langle \psi_0^\dagger(\vec{x}', t) \psi_0(\vec{x}, t) \rangle] \nabla' \psi_0(\vec{x}', t) + \int_{r < a} \langle \nabla' \psi_0^\dagger(\vec{x}', t) \cdot \nabla' \psi_0(\vec{x}', t) \rangle \psi_0(\vec{x}, t) d^3x', \quad (15a)
\end{aligned}$$

$$\begin{aligned}
S_1(\vec{x})\psi_0^\dagger(\vec{x}, t) &= \lim_{\epsilon \rightarrow 0^+} 2 \int d^3x' \delta(r - a) \left(\frac{\partial}{\partial r} \langle \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \psi_0^\dagger(\vec{x}', t) \right)_{r=a+\epsilon} \\
&+ \int_{r < a} d^3x' \nabla' \langle \psi_0(\vec{x}', t) \psi_0(\vec{x}, t) \rangle \cdot \nabla' \psi_0^\dagger(\vec{x}', t). \quad (15b)
\end{aligned}$$

In Eq. (15a) we introduced the chemical potential μ to assure particle conservation. Similarly, we obtain an equation for $V_p^*(\vec{x}, t)$

$$\begin{aligned}
-i \frac{\partial}{\partial t} [V_p^*(\vec{x}, t) e^{iE_p t}] &= -(h - \mu) V_p^*(\vec{x}, t) e^{iE_p t} \\
&+ S_1(\vec{x}) U_p^*(\vec{x}, t) e^{iE_p t}. \quad (16)
\end{aligned}$$

Equations (14) and (16) can be combined in matrix form yielding

$$\left(E_p + i \frac{\partial}{\partial t} \right) \begin{pmatrix} U_p^*(\vec{x}, t) \\ V_p^*(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} h - \mu & -S_1(\vec{x}) \\ S_1^*(\vec{x}) & -h + \mu \end{pmatrix} \begin{pmatrix} U_p^*(\vec{x}, t) \\ V_p^*(\vec{x}, t) \end{pmatrix}. \quad (17)$$

This is the Bogoliubov equation for the superfluid phase. In the absence of an external perturbation, we may assume $U_p^*(\vec{x}, t)$ and $V_p^*(\vec{x}, t)$ independent of t . In this case, we shall express the transformation function in terms of a complete set of orthonormal wave functions

$$U_p^*(\vec{x}) = a_p^* \phi_p^*(\vec{x}), \quad V_p^*(\vec{x}) = b_p^* \phi_p^*(\vec{x}) \quad (18)$$

such that

$$a_p^2 - b_p^2 = 1. \quad (19)$$

In Eq. (18), the choice of the complete orthonormal set $\phi_p^*(\vec{x})$ should depend on the geometrical shape of the system, and should be chosen as the eigenfunctions of h . This method is therefore valid also for liquid films. With this choice of $\phi_p^*(\vec{x})$, we define

$$\int d^3x \phi_p^*(\vec{x}) (h - \mu) \phi_p^*(\vec{x}) \equiv S_2(\vec{p}) \quad (20)$$

and

$$\int d^3x \phi_p^*(\vec{x}) S_1(\vec{x}) \phi_p^*(\vec{x}) \equiv S_1(\vec{p}), \quad (21)$$

then Eq. (17) reduces to

$$\begin{aligned}
E_p a_p^* &= S_2(\vec{p}) a_p^* - S_1(\vec{p}) b_p^*, \\
E_p b_p^* &= S_1^*(\vec{p}) a_p^* - S_2(\vec{p}) b_p^*. \quad (22)
\end{aligned}$$

From Eq. (22) it follows that

$$E_p = [S_2^2(\vec{p}) - S_1^2(\vec{p})]^{1/2}, \quad (23)$$

which agrees with the solution one obtains at zero temperature.^{16,17} The transformation coefficients are

$$b_p^2 = \frac{1}{2} \left(\frac{S_2(\vec{p})}{E_p} - 1 \right), \quad a_p^2 = \frac{1}{2} \left(\frac{S_2(\vec{p})}{E_p} + 1 \right), \quad (24)$$

where b_p^2 is the single-particle distribution of the condensed state in momentum space. Notice that b_p^2 is proportional to p^{-1} for values of p approaching zero, i.e., $p^2 b_p^2$ vanishes as p approaches zero. This means that, although the single-particle distribution is divergent at low momenta, the phase-space factor is enough to curb the amount of Bose-Einstein condensation implied by the divergence of the single-particle distribution. On the other hand, $p^2 b_p^2$ has a maximum at the momentum value corresponding to that of the roton dip. The maximum number of particles are therefore concentrated near the momentum range of the roton dip.

In order to study the thermodynamic properties of our boson system, it is necessary for us to obtain $S_2(p)$ and $S_1(p)$ analytically. It is one of the advantages of the present approach that such analytic solutions can be found.

From Eq. (20), we have

$$(h - \mu) \phi_p^*(\vec{x}) = S_2(\vec{p}) \phi_p^*(\vec{x}), \quad (25)$$

where $\phi_p^*(\vec{x})$ is supposedly an eigenfunction of $(h - \mu)$. To solve for $\phi_p^*(\vec{x})$, therefore, we must solve Eq. (15a). With the help of Eqs. (8) and (25) we obtain the following differential equation for $\phi_p^*(\vec{x})$:

$$S_2(\vec{k})\phi_{\vec{k}}(\vec{x}) = -(\nabla^2 + \mu)\phi_{\vec{k}}(\vec{x}) + \lim_{\epsilon \rightarrow 0^+} \sum' \Gamma_2(\vec{p}) \frac{2}{a} \int d^3x' \delta(r-a) \left(\frac{\partial}{\partial r} r [\phi_{\vec{p}}^*(\vec{x}')\phi_{\vec{p}}(\vec{x}')\phi_{\vec{k}}(\vec{x}) + \phi_{\vec{p}}^*(\vec{x}')\phi_{\vec{k}}(\vec{x}')\phi_{\vec{p}}(\vec{x})] \right)_{r=a+\epsilon} \\ + \sum' \Gamma_2(\vec{p}) \int_{r < a} d^3x' \{ [\nabla' \phi_{\vec{p}}^*(\vec{x}') \cdot \nabla' \phi_{\vec{p}}(\vec{x}')] \phi_{\vec{k}}(\vec{x}) + [\nabla' \phi_{\vec{p}}^*(\vec{x}') \cdot \nabla' \phi_{\vec{k}}(\vec{x}')] \phi_{\vec{p}}(\vec{x}) \}, \quad (26)$$

where

$$\Gamma_2(\vec{p}) = (a_p^2 + b_p^2) \langle \alpha_p^* \alpha_p \rangle + b_p^2. \quad (27)$$

A first-order approximation to $\phi_{\vec{p}}(\vec{x})$ is the plane wave $V^{-1/2} \exp(i\vec{p} \cdot \vec{x})$. In fact, we expect the single-plane-wave approximation as the eigenfunction of $(\hbar - \mu)$ to be rather good, since $(\hbar - \mu)$ is essentially the Hartree-Fock Hamiltonian of a hard-sphere system, and its eigenfunctions should be close to plane waves with an effective-mass correction to the single-particle spectrum. We shall therefore treat $\phi_{\vec{p}}(\vec{x})$ as a single plane wave. From Eq. (26) we therefore obtain

$$S_2(\vec{k}) = k^2 \left[1 - \frac{4}{3} \pi a^3 \rho \right] - \mu \\ + \frac{4}{\pi a^2} \left(F j_0(ka) + B_0 \cos(ka) - \sum_{l=1}^{\infty} (4l+1) B_{2l} j_{2l}(ka) \right), \quad (28)$$

where the constants B_{2l} are defined with $x = ka$ as

$$B_{2l} = \int_0^{\infty} dx x^2 \Gamma_2 \left(\frac{x}{a} \right) j_{2l}(x) \quad (29)$$

and

$$F = \sum_{l=0}^{\infty} (4l+1) (-1)^l B_{2l}. \quad (30)$$

Since we have assumed zero amount of single-particle condensation, the chemical potential μ must be determined self-consistently by requiring

$$\frac{1}{V} \sum' \Gamma_2(\vec{p}) = \rho. \quad (31)$$

Under the plane-wave approximation we obtain from Eq. (15b)

$$S_1(\vec{k}) = -\frac{4}{\pi a^2} \left(D j_0(ka) - \sum_{l=1}^{\infty} (4l+1) \bar{E}_{2l} j_{2l}(ka) + E_0 \cos(ka) - E_0 j_0(ka) \right), \quad (32)$$

where E_{1l} and \bar{E}_{1l} are given by

$$E_{2l} = \lim_{\epsilon \rightarrow 0^+} \int_0^{\infty} dx x^2 \Gamma_1 \left(\frac{x(1-\epsilon)}{a} \right) j_{2l}(x), \quad (33)$$

$$\bar{E}_{2l} = \lim_{\epsilon \rightarrow 0^+} \int_0^{\infty} dx x^2 \Gamma_1 \left(\frac{x(1+\epsilon)}{a} \right) j_{2l}(x),$$

$$D = \sum_{l=0}^{\infty} (4l+1) (-1)^l E_{2l}, \quad (34)$$

and

$$\Gamma_1(\vec{p}) = a_p^2 b_p^2 (2 \langle \alpha_p^* \alpha_p \rangle + 1). \quad (35)$$

For more details about the derivation of Eqs. (28) and (32) we refer the reader to Ref. 17.

III. CALCULATION OF THE TRANSITION TEMPERATURE T_{λ}

It is physically clear that at zero temperature no quasiparticle is excited and therefore all the particles are in the pair condensate state, i.e., $\sum_{\vec{k}} b_{\vec{k}}^2 = N$ [see Eq. (24)]. But at temperatures just above the critical temperature we expect all the particles to be excited from that condensed state. This implies $b_{\vec{p}}^2(T_{\lambda})$ must be equal to zero such that the excitation energy becomes that of the Hartree-Fock solution. In other words, the off-diagonal pairing factor $S_1(\vec{p})$ vanishes. Under this condition we can evaluate T_{λ} and the chemical potential μ which is given by

$$\frac{1}{V} \sum' \frac{1}{e^{S_2(\vec{p})/k_B T_{\lambda}} - 1} = \rho. \quad (36)$$

From Eqs. (28) and (29), we see that the dominant contribution to B_{2l} has to come from the low-momenta region. At small momenta $S_2(\vec{p})$ is roughly given by

$$S_2(\vec{p}) \sim (1-\alpha)p^2 - \mu, \quad (37)$$

where

$$\alpha = \frac{4}{3} \pi a^3 \rho. \quad (38)$$

The constants B_{2l} near T_{λ} can be evaluated by successive iterations. The first iteration using Eq. (37) as an input gives

$$B_{2l} = \left(\frac{\pi}{2} \right)^{1/2} \times \sum_{n=1}^{\infty} \frac{\Gamma(l + \frac{1}{2})}{2^{2l+3/2} [n(1-\alpha)]^{l+1} \Gamma(2l + \frac{3}{2})} \times e^{n\mu/k_B T_{\lambda}} M \left(l + \frac{1}{2}, 2l + \frac{3}{2}, -\frac{a^2 k_B T_{\lambda}}{4(1-\alpha)n} \right). \quad (39)$$

Here $M(a, b, c)$ stands for the confluent hypergeometric function.²⁹ To estimate B_{2l} , we set $a^2 \mu \simeq -5$, $T_{\lambda} \simeq 2$ K. We find that B_{2l} is quite small for all values of l .

Substituting Eq. (24) in Eq. (35) and then in (33), we obtain an infinite set of coupled integral equa-

tions

$$E_{2l} = \lim_{\epsilon \rightarrow 0^+} \int_0^{\infty} dx x^2 \frac{S_1}{2E_p} \coth\left(\frac{E_p}{2k_B T}\right) j_{2l}(x(1-\epsilon)),$$

$$\bar{E}_{2l} = \lim_{\epsilon \rightarrow 0^+} \int_0^{\infty} dx x^2 \frac{S_1}{2E_p} \coth\left(\frac{E_p}{2k_B T}\right) j_{2l}(x(1+\epsilon)).$$
(40)

From our initial assumption that there is no single-particle zero-momentum condensation and the boundary condition that the thermoaverage $\langle \psi_0(\vec{x}, t) \psi_0(\vec{x}', t) \rangle$ vanishes at the hard-sphere diameter, i.e., for $|\vec{x} - \vec{x}'| = a$, we conclude¹⁷ that the constants \bar{E}_{2l} and E_{2l} are equal, and that E_0 is equal to zero. This final restriction on E_0 is es-

entially Lieb's subsidiary condition.²⁰ The limiting process involved in Eq. (40) actually arises only because of our choice of $\phi(\vec{x})$ as a plane wave and not imposing the Lieb's subsidiary condition. In obtaining the actual Hartree-Fock eigenfunctions of $(h - \mu)$, we must also put in the subsidiary condition. Therefore, although we should not really accept the thermoaverage $\langle \psi_0(\vec{x}, t) \psi_0(\vec{x}', t) \rangle$ to satisfy the temperature independent boundary condition, the limiting processes and integration can still be interchanged because no actual undamped oscillating term exists in the off-diagonal pairing factor $S_1(\vec{p})$.

Substituting Eq. (32) and (34) into Eq. (40), we obtain

$$E_{2l} = - \int_0^{\infty} dx x^2 \frac{1}{2E_p} \coth\left(\frac{E_p}{2k_B T}\right) j_{2l}(x) \frac{4}{\pi} \left(E_0 \cos x - \sum_{m=0}^{\infty} (4m+1) E_{2m} [j_{2m}(x) - (-1)^m j_0(x)] \right). \quad (41)$$

To solve for T_λ , we have to solve Eq. (41) together with the constraint given by Eq. (36).

Equation (41) can be written more formally as

$$E_{2l} = \sum_{l'=0}^{\infty} M_{ll'} E_{2l'}, \quad (42)$$

where

$$M_{ll'} = \frac{4}{\pi a^2} (4l'+1) \int_0^{\infty} dx x^2 \frac{1}{2E_p} \coth\left(\frac{E_p}{2k_B T_\lambda}\right) (j_{2l}(x) \{ [j_{2l'}(x) - (-1)^{l'} j_0(x)] - \delta_{l'l} \cos(x) \}). \quad (43)$$

A simple estimation is to ignore B_{2l} in $S_2(\vec{p})$, and approximate $S_2(\vec{p})$ by Eq. (37). Thus, $M_{ll'}$ has the following form:

$$M_{ll'} = \frac{2}{\pi} (4l'+1) (-1)^{l'} \left(C_{ll'} (-1)^{l'} - C_{l_0} - \sum_{m=0}^{\infty} (4m+1) (-1)^{m+l'} C_{lm} \delta_{l'_0} \right), \quad (44)$$

where

$$C_{lm} = (-1)^{m-l} \frac{\pi}{2} I_{2m+1/2} [-\mu a^2 / (1-\alpha)]^{1/2} K_{2l+1/2} [-\mu a^2 / (1-\alpha)]^{1/2}$$

$$+ \frac{\pi}{1-\alpha} \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} (-1)^s \left(\frac{-\mu a^2}{1-\alpha} \right)^{l+m+s+1/2} 2^{-2(l+m+s+1)}$$

$$\times \frac{\Gamma(l+m+s+\frac{3}{2}) \Gamma(2l+2m+2s+2) \Gamma(-l-m-s-\frac{1}{2}, n\mu/k_B T_\lambda)}{S! \Gamma(2m+s+\frac{3}{2}) \Gamma(2l+s+\frac{3}{2}) \Gamma(2l+2m+s+2)} \quad (45)$$

for $m \geq l-1$. For $m < l-1$ we interchange l and m in Eq. (45). I_m and K_m are hyperbolic Bessel functions.³⁰ $\Gamma(a, b)$ is the incomplete γ function.³¹

To solve now for T_λ we observe that for $T > T_\lambda$ we assumed $S_1(\vec{p}) \equiv 0$. Lowering the temperature through T_λ an on set of long-range ordering is expected, i.e., $S_1(\vec{p}) \neq 0$. This implies a nontrivial solution of Eq. (42), which means

$$\text{Det} |M - I| = 0. \quad (46)$$

With Eq. (46) and the constraint given in (36),

which after integrations takes the form

$$(1-\alpha)^{-3/2} \sum_{n=1}^{\infty} \left(\frac{k_B T_\lambda}{n} \right)^{3/2} e^{\mu n / k_B T_\lambda} = 8\pi^{3/2} \rho, \quad (47)$$

we are able to solve for T_λ . We find under these approximations $a^2 \mu(T_\lambda) = -6$ and $T_\lambda = 2.5^\circ\text{K}$. The size of the core is taken as 2.17 \AA and the density of the liquid as $(3.6)^{-3} \text{ \AA}^{-3}$. The choice of the size is governed by the best fit to the excitation spectrum,¹⁷ particularly the position and amplitude of the roton dip. By increasing a we decrease T_λ ,

which can be easily seen from Eq. (47) if we write it in a slightly different way

$$8\pi^{3/2}\rho(1-\alpha)^{3/2}\frac{(-\mu)^{-1/2}}{k_B T_\lambda} \cong \int_{-\mu/k_B T_\lambda}^{\infty} dx x^{-3/2} e^{-x} \\ = \Gamma\left(-\frac{1}{2}, -\frac{\mu}{k_B T_\lambda}\right). \quad (48)$$

Hence, by increasing a , we decrease μ and T_λ simultaneously. In fact, if a is increased to values larger than 2.2 \AA , T_λ drops to zero very fast and beyond values for which $\alpha > 1$, there is no solution for T_λ . A better approximation to $S_2(p)$ than that given by Eq. (37) can be performed numerically with Eqs. (28) and (41). Under this improved approximation, we can solve numerically for T_λ again. For $a = 2.17 \text{ \AA}$, we obtain $T_\lambda = 2.3^\circ \text{K}$ in slightly better agreement with experiment.

IV. SOLUTION OF THE TRANSITION TEMPERATURE WHEN ATTRACTIVE TAIL INTERACTION IS ADDED

If we consider a square-well attractive tail in the interaction potential, the model Hamiltonian (3) is modified by adding the following term:

$$V = \int d^3x d^3x' \psi^*(\vec{x}, t) \psi(\vec{x}', t) \\ \times \bar{V}(|\vec{x} - \vec{x}'|) \psi(\vec{x}', t) \psi(\vec{x}, t), \quad (49)$$

where

$$\bar{V}(|\vec{x} - \vec{x}'|) = \begin{cases} -\bar{W} & \text{if } a < |\vec{x} - \vec{x}'| < b, \\ 0 & \text{otherwise,} \end{cases} \quad (50)$$

where $\bar{W} = Wa^2/6.06$ is dimensionless and can be converted to $^\circ \text{K}$ by multiplying with $6.06/a^2$ and inserting the diameter a in \AA .

Going through the same procedure with the new model Hamiltonian as in Sec. III (detailed calculations are presented in Ref. 32), we obtain a new critical equation similar to Eq. (46),

$$\text{Det}|M' - I| = 0, \quad (51)$$

where M' now also contains the terms resulting from the decomposition of Eq. (50) into spherical Bessel functions. The determinant on the left side is a function of temperature T and the chemical potential μ only. Therefore, T_λ and μ can be found by satisfying Eqs. (51) and (36) simultaneously.

For the numerical calculation, we start the iteration with

$$S_2(\vec{k}) = (1 - \alpha)a^2 k^2 - \mu a^2. \quad (52)$$

We define

$$\eta \equiv \text{Det}|M' - I|. \quad (53)$$

Both η and ρ [see Eq. (36)] are calculated with different sets of (T_λ, μ) (Figs. 2 and 3, respective-

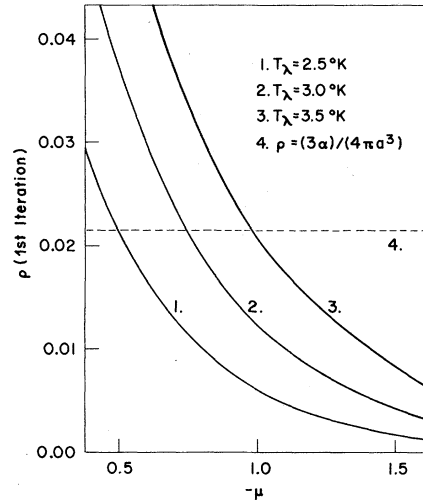


FIG. 2. Dependence of ρ on T_λ and μ .

ly). The result of the first iteration is

$$T_\lambda = 3.43^\circ \text{K}, \quad \mu = -0.949 \text{ \AA}^{-2}.$$

In the following iterations, the exact form of $S_2(k)$ is used employing the results from the previous iteration.

From Fig. 4, we can see the convergence of the result of each iteration. Therefore, for the case specified by $a = 2.073 \text{ \AA}$, $b = 3.5 \text{ \AA}$, and $W = 1.41^\circ \text{K}$, the transition temperature T_λ is determined to be 3.78°K .

For the case with $W = 0.846^\circ \text{K}$ and $W = 0.282^\circ \text{K}$, we obtain $T_\lambda = 3.83$ and 3.92°K , respectively. Within the limit of computational accuracy, the dependence of the transition temperature on the attractive well depth W is not conclusive from this self-consistent-field method approach.

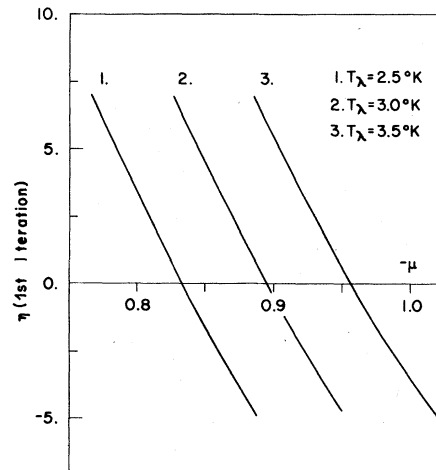


FIG. 3. Dependence of η on T_λ and μ .

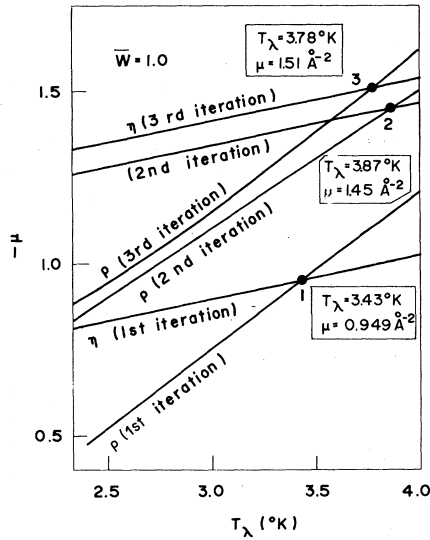


FIG. 4. All the points along the lines of ρ and η are the values of T_λ and μ which satisfy Eqs. (36) and (53), respectively. Therefore, the values at the intersection 1, 2, 3 satisfy Eqs. (36) and (53) simultaneously.

V. NORMAL FLUID DENSITY ρ_N AS A FUNCTION OF TEMPERATURE

According to Tisza's two-fluid model,² the amount of the normal fluid is zero at $T=0$ °K, and increases as temperature is increased. The transition temperature is, therefore, the temperature at which all the superfluid part is depleted, that is, $\rho_n/\rho=1$. The temperature dependence is found empirically by Andrikonikashvili.³³

From Eq. (13), we have

$$\rho_n = \sum_k' \langle \alpha_k^* \alpha_k \rangle = \sum_k' \frac{1}{e^{E_k/k_B T} - 1}. \quad (54)$$

Using the energy spectrum at $T=0$ as an approximation, ρ_n can be calculated numerically at temperature $T \leq T_\lambda$. The numerical calculation has been performed in Ref. 17 and is used here to calculate the curve $\rho_n T$ with a specified attractive well depth \bar{W} . The result is plotted in Fig. 5. Qualitatively, it is similar to the experimental curve, except the change of slope is too small to be noticeable in our case, where as experimentally it is very clear. Due to the lack of a sufficient roton dip in the energy spectrum E_k , T_λ found from $\rho_n/\rho=1$ is 5.2 °K, much larger than the experimental value 2.17 °K. This numerical discrepancy is also partially due to our choice of the core radius $a=2.073$ Å in the calculation of E_k , instead of 2.17 Å as discussed previously in Sec. III. From Eq. (48), we can see that the change of T_λ as a function of core size can be estimated roughly

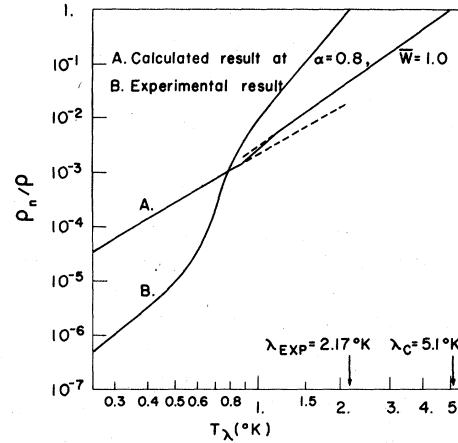


FIG. 5. Normalized normal fluid density ρ_n/ρ vs T_λ .

by

$$\frac{T_\lambda}{T_\lambda'} = \frac{(1-\alpha)^{3/2} (-\mu)^{-1/2} \Gamma(-\frac{1}{2}, -\mu'/k_B T_\lambda')}{(1-\alpha')^{3/2} (-\mu')^{-1/2} \Gamma(-\frac{1}{2}, -\mu/k_B T_\lambda)}. \quad (55)$$

Assuming

$$\alpha=0.8 \quad (a=2.073 \text{ \AA}), \quad T_\lambda=5.2 \text{ }^\circ\text{K},$$

we obtain

$$\alpha'=0.9 \quad (a=2.156 \text{ \AA}), \quad T_\lambda' \sim 2.3 \text{ }^\circ\text{K},$$

which brings the value for T_λ closer to the one obtained in Sec. III.

T_λ is also found for different values of \bar{W} , and plotted in Fig. 6. As \bar{W} is decreased to 0.282 °K, T_λ goes down to 4.0 °K. This behavior indicates that the stronger the attractive interactions between He atoms are, the harder it will be to excite the particles out of the pair condensed state.

VI. RESPONSE TO EXTERNAL PERTURBATION

In this section we shall study the responses of our system to an external perturbation. To be able to pursue this task we must first separate our eigenfunction of the unperturbed Hamiltonian \hat{H} given by Eq. (17) into a normal and a superfluid component. Before doing that, let us investigate

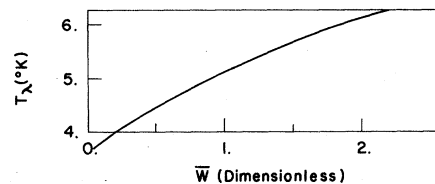


FIG. 6. T_λ vs attractive well depth \bar{W} .

Eq. (31). We have

$$\Omega^{-1} \sum_{\vec{p}}' [(a_p^2 + b_p^2) \langle \alpha_{\vec{p}}^* \alpha_{\vec{p}} \rangle + b_p^2] = \rho. \quad (56)$$

According to Landau³⁴ the normal component is defined by

$$\rho_N = \Omega^{-1} \sum_{\vec{p}}' \langle \alpha_{\vec{p}}^* \alpha_{\vec{p}} \rangle. \quad (57)$$

Hence we can obtain

$$\rho_S = \Omega^{-1} \sum_{\vec{p}}' (2b_p^2 \langle \alpha_{\vec{p}}^* \alpha_{\vec{p}} \rangle + b_p^2). \quad (58)$$

Let us denote the field operator as a four-component operator

$$\psi_0(\vec{x}) = \begin{pmatrix} \psi_0^N(\vec{x}) \\ \psi_0^S(\vec{x}) \end{pmatrix}, \quad (59)$$

where the superscripts N and S denote the normal and superfluid component, respectively, such that $\psi_0^N(\vec{x})$ and $\psi_0^S(\vec{x})$ are each a two-component operator given by

$$\psi_0^N(\vec{x}) = \sum_{\vec{p}}' C_{\vec{p}}^{\dagger} \begin{pmatrix} U_{\vec{p}}(\vec{x}) \alpha_{\vec{p}}^{\dagger} \\ -V_{-\vec{p}}^*(\vec{x}) \alpha_{-\vec{p}}^* \end{pmatrix}$$

and

$$\psi_0^S(\vec{x}) = \sum_{\vec{p}}' D_{\vec{p}}^{\dagger} \begin{pmatrix} U_{\vec{p}}(\vec{x}) \alpha_{\vec{p}}^{\dagger} \\ -V_{-\vec{p}}^*(\vec{x}) \alpha_{-\vec{p}}^* \end{pmatrix}.$$

With these definitions we obtain

$$\langle \psi_0^{\dagger}(\vec{x}) \psi_0(\vec{x}) \rangle = \langle \psi_0^{N\dagger}(\vec{x}) \psi_0^N(\vec{x}) + \psi_0^{S\dagger}(\vec{x}) \psi_0^S(\vec{x}) \rangle. \quad (61)$$

From Eqs. (60), (61), (57), and (58) we deduce

$$C_{\vec{p}}^2(T) = f_T(\vec{p}) / \Gamma_2(\vec{p}) \quad (62)$$

and

$$D_{\vec{p}}^2(T) = [2b_p^2 f_T(\vec{p}) + b_p^2] / \Gamma_2(\vec{p}), \quad (63)$$

where $f_T(\vec{p})$ is the Bose distribution

$$f_T(\vec{p}) = 1 / (e^{\beta \epsilon_p} / k_B T - 1). \quad (64)$$

Using the notation [see Eq. (17)]

$$\hat{H} \equiv \begin{pmatrix} h - \mu & -S_1(\vec{x}) \\ S_1^*(\vec{x}) & -h + \mu \end{pmatrix}, \quad (65)$$

we have for the unperturbed system

$$\hat{H} \begin{pmatrix} U_{\vec{p}}(\vec{x}) \\ V_{\vec{p}}(\vec{x}) \end{pmatrix} = E_{\vec{p}} \begin{pmatrix} U_{\vec{p}}(\vec{x}) \\ V_{\vec{p}}(\vec{x}) \end{pmatrix}. \quad (66)$$

We can construct a complete set of orthonormal functions

$$\Phi_{\vec{p}}^N(\vec{x}) = N_{\vec{p}}^{\dagger} \begin{pmatrix} U_{\vec{p}}(\vec{x}) \\ V_{\vec{p}}(\vec{x}) \\ 0 \\ 0 \end{pmatrix} \quad (67)$$

and

$$\Phi_{\vec{p}}^S(\vec{x}) = N_{\vec{p}}^{\dagger} \begin{pmatrix} 0 \\ 0 \\ U_{\vec{p}}(\vec{x}) \\ V_{\vec{p}}(\vec{x}) \end{pmatrix},$$

where with Eq. (18)

$$N_{\vec{p}}^{-2} = a_p^2 + b_p^2. \quad (68)$$

Both $\Phi_{\vec{p}}^N(\vec{x})$ and $\Phi_{\vec{p}}^S(\vec{x})$ are eigenfunctions of the 4×4 Hamiltonian

$$H = \begin{pmatrix} \hat{H} & 0 \\ 0 & \hat{H} \end{pmatrix}. \quad (69)$$

It is now easy to study the corrections to $\Phi_{\vec{p}}^N(\vec{x})$ and $\Phi_{\vec{p}}^S(\vec{x})$ if we have an external perturbation H_I

$$H_I = \begin{pmatrix} \hat{H}_I^N(t) & 0 \\ 0 & \hat{H}_I^S(t) \end{pmatrix}. \quad (70)$$

Because of the block diagonal form of (70) and since $\Phi_{\vec{p}}^N(\vec{x})$ form a complete orthonormal set, we can express the perturbed wave functions $\Phi_{\vec{p}}^N(\vec{x}, t)$ and $\Phi_{\vec{p}}^S(\vec{x}, t)$ in the following form

$$\Phi_{\vec{p}}^N(\vec{x}, t) = \Phi_{\vec{p}}^N(\vec{x}) + \sum_{\vec{k}}' A_{\vec{p}\vec{k}}^{\dagger}(t) \Phi_{\vec{k}}^N(\vec{x}) \quad (71)$$

and

$$\Phi_{\vec{p}}^S(\vec{x}, t) = \Phi_{\vec{p}}^S(\vec{x}) + \sum_{\vec{k}}' B_{\vec{p}\vec{k}}^{\dagger}(t) \Phi_{\vec{k}}^S(\vec{x}), \quad (72)$$

where

$$A_{\vec{p}\vec{k}}^{\dagger} = \frac{\langle \Phi_{\vec{k}}^N | \hat{H}_I^N | \Phi_{\vec{p}}^N \rangle}{E_{\vec{p}} - E_{\vec{k}}}, \quad (73)$$

$$B_{\vec{p}\vec{k}}^{\dagger} = \frac{\langle \Phi_{\vec{k}}^S | \hat{H}_I^S | \Phi_{\vec{p}}^S \rangle}{E_{\vec{p}} - E_{\vec{k}}}.$$

As for an application of the Bogoliubov equations derived for the normal and superfluid components, we shall consider the velocity flow of the two components, respectively, such that

$$\hat{H}_I^N = \begin{pmatrix} -i \vec{v}^N(\vec{x}) \cdot \vec{V} & 0 \\ 0 & -i \vec{v}^N(\vec{x}) \cdot \vec{V} \end{pmatrix} \quad (74)$$

and

$$\hat{H}_I^S = \begin{pmatrix} -i\vec{v}^S(\vec{x}) \cdot \vec{V} & 0 \\ 0 & -i\vec{v}^S(\vec{x}) \cdot \vec{V} \end{pmatrix}. \quad (75)$$

In constructing the perturbation H_I , we have assumed that there is a net translational kinetic

energy of the system given by

$$E_T = \int d^3x \langle \psi^\dagger(\vec{x}) \hat{H}_I \psi(\vec{x}) \rangle = \frac{1}{2} \Omega [\rho_S (v^S)^2 + \rho_n (v^N)^2]. \quad (76)$$

$A_{\vec{p}\vec{k}}$ and $B_{\vec{p}\vec{k}}$ can now be obtained explicitly, we have

$$A_{\vec{p}\vec{k}} = [\vec{p} \cdot \vec{v}^N (\vec{p} - \vec{k}) (a_{\vec{k}}^2 + b_{\vec{k}}^2)^{-1/2} (a_{\vec{p}}^2 + b_{\vec{p}}^2)^{-1/2} / (E_{\vec{p}} - E_{\vec{k}})] (a_{\vec{k}} a_{\vec{p}} + b_{\vec{k}} b_{\vec{p}}), \quad (77)$$

$$B_{\vec{p}\vec{k}} = [\vec{p} \cdot \vec{v}^S (\vec{p} - \vec{k}) (a_{\vec{k}}^2 + b_{\vec{k}}^2)^{-1/2} (a_{\vec{p}}^2 + b_{\vec{p}}^2)^{-1/2} / (E_{\vec{p}} - E_{\vec{k}})] (a_{\vec{k}} a_{\vec{p}} + b_{\vec{k}} b_{\vec{p}}).$$

The current operator of our system is given by

$$\vec{J}(\vec{x}) = \frac{1}{2} i \{ \psi_0^\dagger(\vec{x}) \vec{v} \psi_0(\vec{x}) - [\vec{v} \psi_0^\dagger(\vec{x})] \psi_0(\vec{x}) \}. \quad (78)$$

Applying Eqs. (59), (60), (67), (71), and (72) it is straightforward to obtain the Fourier transform of $\vec{J}(\vec{x})$

$$\vec{J}(\vec{q}) = \vec{J}^N(\vec{q}) + \vec{J}^S(\vec{q}), \quad (79)$$

where

$$\vec{J}^N(\vec{q}) = \sum_{\vec{q}} C_{\vec{q}}^2 a_{\vec{q}}^* a_{\vec{q}+\vec{I}} A_{\vec{q}, \vec{q}+\vec{I}} \frac{N_{\vec{q}+\vec{I}}}{N_{\vec{q}}} (\vec{q} + \vec{I}) \alpha_{\vec{q}}^* \alpha_{\vec{q}} + C_{\vec{q}}^2 b_{-\vec{q}} b_{-\vec{q}-\vec{I}} A_{\vec{q}, \vec{q}+\vec{I}} \frac{N_{\vec{q}+\vec{I}}}{N_{\vec{q}}} (\vec{q} + \vec{I}) \alpha_{-\vec{q}} \alpha_{-\vec{q}}^* + C_{\vec{q}}^2 A_{\vec{q}, \vec{q}-\vec{I}} \frac{N_{\vec{q}-\vec{I}}}{N_{\vec{q}}} a_{\vec{q}-\vec{I}}^* a_{\vec{q}} \alpha_{\vec{q}}^* \alpha_{\vec{q}} \vec{q} + C_{\vec{q}}^2 A_{\vec{q}, \vec{q}-\vec{I}} \frac{N_{\vec{q}-\vec{I}}}{N_{\vec{q}}} b_{-\vec{q}+\vec{I}} b_{-\vec{q}} \alpha_{-\vec{q}} \alpha_{-\vec{q}}^* \vec{q}, \quad (80)$$

and for $\vec{J}^S(\vec{q})$ we just need to exchange $C_{\vec{k}}^2$ with $D_{\vec{k}}^2$. Hence, for the persistent current components we take the thermal-averages of $\vec{J}^N(0)$ and $\vec{J}^S(0)$

$$\langle \vec{J}^N(0) \rangle = \lim_{\vec{q} \rightarrow 0} \sum_{\vec{k}} \vec{q} \left([\vec{v}^N(-\vec{q}) \cdot \vec{k}] k / \frac{\partial E_{\vec{k}}}{\partial k} (\vec{k} \cdot \vec{q}) \right) f_T(k) \quad (81)$$

and

$$\langle \vec{J}^S(0) \rangle = \lim_{\vec{q} \rightarrow 0} \sum_{\vec{k}} \vec{q} \left([\vec{v}^S(-\vec{q}) \cdot \vec{k}] k / \frac{\partial E_{\vec{k}}}{\partial k} (\vec{k} \cdot \vec{q}) \right) \times [2f_T(k) b_{\vec{k}}^2 + b_{\vec{k}}^2]. \quad (82)$$

Investigating Eqs. (81) and (82) we observe that the perpendicular components of $\vec{J}^N(0)$ to \vec{v}^N and $\vec{J}^S(0)$ to \vec{v}^S vanish and the parallel components are always in the same direction as the flow velocities \vec{v}^N and \vec{v}^S , respectively, and are given by

$$\langle \vec{J}_{\parallel}^N(0) \rangle = \vec{v}^N \sum_{\vec{k}} \left(k / \frac{\partial E_{\vec{k}}}{\partial k} \right) f_T(k), \quad (83)$$

$$\langle \vec{J}_{\parallel}^S(0) \rangle = \vec{v}^S \sum_{\vec{k}} \left(k / \frac{\partial E_{\vec{k}}}{\partial k} \right) \frac{1}{2} \left(\frac{S_2(k)}{E_k} - 1 \right) \times [1 + 2f_T(k)]. \quad (84)$$

For low temperature $T \rightarrow 0$, we can approximate $E_{\vec{k}}$ by

$$E_{\vec{k}} \cong Ck. \quad (85)$$

Thus,

$$\lim_{T \rightarrow 0} \langle \vec{J}_{\parallel}^N(0) \rangle = \vec{v}^N \frac{\pi^2 (kT)^4}{30C^5}. \quad (86)$$

Equation (86) is in agreement with (67.3) given in Ref. 34, except for a factor of $\frac{3}{4}$.³⁵ On the other hand,

$$\lim_{T \rightarrow 0} \langle \vec{J}_{\parallel}^S(0) \rangle = \vec{v}^S [I - (\mu/C^5)K(kT)^3], \quad (87)$$

where

$$I = \sum_{\vec{k}} \left(k / \frac{\partial E_{\vec{k}}}{\partial k} \right) \frac{1}{2} \left(\frac{S_2(k)}{E_k} - 1 \right),$$

$$K = \frac{1}{2\pi^2} \int_0^\infty dx x^2 f(x), \quad (88)$$

$$f(x) = 1/(e^x - 1).$$

We would like to point out here that our results given by Eqs. (83) and (84) are in fact quite instructive. At exactly zero temperature we see that the total current is given by the superfluid component alone and I is simply the average value of the momentum of all the particles over the sound velocity amplitude. But as temperature T surpasses T_λ , the superfluid current component vanishes as $S_1(k)$ vanishes and the current is purely normal in character.

In discussing the temperature dependence of the superfluid current component near zero temperature, we must bear in mind that the chemical potential μ itself is also temperature dependent

such that $\rho_S(T) + \rho_N(T)$ is kept temperature independent. Therefore the constant I we have in Eq. (88) is actually temperature dependent too.

VII. CONCLUSION

Based on the hard-sphere model we calculated the transition temperature T_λ , the normal fluid density $\rho_N(T)$ and the persistent super- and normal-current components \vec{J}^S and \vec{J}^N in the presence of an external perturbation. Using the onset of off-diagonal long-range order as the criterion for the occurrence of T_λ , our model, which contains the diameter a as a sole parameter, gives rise to $T_\lambda = 2.3^\circ\text{K}$ in remarkably good agreement with the experimental value of 2.17°K . The choice of diameter $a = 2.17 \text{ \AA}$ agrees with the best fit of the excitation spectrum resulting from this model. Such a choice of diameter is also compatible with inelastic neutron scattering data. For increasing diameter a , corresponding to increasing density of the system, T_λ decreases in qualitative agreement with the experiment.

As far as the numerical calculations of $\rho_N(T)$ are concerned, we found that our results only

exhibit some qualitative features of the experimental curve ρ_N/ρ vs T . Equation (54) reveals immediately that the absence of a sufficiently strong roton dip in the calculation of the excitation spectrum E_k , a deficiency that can be traced back to the pair approximation,¹⁷ is the origin for this lack of a more quantitative agreement. The calculation of the super and normal current components is based on strict particle conservation [Eqs. (56)–(58)]. If the current density were to be expressed as

$$\vec{J}(T) = \vec{v}^N \rho^N + \vec{v}^S \rho^S, \quad (89)$$

where ρ^S and ρ^N are defined by Eqs. (57) and (58), instead of 67.3 (Ref. 34), then it would follow that the temperature dependence of \vec{J}^N and \vec{J}^S are the same as ρ^N and ρ^S . However, this is not the case, since \vec{J}^N is not proportional to $\vec{v}^N \rho_N$. ρ^N is proportional to T^3 where as \vec{J}^N is proportional to T^4 . Therefore Eq. (89) is not consistent with the particle-conservation equation (56). We must therefore be very careful as to our interpretation of experimental results obtained from current measurements.

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