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Ferromagnetic resonance in metallic multilayers. I. Formal treatment*

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We present here a method for calculating the power absorbed in the ferromagnetic resonance of a sample consisting of metallic multilayers, with arbitrary directions of the applied static field. It is presented in as general a form as possible and then a number of simplifications are introduced to make the numerical calculation reasonable. These simplifications are shown to be justified and applicable for a large class of problems. In a following paper, the method is applied to a particular problem, and a semiquantitative model of the resonance behavior is presented.

I. INTRODUCTION

This is the first of two articles on the theoretical study of ferromagnetic resonance (FMR) in isotropic metallic magnetic multilayers, with applied static field ($\vec{H}_{app}$) at an arbitrary angle with respect to the film normal. Our work is a logical extension of Hoffmann's study,[1] in which he considered primarily the case in which $\vec{H}_{app}$ is normal to the film surface. There is reason to believe that such investigations of the effect of interfacial coupling upon resonance absorption spectra will aid in clarifying the troubling problem of the magnetic behavior of the surfaces. The possibility of analyzing resonance spectra for arbitrary field directions, and under arbitrary conditions of coupling strength, should contribute substantially to the understanding of this problem.

For the analysis of the properties of the individual layers, we use the approximate method which we have reported earlier.[2] This permits us to simplify both the mathematical description and the numerical solution of the problem. The continuity equations for the rf magnetization at the interfaces between layers are those derived by Hoffmann,[1] modified to make them consistent with the approximate calculation. Finally, an expression is derived from Poynting's theorem for the power absorbed in terms of surface impedances.

In the following paper[3] we solve the problem of FMR for a sample consisting of three layers with similar properties. This is the simplest problem to which the method can be applied. Based on the results of this calculation, we are able to describe a method for deducing the coupling strength from absorption spectra. We show that it is possible, from a relatively simple semiquantitative model, to interpret and understand the numerical predictions obtained from the calculations.

The method described here is applicable to a large number of different problems. It may be used for predicting the behavior of metallic multilayers with ferromagnetic[1] or with antiferromagnetic[4] interfacial coupling. It may be used for investigating resonance in metallic films with ferrimagnetic surface layers,[5] or with inhomogeneous ferromagnetic surface layers.[6-8] Finally, one can develop a simplified version, neglecting conductivity, which may be applied to the ferrimagnetic resonance in garnet films, with surface layers different from the bulk.[9-11] In all of these problems, application of the method described here would permit a more exact and more complete calculation (of linewidth, for example) than has been possible in the past. The semiquantitative model of the following paper[3] should yield insights into all of these problems. At present, we are using the method to calculate transmission through thin bilayers, for which interesting exchange effects might be expected.[12]

II. THEORETICAL SITUATION

Consider a ferromagnetic thin-film sample consisting of N metallic layers, infinite in the x-y plane. Each of the layers consists of isotropic material. The spatial configuration is shown in Fig. 1.

The solution of the FMR problem for such a sample consists first of the simultaneous solution of the Landau-Lifschitz (LL) equation and Maxwell's equations to obtain the dispersion relations. Then the relative amplitudes of the different components are obtained from the continuity equations for the electromagnetic (EM) fields and the rf magnetization at the outer surfaces. There are two difficulties in this procedure: First, the LL equation and Maxwell equations form a set of differential equations whose coefficients are physical variables (dc magnetization, exchange, conductivity, etc.), and are difficult, if not impossible to solve analytically. It is therefore necessary to solve the problem numerically, point by point. Second, the values of the physical variables (mentioned above) at the in-

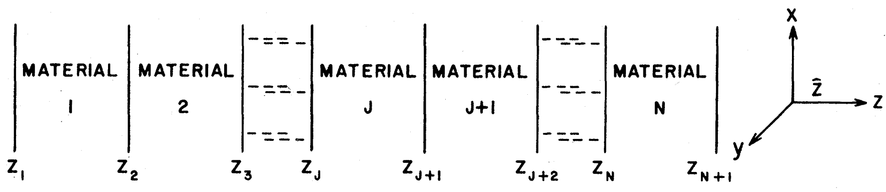

FIG. 1. Spatial configuration of the sample.

terfaces between layers, and the behavior of the solutions as a function of the z coordinate depend upon the interdiffusion between layers; this is difficult to determine once the films have been produced. However, Hoffmann[1] has shown that when this interdiffusion is small it is possible to divide the problem into as many subproblems as there are layers. In fact, when this condition is fulfilled, the coefficients appearing in the LL and Maxwell equations for the jth layer may be treated, to first order as constant, even near the interfaces of this layer ($z = z_j$ and $z = z_{j+1}$) as shown in Fig. 1. In this case, the problem reduces to the solution of the equations of motion for each of the N layers, followed by solution of the continuity equations at each of the $N-1$ interfaces and the exterior surfaces to fix the amplitudes of the different waves.

Even in this situation, there remains one difficulty: despite weak interdiffusion, the direction of the static magnetization varies near the interface, and the Hoffmann model fails if this variation is large. We shall return to this question, in order to quantify the limits of validity, in Sec. IV.

The special case, of small diffusion and of small variation of magnetization direction within a layer, is still of considerable interest. Under these conditions one can directly apply the methods previously developed for FMR in monolayer films, either exact[13] or approximate,[2] to each layer. Further, from the absorption spectrum, it is possible to obtain, by a fairly simple method, an understanding of the behavior of the rf magnetization near the interface. This will be demonstrated in the following paper.[3] Thus, this article is devoted to the problem in the case of weak interdiffusion.

III. DISPERSION RELATIONS

A. Exact method

Consider the jth layer of the sample of Fig. 1. The Maxwell equations for this layer are

$$\vec{\nabla} \times \vec{e}_j = -\frac{1}{c} \frac{\partial(\vec{H}_j + 4\pi\vec{M}_j)}{\partial t} , \tag{1}$$

$$\vec{\nabla} \times \vec{H}_j = (4\pi\sigma_j/c)\vec{e}_j , \tag{2}$$

$$\vec{\nabla} \cdot (\vec{H}_j + 4\pi\vec{M}_j) = 0 , \tag{3}$$

$$\vec{\nabla} \cdot \vec{e}_j = 0 . \tag{4}$$

$\vec{H}_j$ and $\vec{M}_j$ represent the internal field and the magnetization in this layer. The electric field is written as $\vec{e}_j$ to indicate that the static component is assumed to be zero; σ_j is the conductivity of the layer. In these equations we have neglected the dielectric, displacement terms in comparison with the conduction terms (low-frequency approximation). The LL equation may be written

$$\frac{1}{\gamma_j} \frac{\partial \vec{M}_j}{\partial t} = \vec{M}_j \times \left[\vec{H}_j + \frac{2A_j}{M_j^2} \nabla^2 \vec{M}_j - \frac{\lambda_j}{\gamma_j M_j^2} \vec{M}_j \times \left(\vec{H}_j + \frac{2A_j}{M_j^2} \nabla^2 \vec{M}_j \right) \right] . \tag{5}$$

Equation (5) is written with the convention $\gamma_j < 0$. A_j is the exchange constant and λ_j the damping constant of the jth layer. The spatial orientation of the fields and of the magnetization in a layer, far from any interface, is shown in Fig. 2. It is easy to show from Eq. (5) that the static components of $\vec{H}_j$ the internal field and of $\vec{M}_j$, which we call $\vec{H}_{0j}$, and $\vec{M}_{0j}$, are parallel. We take them as lying in the x-z plane. By writing the demagnetizing field in terms of the static magnetization we see that the applied and internal fields are related by

$$\vec{H}_{0j} = \vec{H}_{app} - (4\pi M_{0j} \cos\theta_{0j})\hat{z} , \tag{6}$$

and that

$$2\pi M_{0j} \sin 2\theta_{0j} + H_{app} \sin(\alpha - \theta_{0j}) = 0 , \tag{7}$$

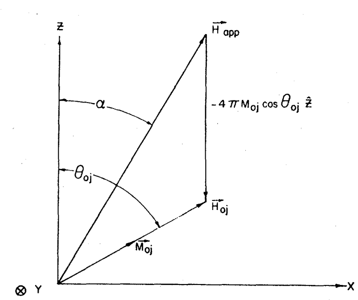

FIG. 2. Orientation of the static fields. $\hat{z}$ is the unit vector in the direction normal to the sample plane. α is the angle between the applied field $\vec{H}_{app}$ and the normal.

where the angles, θ_{0j} and α, are as shown in Fig. 2.

For the infinite plate, propagation may be assumed to be confined to the z direction, so that

$$\vec{M}_j = \vec{M}_{0j} + \vec{m}_j \ , \quad (8)$$

$$\vec{H}_j = \vec{H}_{0j} + \vec{h}_j \ , \quad (9)$$

where

$$\vec{m}_j = \vec{m}_{0j}\, e^{i(\omega t - kz)} \ ,$$

$$\vec{h}_j = \vec{h}_{0j}\, e^{i(\omega t - kz)} \ .$$

We assume, in what follows, that $\vec{m}_j$ and $\vec{h}_j$ are small compared to $\vec{M}_{0j}$ and $\vec{H}_{0j}$ (small-signal approximation). It follows from this[2] that $\vec{m}_j$ is normal to $\vec{M}_{0j}$, so that only two of its three components are independent. Substituting the values from Eqs. (6)–(9) into Eqs. (1)–(5), and neglecting second-order terms,[2] one obtains

$$m_j^n = -Q_j h_j^n, \quad n = x, y \ , \quad (10a)$$

$$m_j^z = -h_j^z/4\pi \ , \quad (10b)$$

where

$$Q_j = (1/4\pi)[1 - \tfrac{1}{2} i \delta_j^2 k_j^2] \quad (11)$$

and

$$\delta_j^2 = c^2/2\pi\sigma_j\omega \ .$$

Eliminating $\vec{h}$ by means of Eqs. (10), Eq. (5) reduces to

$$\underline{G}^j \begin{pmatrix} m_j^\theta \\ m_j^\phi \end{pmatrix} = 0 \ . \quad (12)$$

In Eq. (12)

$$\underline{G}^j = \begin{pmatrix} c_j \sin^2\theta_{0j} + b_j \cos^2\theta_{0j} & a_j + i\Omega_j \\ -(d_j \sin^2\theta_{0j} + a_j \cos^2\theta_{0j} + i\Omega_j) & b_j \end{pmatrix} \ . \quad (13)$$

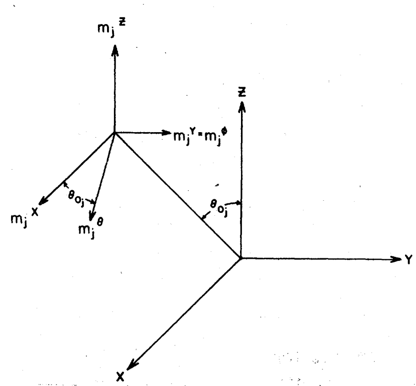

FIG. 3. Cartesian and polar components of the rf magnetization.

The terms a_j, b_j, c_j, d_j, and Ω_j in Eq. (13) are defined in Appendix A. The terms m_j^θ and m_j^ϕ are the polar components shown in Fig. 3.

The system of homogeneous equations, Eq. (12), has a solution only when the determinant of $\underline{G}^j$ is zero. From this condition, one obtains a dispersion relation which is quartic in k^2, except at $\alpha = 0$, where it decouples into two biquadratic equations, and at $\alpha = 90°$, where it separates into a cubic and a linear equation in k^2.

The four pairs of k values for each frequency, obtained from the dispersion relation, may be reintroduced into Eq. (12) to give the relationship between m_j^θ and m_j^ϕ. Then Eqs. (2) and (10) yield the electric and magnetic fields associated with each solution. This constitutes the exact method.[13]

B. Approximate method

In the absence of conductivity and of damping, G^j reduces to[2]

$$\underline{G}_0^j = \begin{pmatrix} H_{0j} + 2A_j k_j^2/M_{0j} + 4\pi M_{0j} \sin^2\theta_{0j} & i\Omega_j \\ -i\Omega_j & H_{0j} + 2A_j k_j^2/M_{0j} \end{pmatrix} . \quad (14)$$

The matrix $\underline{G}_0^j$ can be diagonalized by the same unitary transformation $\underline{U}^j$ which permitted Kobayashi *et al.*[14] to obtain the uniform precession mode for the magnetoeleastic insulator. The elements of $\underline{U}^j$ are

$$U_{11}^j = U_{22}^j = \Omega_j/(2\Omega_{0j}^2 + 4\pi\Omega_{0j} M_{0j} \sin^2\theta_{0j})^{1/2} \ , \quad (15a)$$

$$U_{12}^j = U_{21}^j = i\Omega_j/(2\Omega_{0j}^2 - 4\pi\Omega_{0j} M_{0j} \sin^2\theta_{0j})^{1/2} \ . \quad (15b)$$

In Eqs. (15)

$$\Omega_{0j}^2 = \Omega_j^2 + (2\pi M_{0j} \sin^2\theta_{0j})^2 \ .$$

If we apply the transformation $\underline{U}^j$ to $\underline{G}^j$, the off-diagonal elements of the matrix,

$$\underline{S}^j = \underline{U}^{j\dagger} \underline{G}^j \underline{U}^j$$

are generally small compared with the diagonal elements.[2] Thus, in taking the determinant of $\underline{S}^j$, we may neglect the off-diagonal elements. Then each of the diagonal elements yields a dispersion relation which is biquadratic in k:

$$\pi_{1j} k_j^4 + \pi_{2j} k_j^2 + \pi_{3j} = 0 \ , \quad (16)$$

$$\pi_{4j} k_j^4 + \pi_{5j} k_j^2 + \pi_{6j} = 0 \ . \quad (17)$$

The terms π_{ij} of Eqs. (16) and (17) are given in Appendix B. Equation (16) is the dispersion relation for the branches with the resonant sense of polarization, Eq. (17) for the nonresonant sense. We refer to these as μ_j^+ and μ_j^-, respectively. The relation between $\vec{m}$ and $\vec{\mu}$ is given by

$$\begin{pmatrix} m_j^\theta \\ m_j^\phi \end{pmatrix} = \underline{U}^j \begin{pmatrix} \mu_j^+ \\ \mu_j^- \end{pmatrix} . \qquad (18)$$

Each of the dispersion relations, Eqs. (16) and (17) gives rise to four roots, in two pairs, for positive and negative propagation directions. We shall refer to the roots of Eq. (16) for propagation in the positive z direction as k_{1j}^+ and k_{2j}^+, and to those of Eq. (17) as k_{1j}^- and k_{2j}^-. The rf magnetization in layer j (omitting the time dependence here, and in what follows) may then be described by

$$\mu_j^\pm = \sum_{n=1}^{2} \mu_{nj}^\pm$$

$$= \sum_{n=1}^{2} [C_{nj}^\pm \cos(k_{nj}^\pm z) + C_{n+2,j}^\pm \sin(k_{nj}^\pm z)] . \qquad (19)$$

The eight coefficients $C_{pj}^\pm (1 \leq p \leq 4)$, are to be determined from the simultaneous solution of the EM continuity equations, the continuity equations for the different magnetization polarizations, and the boundary conditions on the magnetization at the external surfaces. We may now combine Eqs. (18) and (19), and make use of the relation between polar components of $\vec{m}$ evident from Fig. 3, to solve Eqs. (10) for the magnetic field in terms of the resonant and nonresonant polarized components of the magnetization:

$$h_j^x = -\cos\theta_{0j} \sum_{n=1}^{2} \left(\frac{U_{11}^j \mu_{nj}^+}{Q_{nj}^+} + \frac{U_{12}^j \mu_{nj}^-}{Q_{nj}^-} \right) , \qquad (20)$$

$$h_j^y = -\sum_{n=1}^{2} \left(\frac{U_{21}^j \mu_{nj}^+}{Q_{nj}^+} + \frac{U_{22}^j \mu_{nj}^-}{Q_{nj}^-} \right) , \qquad (21)$$

$$h_j^z = 4\pi \sin\theta_{0j} (U_{11}^j \mu_j^+ + U_{12}^j \mu_j^-) , \qquad (22)$$

where

$$Q_{nj}^\pm = (1/4\pi)[1 - \tfrac{1}{2} i \delta_j^2 (k_{nj}^\pm)^2] .$$

Then, with Eq. (2), we may solve for the electric field $\vec{e}_j$:

$$e_j^x = \frac{-c}{4\pi\sigma_j} \frac{\partial h_j^y}{\partial z} , \qquad (23)$$

$$e_j^y = \frac{c}{4\pi\sigma_j} \frac{\partial h_j^x}{\partial z} , \qquad (24)$$

$$e_j^z = 0 . \qquad (25)$$

Thus, the problem is solved for the bulk of a given layer. However, before going further, it is worthwhile to point out the limitation of the approximate technique. At antiresonance (FMAR), the off-diagonal matrix elements of $\underline{S}^j$ are no longer negligible compared to the diagonal elements,[2] and one must resort to the exact method for accurate solutions.[15] Outside of this region, we have found that the approximate method yields excellent agreement with the exact method, with a factor of 7 or 8 less numerical computation.[2] Given this fact, and the difficulties in numerical calculation associated with the problem considered here, the remainder of this and the following paper assumes use of the approximate method.

IV. CONTINUITY EQUATIONS FOR THE MAGNETIZATION

For the case of weak interdiffusion, that is, the case when the magnetization $|M|$ remains essentially constant almost to the interface, Hoffmann has obtained a set of continuity equations for $\vec{M}$ at each interface from enthalpy minimization considerations:

$$\frac{A_j}{M_j^2} \vec{M}_j \times \frac{\partial \vec{M}_j}{\partial z}\bigg|_{z_{j+1}} - \frac{A_{j,j+1}}{M_j M_{j+1}} \vec{M}_j \times \vec{M}_{j+1} + \beta_j (\vec{M}_j \cdot \hat{z})(\vec{M}_j \times \hat{z}) = 0 , \qquad (26)$$

$$\frac{-A_{j+1}}{M_{j+1}^2} \vec{M}_{j+1} \times \frac{\partial \vec{M}_{j+1}}{\partial z}\bigg|_{z_{j+1}} - \frac{A_{j,j+1}}{M_j M_{j+1}} \vec{M}_{j+1} \times \vec{M}_j + \beta_{j+1} (\vec{M}_{j+1} \cdot \hat{z})(\vec{M}_{j+1} \times \hat{z}) = 0, \quad 1 \leq j \leq N-1 . \qquad (27)$$

The first term of each of these equations represents the torque due to the gradient in the direction of $\vec{M}$ near the interface, so that the exchange constant is that of the material concerned. The second torque is due to the exchange coupling between layers, and contains the coupling constant $A_{j,j+1}$. The last term represents the torque due to the interfacial (surface) anisotropy, and contains the term β, defined as

$$\beta = -K_s / M^2 .$$

Since we wish to employ the approximate calculation for the rf magnetization, we need to transform Eqs. (26) and (27) into the coordinate system of the resonant and nonresonant polarizations. This poses a problem: because of the interfacial anisotropy and the coupling between layers, the magnetization direction varies near the interface even in the case of small interdiffusion. There are two serious consequences of this fact: First, a completely rigorous solution of the LL equation should include the exchange interaction arising from the variation of magnetization direction. This makes the characteristic wave vectors functions of position within each layer. Second, it is not obvious *a priori* that the approximate calculation can be applied in the surface layer, that is, that the rf magnetization can be described simply as the sum of a resonant and a nonresonant wave. It can be proved,[16] however, that the exchange modifies only the diagonal terms in $G^j(z)$ [Eq. (13)]; it introduces no additional off-diagonal terms. In fact, for each point in space, the wave vectors are given by the approximate calculation if we use the transformation of Eq. (15) with $\theta_j(z)$ replacing θ_{0j}. This of course implies that, for the region of the interface, the coefficients $C^{\pm}_{nj}$ in Eq. (19) must be replaced by $C^{\pm}_{pj}(z)$ when the k's are replaced by $k^{\pm}_{pj}(z)$ $(1 \le p \le 4)$. In order to make this distinction clear, we designate the waves in the interface region by $\xi^{\pm}_j$, and those in the bulk of a layer, where $\theta=\theta_{0j}$, by $\mu^{\pm}_j$. Solution of the continuity problem thus requires transformation of Eqs. (26) and (27) into equations for $\xi^{\pm}_j$. This is done after transformation into polar coordinates, with Eqs. (18), in which θ_{0j} is replaced by $\theta_j(z)$. Neglecting second-order terms, and making use of $\vec{m}_j \cdot \vec{M}_{0j}=0$, we find

$$[\underline{\underline{A}}\,\underline{U}^j(z_{j+1})+\underline{\underline{B}}\,\partial\underline{U}^j(z_{j+1})]\begin{pmatrix}\xi^+_j\\ \xi^-_j\end{pmatrix}+\underline{\underline{B}}\,\underline{U}^j(z_{j+1})\begin{pmatrix}\partial\xi^+_j\\ \partial\xi^-_j\end{pmatrix}+\underline{\underline{C}}\,\underline{U}^{j+1}(z_{j+1})\begin{pmatrix}\xi^+_{j+1}\\ \xi^-_{j+1}\end{pmatrix}=0\ , \tag{28}$$

$$[\underline{\underline{D}}\,\underline{U}^{j+1}(z_{j+1})+\underline{\underline{E}}\,\partial\underline{U}^{j+1}(z_{j+1})]\begin{pmatrix}\xi^+_{j+1}\\ \xi^-_{j+1}\end{pmatrix}+\underline{\underline{E}}\,\underline{U}^{j+1}(z_{j+1})\begin{pmatrix}\partial\xi^+_{j+1}\\ \partial\xi^-_{j+1}\end{pmatrix}+\underline{\underline{F}}\,\underline{U}^j(z_{j+1})\begin{pmatrix}\xi^+_j\\ \xi^-_j\end{pmatrix}=0\ , \tag{29}$$

where $\partial\equiv\partial/\partial z$. $\underline{A},\ldots,\underline{F}$ are diagonal matrices whose elements are given in Appendix C. The elements of the derivatives of $\underline{U}(z)$ are given in Appendix D.

A complete rigorous solution of the problem requires solving the dispersion relations point-by-point, followed by the point-by-point solution for $\xi^{\pm}_j$. That is, one finds amplitude functions $C^{\pm}_{pj}(z)$ $(1\le p\le 4)$ such that the waves $\xi^{\pm}_j$ simultaneously satisfy Eqs. (28) and (29) at all the interfaces. This can be done, in principle, but is exceedingly difficult. The problem can, however, be greatly simplified if we make two assumptions (whose validity we shall discuss later). First, we assume that the thickness D_j, over which $\theta_j(z)$ is different from θ_{0j}, obeys the relation

$$D_j \ll d_j\ ,$$

where d_j is the total thickness of the jth layer. Second, θ_{ij}, the value of $\theta_j(z)$ at the interface, obeys

$$|\theta_{ij}-\theta_{0j}|\ \text{ small}.$$

When these two conditions are fulfilled, the magnetization direction is constant almost everywhere and assumes values near the bulk value at the interfaces. In this case, we can take $\xi^{\pm}_j$ as equal to $\mu^{\pm}_j$ to first order, and Eqs. (28) and (29) become

$$[\underline{\underline{A}}\,\underline{U}^j(z_{j+1})+\underline{\underline{B}}\,\partial\underline{U}^j(z_{j+1})]\begin{pmatrix}\mu^+_j\\ \mu^-_j\end{pmatrix}+\underline{\underline{B}}\,\underline{U}^j(z_{j+1})\begin{pmatrix}\partial\mu^+_j\\ \partial\mu^-_j\end{pmatrix}+\underline{\underline{C}}\,\underline{U}^{j+1}(z_{j+1})\begin{pmatrix}\mu^+_{j+1}\\ \mu^-_{j+1}\end{pmatrix}=0\ , \tag{30}$$

$$[\underline{\underline{D}}\,\underline{U}^{j+1}(z_{j+1})+\underline{\underline{E}}\,\partial\underline{U}^{j+1}(z_{j+1})]\begin{pmatrix}\mu^+_{j+1}\\ \mu^-_{j+1}\end{pmatrix}+\underline{\underline{E}}\,\underline{U}^{j+1}(z_{j+1})\begin{pmatrix}\partial\mu^+_{j+1}\\ \partial\mu^-_{j+1}\end{pmatrix}+\underline{\underline{F}}\,\underline{U}^j(z_{j+1})\begin{pmatrix}\mu^+_j\\ \mu^-_j\end{pmatrix}=0 \tag{31}$$

at the interfaces. We note that while $\xi^{\pm}_j$ have been replaced by $\mu^{\pm}_j$ in Eqs. (30) and (31) the values of $\theta_j(z)$ and of $\theta_{j+1}(z)$ in the matrices $\underline{A}$-$\underline{F}$ are those at the interface. These are obtained from the solution of the equation for the equilibrium magnetization direction

$$-2A_p\,\frac{\partial^2\theta_p(z)}{\partial z^2}=2\pi M^2_{0p}\sin 2\theta_p(z)+M_{0p}H_{\text{app}}\sin[\alpha-\theta_p(z)]\ , \tag{32}$$

where $p=j,j+1$. Equation (32) is subject to two boundary conditions: $\theta_j(z)$ is equal to θ_{0j} far from the interface, and simultaneously satisfies Eqs. (26) and (27) at the interface.

Let us now consider the validity of the two assumptions made above on the size of D and of θ. For simplicity, we ignore the interfacial anisotropy term, which we assume will be less important than the exchange coupling term.[1] When two adjacent layers have the same magnetization, the gradient at the interface is zero, and the two hypotheses are automatically satisfied. When the applied field is parallel or perpendicular to the film normal, the magnetization in each is parallel to the field, and they are again satisfied automatically. Finally, when adjacent layers have similar values for the magnetization $M_{0j}\sim M_{0,j+1}$, or when the applied field is strong (for arbitrary θ), then the bulk magnetization directions of adjacent layers are close to each other [see Eq. (7)]. We then expect the magnetization gradients to be small, and two conditions to be satisfied, at least to first order. This will be discussed further below, and in Paper II.[3] In all other cases, for example, when the magnetizations in adjacent layers are dissimilar, and the field is small but θ is close to 0° or 90°, we have not found an *a priori* criterion for determining if the assumptions are good. It is necessary in these cases to solve Eq. (32) and determine $\theta_j(z)$. If the hypotheses are satisfied, one may use Eqs. (30) and (31); if not, one must solve Eqs. (28) and (29).

A. Decoupling the continuity equations

Let us now assume that the above hypotheses hold and that Eqs. (30) and (31) apply. In this case, the problem can be further simplified. Inspection of these equations indicates that the resonant and nonresonant senses of polarization, μ^+ and μ^-, are coupled. However, if the magnetizations of adjacent regions are similar, or if a strong field is applied, as indicated above, the magnetization directions satisfy

$$\theta_{ij} \simeq \theta_{i,j+1} \simeq \theta_{0j} \simeq \theta_{0,j+1} , \tag{33}$$

$$\underline{U}^j(z_{j+1}) \simeq \underline{U}^{j+1}(z_{j+1}) \simeq \underline{U}^j \simeq \underline{U}^{j+1} , \tag{34}$$

and Eqs. (30) and (31) reduce to

$$\begin{pmatrix} \mu_j^+ \\ \mu_j^- \end{pmatrix} + \frac{A_j}{A_{j,j+1}} \begin{pmatrix} \partial\mu_j^+ \\ \partial\mu_j^- \end{pmatrix} - \frac{M_{0j}}{M_{0,j+1}} \begin{pmatrix} \mu_{j+1}^+ \\ \mu_{j+1}^- \end{pmatrix} = 0 , \tag{35}$$

$$\begin{pmatrix} \mu_{j+1}^+ \\ \mu_{j+1}^- \end{pmatrix} - \frac{A_{j+1}}{A_{j,j+1}} \begin{pmatrix} \partial\mu_{j+1}^+ \\ \partial\mu_{j+1}^- \end{pmatrix} - \frac{M_{0,j+1}}{M_{0j}} \begin{pmatrix} \mu_j^+ \\ \mu_j^- \end{pmatrix} = 0 . \tag{36}$$

The continuity equations for the two polarizations are decoupled and may be treated separately. We note that when the coupling $A_{j,j+1}$ between layers j and $j+1$ is weak these reduce to

$$\partial\mu_p^\pm = 0, \quad p = j, j+1 . \tag{37}$$

The spins are free at the interface. When the coupling is strong, the waves are in phase and

$$M_{0,j+1}\mu_j^\pm = M_{0j}\mu_{j+1}^\pm . \tag{38}$$

From Eqs. (35) and (36) we find that the gradients of the rf magnetization are related in a fashion completely independent of the coupling:

$$\frac{A_j}{M_{0j}}\, \partial\mu_j^\pm = \frac{A_{j+1}}{M_{0,j+1}}\, \partial\mu_{j+1}^\pm . \tag{39}$$

The derivatives of the rf magnetization waves have the same sign at the interfaces.

There exists one other case for which the two polarizations are decoupled. When the interfacial anisotropy β is large, the first terms of Eqs. (30) and (31) dominate the others. The continuity equations then reduce to

$$\mu_p^\pm = 0, \quad p = j, j+1 . \tag{40}$$

The spins are pinned at the interface $z = z_{j+1}$. [This result can also be obtained directly from Eqs. (26) and (27).]

B. Continuity equations at the outer surfaces

At the external surfaces of the sample the continuity conditions are given by Eqs. (30) and (31), with $A_{01} = 0$, $A_{N,N+1} = 0$:

$$[\underline{D}\underline{U}^1(z_1) + \underline{E}\, \partial\underline{U}^1(z_1)] \begin{pmatrix} \mu_1^+ \\ \mu_1^- \end{pmatrix} + \underline{E}\underline{U}^1(z_1) \begin{pmatrix} \partial\mu_1^+ \\ \partial\mu_1^- \end{pmatrix} = 0 , \tag{41}$$

$$[\underline{A}\underline{U}^N(z_{N+1}) + \underline{B}\, \partial\underline{U}^N(z_{N+1})] \begin{pmatrix} \mu_N^+ \\ \mu_N^- \end{pmatrix} + \underline{B}\underline{U}^N(z_{N+1}) \begin{pmatrix} \partial\mu_N^+ \\ \partial\mu_N^- \end{pmatrix} = 0 . \tag{42}$$

These equations are equivalent to those which have been given previously in the literature[1,17] for single films. They differ only in that they are written in terms of μ^+ and μ^-. The only nonzero terms remaining in matrices $\underline{A}$ and $\underline{D}$ (see Appendixes C and D) are those proportional to β, the surface anisotropy. When β is large, the first terms of Eqs. (41) and (42) are dominant, and the spins are pinned at the external surfaces. If β is zero, the external spins are free. Except in these two limits, μ^+ and μ^- are coupled at the external surfaces. Clearly, the value of β may be different for each of the external surfaces.

V. SOLUTION OF THE PROBLEM: ABSORBED POWER SPECTRUM

In order to write an expression for the power absorbed by the specimen, one must know the various components of electric and magnetic field throughout the sample. That is, one needs the different coefficients $C_{pj}^\pm$ ($1 \leq p \leq 4$; $1 \leq j \leq N$) of Eq. (19). In order to obtain these, one simultaneously solves the equations of continuity for the EM fields and for the magnetization waves of both polarizations at the external surfaces and at all the interfaces. In general, this is a difficult problem, and we shall solve it in the special case of complete symmetry about a central plane, with completely free or completely pinned spins at the outer surfaces. By complete symmetry we imply symmetrical excitation, a sample consisting of an odd number of layers, with layer j identical with layer $N-j+1$, and interfacial conditions identical at $z=z_j$ and $z=z_{N-j+2}$.

Under these assumptions, only one half of the continuity equations need be solved, as the other half are identical and are solved trivially. The equations which remain are the continuity equations for the transverse electric and magnetic fields at the interfaces[18] [see Eqs. (20), (21), (23), and (24)]:

$$h_j^x = h_{j+1}^x, \quad h_j^y = h_{j+1}^y,$$
$$e_j^x = e_{j+1}^x, \quad e_j^y = e_{j+1}^y, \tag{43}$$
$$z = z_{j+1}, \quad 1 \leq j \leq \tfrac{1}{2}(N-1) ;$$

and at outside surface:

$$h_0^x = h_1^x , \tag{44}$$
$$h_0^y = h_1^y , \tag{45}$$
$$e_0^x = e_1^x , \tag{46}$$
$$e_0^y = e_1^y , \tag{47}$$
$$z = z_1 ,$$

plus the equations of continuity for the two polarizations of the magnetization at the interfaces and at the surfaces.

In all, there are $2(N+1)$ equations for the EM field, and $2N$ equations for the magnetization, giving a total of $2(2N+1)$. As for the number of unknowns, there are $4(N-1)$ values of $C_{pj}^{\pm}$ to be determined for the first $\frac{1}{2}(N-1)$ layers, and 4 for the central layer (due to symmetry, only the even solutions are kept for this layer). Finally, there are four unknowns $(h_0^x, h_0^y, e_0^x, e_0^y)$ at the surface, giving a total of $4(N+1)$. That is, there are two arbitrary amplitudes which we may specify.

Let us now introduce a set of quantities which play the role of surface impedances[19]:

$$R_{xy} = -e_0^x / h_0^y , \tag{48}$$

$$R_{yx} = e_0^y / h_0^x . \tag{49}$$

With Eqs. (48) and (49), we may eliminate the two electric components (e_0^x and e_0^y) from Eqs. (46) and (47). Then, from the group of continuity equations [Eqs. (30), (31), and (41)–(47)], one obtains the coefficient $C_{pj}^{\pm}$ in terms of h_0^x, h_0^y, R_{xy}, and R_{yx}. Using these coefficients in Eqs. (20) and (21) along with Eqs. (44) and (45), one obtains two homogeneous equations for h_0^x and h_0^y. Setting the determinant of this system of equations to zero, we arrive finally at an equation of the form

$$R_{xy} R_{yx} + \alpha_1 R_{xy} + \alpha_2 R_{yx} + \alpha_3 = 0 . \tag{50}$$

When $R_{xy} = R_{yx} = R$, there is conservation of the polarization of the EM mode as it is reflected at the sample surface.[13] R is a surface impedance, in the same sense in which the term is generally defined.[18] In this case, Eq. (50) is quadratic in R and has two solutions, R^+ and R^-. These are the characteristic impedances of the system. To each of these, there corresponds a characteristic polarization of the EM wave in vacuum. These two polarizations are linearly independent,[19] and it is possible to treat any incident polarization as a linear combination of the characteristic polarizations. Let us be more explicit: let $\vec{h}_i^+$ and $\vec{h}_i^-$ be the values of the incident magnetic fields in the waves corresponding to R^+ and R^-, let P^+ be the power absorbed when the incident field is $\vec{h}_i^+$ and let P^- be the power absorbed from $\vec{h}_i^-$. Since the two polarizations are linearly independent, the power absorbed when the incident field is

$$\vec{h}_i = a\vec{h}_i^+ + b\vec{h}_i^- ,$$

given by the sum of the power absorbed separately from fields of intensity $aa^*|h_i^+|^2$ and $bb^*|h_i^-|^2$, that is,

$$P = aa^* P^+ + bb^* P^- . \tag{51}$$

The problem of calculating the resonance spectra is thus reduced to the problem of obtaining P^+ and P^-, that is, to a solution of the equations of continuity with the characteristic impedances of the system. Let us therefore, obtain explicit expressions for P^+ and P^-.

To the two surface impedances R^+ and R^- there correspond two series of coefficients $(C_{pj}^{\pm})^+$ and $(C_{pj}^{\pm})^-$. Introducing these coefficients into Eqs. (20), (21), (23), and (24), we obtain the magnetic and electric fields everywhere inside the sample. From Poynting's theorem, one obtains expressions for the power absorbed by the sample, and by each of the layers. We find[20] that the power absorbed by the whole sample is given by

$$P^{\pm} = (c/\pi)|\vec{h}_i^{\pm}|^2 \operatorname{Re}(R^{\pm}) . \tag{52}$$

The power absorbed by the jth layer [and by its mirror image, the $(N-j+1)$th layer] is

$$P_j^{\pm} = (c/\pi)|\vec{h}_i^{\pm}|^2 \operatorname{Re}(|\lambda^{\pm}|^2 R_{yx}^{\pm} + |\lambda_y^{\pm}|^2 R_{xy}^{\pm})\Big|_{z=z_{j+1}}^{z=z_j},$$
$$1 \leq j \leq \tfrac{1}{2}(N-1) . \tag{53}$$

For the central layer

$$P_{(N+1)/2}^{\pm} = (c/\pi)|\vec{h}_i^{\pm}|^2 \operatorname{Re}(|\lambda_x^{\pm}|^2 R_{yx}^{\pm} + |\lambda_y^{\pm}|^2 R_{xy}^{\pm})\big|_{z=z_{(N+1)/2}} . \tag{54}$$

The coefficients $\lambda_x^{\pm}$, $\lambda_y^{\pm}$, and the R's are defined in Appendix E. We note finally that Eqs. (52)–(54) are not exact. In going from Poynting's theorem to these final equations we have neglected the surface and interfacial impedances in comparison with the intrinsic impedance of vacuum. This is always a good approximation, since R never exceeds a few ohms,[21] while the vacuum impedance is 367 Ω.[19] Equations (52)–(54) have the desirable feature of being relatively simple and easily used in numerical calculations.

A special case. Let us consider the special case in which the interface anisotropies are negligible, and one of the two conditions, strong applied field, or similar neighboring layers, are satisfied. As we have shown in Sec. V, the two polarizations are then decoupled and may be treated separately. The nonresonant polarizations will be only weakly excited,[14] and it is possible to neglect them without introducing any inconsistency in the continuity equations. In this case, Eqs. (20), (21), (23), and (24) yield

$$\frac{h_j^x}{h_j^y} = -\frac{e_j^y}{e_j^x} = \frac{U_{11}^j(z)\cos\theta_j(z)}{U_{21}^j(z)}\,, \quad 1 \leq j \leq \tfrac{1}{2}(N+1)\,. \tag{55}$$

When we take account of Eqs. (33) and (34), this implies

$$\frac{h_j^x}{h_j^y} \simeq \frac{h_{j+1}^x}{h_{j+1}^y}\,, \quad \frac{e_j^x}{e_j^y} \simeq \frac{e_{j+1}^x}{e_{j+1}^y}\,, \quad z = z_{j+1}, \quad 1 \leq j \leq \tfrac{1}{2}(N+1)\,. \tag{56}$$

Equations (56) permit us to eliminate half of the equations of continuity for the EM field. Since we may neglect the continuity equations for the nonresonant polarization, the total number of equations and of unknowns is half of what it was previously. Following the same reasoning as in the previous discussion, we obtain a linear equation in R for a single surface impedance. Then, the absorbed power is

$$P = (c/\pi)\,|\vec{\mathrm{h}}_i|^2\,\mathrm{Re}(R) \tag{57}$$

for the whole sample,

$$P_j = (c/\pi)\,|h_i|^2\,\mathrm{Re}(|\lambda|^2 r|_{z=z_{j+1}}^{z=z_j}), \quad 1 \leq j \leq \tfrac{1}{2}(N-1) \tag{58}$$

for the jth layer, and

$$P_{(N+1)/2} = (c/\pi)\,|h_i|^2\,\mathrm{Re}(|\lambda|^2 r|_{z=z_{(N+1)/2}}) \tag{59}$$

for the central layer. The coefficients λ and r are defined in Appendix F. This is obviously a much simpler numerical problem than the general case.

Whether we are interested in the general or the special case, the power can always be obtained by another approach. Combining Poynting's theorem with Gauss's theorem, one can integrate the energy density to obtain[22]

$$P = \tfrac{1}{2}\omega\,\mathrm{Im}\left(\int_v \vec{\mathrm{m}}\cdot\vec{\mathrm{h}}^*\,dv\right) + \tfrac{1}{2}\sigma\int_v \vec{\mathrm{e}}\cdot\vec{\mathrm{e}}^*\,dv\,.$$

This expression is useful in the interpretation of the behavior of the system, but is disadvantageous for numerical calculations as it requires lengthy numerical integrations.

VI. DISCUSSION

We have presented here a method which permits us to solve, by an approximate treatment, the problem of FMR in metallic multilayers for arbitrary directions of the applied static field. The restrictions on the validity of the approximations and of the method are: first, that the FMR spectrum not include the antiresonance of either of the layers, as the approximate dispersion relations are not valid in this case[2]; second, the interdiffusion between layers must be weak so that Hoffmann's equations[1] will be applicable; third, the static magnetization near the interfaces must obey the conditions, discussed above, of small change and of small region of change (compared to the layer thickness). If the second and, especially, the third conditions are not obeyed, the wave vectors become point-by-point functions of position and a completely different approach must be taken to the problem. We have seen that when the third condition is fulfilled, the magnetization may be described everywhere in a layer in terms of the two polarizations μ^+ and μ^-. If adjacent layers are sufficiently similar, or if the applied field is sufficiently large, one may simply neglect the nonresonant polarization in the solution of the problem. Evidently, the conditions of small change of magnetization direction and of similar layers are closely interrelated if the interfacial coupling is reasonably strong.

Even if the three conditions cited above are obeyed, if adjacent layers are not sufficiently similar a solution involving all of the polarizations becomes very difficult numerically. It is relatively easy to see why this becomes a problem. We can write the set of continuity equations as a matrix equation

$$\underline{\tau}\,\vec{\mathrm{C}} = \vec{\Gamma}\,. \tag{60}$$

In Eq. (60), $\vec{\mathrm{C}}$ is a vector of amplitude coefficients C_{nj}, $\vec{\Gamma}$ is a vector whose components depend upon the fields at the surface of the sample, and τ is a matrix whose elements depend upon the physical properties of the various layers. Solving Eq. (60) is equivalent to finding the inverse of $\underline{\tau}$. In the approximate problem of N layers, $\underline{\tau}$ is a $4N\times 4N$ matrix. In the more complete problem, it is an $8N\times 8N$ matrix, as one takes into account both resonant and nonresonant polarizations. The terms relating to the nonresonant polarization appear in the same rows as those relating to the resonant polarization. These different types of term may differ in size by several orders of magnitude. If one simply attempts to invert the matrix, it is impossible to find an appropriate normalization which avoids overflows or underflows, and the numerical accuracy becomes unacceptable. As a result we have not yet solved such a problem, although it is likely that this can be done by using the solution of the simpler problem as a starting point.[23]

In the following paper,[3] we shall present the solution of a particular simple problem, along with a semiquantitative model explaining the results of the calculation.

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APPENDIX A

The terms in Eq. (13) are defined as follows:

$$a_j = (\lambda_j/\gamma_j)(H_{0j}/M_{0j} + 1/Q_j) + 2A_j\lambda_j k_j^2/\gamma_j M_{0j}^2 ,$$

$$b_j = H_{0j} + M_{0j}/Q_j + 2A_j k_j^2/M_{0j} ,$$

$$c_j = H_{0j} + 4\pi M_{0j} + 2A_j k_j^2/M_{0j} ,$$

$$d_j = (\lambda_j/\gamma_j)(H_{0j}/M_{0j} + 4\pi) + 2A_j\lambda_j k_j^2/\gamma_j M_{0j}^2 ,$$

$$\Omega_j = \omega/\gamma_j .$$

APPENDIX B

The terms in Eqs. (16) and (17) are defined as follows:

$$\pi_{1j} = iA_j\Omega_{0j}\delta_j^2\xi_{1j}/M_{0j} ,$$

$$\pi_{2j} = -[2A_j\Omega_{0j}\xi_{1j}/M_{0j} + (H_{0j} + 2\pi M_{0j}\sin^2\theta_{0j}) \times(\lambda_j\Omega_j/\gamma_j M_{0j} - i\Omega_{0j})\tfrac{1}{2}\delta_j^2 + \tfrac{1}{2}i\Omega_{0j}^2\delta_j^2] ,$$

$$\pi_{3j} = -(H_{0j} + 4M_{0j})(\Omega_{0j} + i\lambda_j\Omega_j/\gamma_j M_{0j}) + \Omega_j^2 ,$$

$$\pi_{4j} = iA_j\Omega_{0j}\delta_j^2\xi_{2j}/M_{0j} ,$$

$$\pi_{5j} = (H_{0j} + 2\pi M_{0j}\sin^2\theta_{0j})(\lambda_j\Omega_j/\gamma_j M_{0j} + i\Omega_{0j}) \times \tfrac{1}{2}\delta_j^2 - 2A_j\Omega_{0j}\xi_{2j}/M_{0j} + \tfrac{1}{2}i\Omega_{0j}^2\delta_j^2 ,$$

$$\pi_{6j} = -(H_{0j} + 4\pi M_{0j})(\Omega_{0j} - i\gamma_j\Omega_j/\gamma_j M_{0j}) - \Omega_j^2 ,$$

where

$$\xi_{1j} = 1 + i\lambda_j\Omega_j/\Omega_{0j}\gamma_j M_{0j} ,$$

$$\xi_{2j} = 1 - i\lambda_j\Omega_j/\Omega_{0j}\gamma_j M_{0j} .$$

APPENDIX C

The diagonal elements of matrices $\underline{A}$-$\underline{F}$ of Eqs. (28) and (29) are given by

$$a_{11} = (A_{j,j+1}/M_{0j})\cos(\theta_{i,j+1} - \theta_{ij}) - \beta_j M_{0j}\cos 2\theta_{ij} ,$$

$$a_{22} = \sin\theta_{ij}[(A_{j,j+1}/M_{0j})\cos(\theta_{ij} - \theta_{i,j+1}) - \beta_j M_{0j}\cos^2\theta_{ij}] ,$$

$$b_{11} = A_j/M_{0j} , \quad b_{22} = (A_j/M_{0j})\sin\theta_{ij} ,$$

$$c_{11} = -(A_{j,j+1}/M_{0,j+1})\cos(\theta_{i,j+1} - \theta_{ij}) ,$$

$$c_{22} = -(A_{j,j+1}/M_{0,j+1})\sin\theta_{ij} ,$$

$$d_{11} = (A_{j,j+1}/M_{0,j+1})\cos(\theta_{i,j+1} - \theta_{ij}) - \beta_{j+1}M_{0,j+1}\cos 2\theta_{i,j+1} ,$$

$$d_{22} = \sin\theta_{i,j+1}[(A_{j,j+1}/M_{0,j+1})\cos(\theta_{i,j+1} - \theta_{ij}) - \beta_{j+1}M_{0,j+1}\cos^2\theta_{i,j+1}] ,$$

$$e_{11} = -A_{j+1}/M_{0,j+1} ,$$

$$e_{22} = -(A_{j+1}/M_{0,j+1})\sin\theta_{i,j+1} ,$$

$$f_{11} = -(A_{j,j+1}/M_{0j})\cos(\theta_{i,j+1} - \theta_{ij}) ,$$

$$f_{22} = -(A_{i,j+1}/M_{0j})\sin\theta_{i,j+1} .$$

APPENDIX D

The derivatives of the matrix $\underline{U}$ in Eqs. (28) and (29) are given by

$$\frac{\partial U_{11}^p(z)}{\partial z} = \frac{\partial U_{22}^p(z)}{\partial z} = -U_{11}^p(z)\left(\frac{\Omega_{0p} + \Lambda_p}{2\Omega_{0p}^2}\right)\frac{\partial\Lambda_p}{\partial z} ,$$

$$\frac{\partial U_{12}^p(z)}{\partial z} = \frac{\partial U_{21}^p(z)}{\partial z} = U_{12}^p(z)\left(\frac{\Omega_{0p} - \Lambda_p}{2\Omega_{0p}^2}\right)\frac{\partial\Lambda_p}{\partial z} ,$$

where

$$\Lambda_p = 2M_{0p}\sin^2\theta_p(z) , \quad p = j, j+1 .$$

APPENDIX E

The terms in Eqs. (53) and (54) [omitting the (+) and (−) superscripts] are defined as

$$R_{yx}|_{z=z_p} = (e_p^y/h_p^x)|_{z=z_p} ,$$

$$R_{xy}|_{z=z_p} = -(e_p^x/h_p^y)|_{z=z_p} ,$$

$$|\lambda_x|^2 = \frac{|\rho|^2}{1+|\rho|^2}\left|\frac{h_p^x|_{z=z_p}}{h_1^x|_{z=z_1}}\right|^2 ,$$

$$|\lambda_y|^2 = \frac{1}{1+|\rho|^2}\left|\frac{h_p^y|_{z=z_p}}{h_1^y|_{z=z_1}}\right|^2 ,$$

where

$$\rho = (h_1^x/h_1^y)|_{z=z_1} .$$

APPENDIX F

The terms in Eqs. (58) and (59) are defined as

$$r|_{z=z_p} = R_{xy}|_{z=z_p} = R_{yx}|_{z=z_p} ,$$

$$|\lambda|^2|_{z=z_p} = (|\lambda_x|^2 + |\lambda_y|^2)|_{z=z_p} .$$

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[1]F. Hoffmann, Phys. Status Solidi 41, 807 (1970); Doctoral thesis (Université de Paris-Sud, 1971) (unpublished).
[2]A. Yelon, G. Spronken. T. Buithieu, R. C. Barker, Y. J. Liu, and T. Kobayashi, Phys. Rev. B 10, 1070 (1974).
[3]G. Spronken, A. Friedmann, and A. Yelon, following paper, Phys. Rev. B 15, 5151 (1977).
[4]B. Waksmann, R. Buder, P. Escudier, C. F. Kooi, and O. Massenet, J. Phys. (Paris) 29, C2-128 (1968).
[5]Z. Frait, D. Fraitova, and R. Gemperle, Czech. J. Phys. B 25, 906 (1975).
[6]P. E. Wigen, C. F. Kooi, M. R. Shanabarger, and T. D. Rossing, Phys. Rev. Lett. 9, 206 (1962).
[7]A. M. Portis, Appl. Phys. Lett. 2, 69 (1963).
[8]C. F. Kooi, P. E. Wigen, M. R. Shanabarger, and J. V. Kerrigan, J. Appl. Phys. 35, 791 (1964).
[9]C. S. Guenzer, C. Vittoria, and H. Lessoff, AIP Conf. Proc. 18, 1292 (1973).
[10]J. T. Yu, R. A. Turk, and P. E. Wigen, Phys. Rev. B 11, 420 (1975).
[11]O. G. Ramer, and C. H. Wilts, Phys. Status Solidi B 73, 443 (1976); 79, 313 (1977).
[12]B. Heinrich, J. F. Cochran, J. H. Liaw, and R. C. Barker, AIP Conf. Proc. 34, 250 (1976).
[13]C. Vittoria, R. C. Barker, and A. Yelon, J. Appl. Phys. 40, 1561 (1969).
[14]T. Kobayashi, R. C. Barker, J. L. Bleustein, and A. Yelon, Phys. Rev. B 7, 3273 (1973).
[15]For another approach, see Y. J. Liu and R. C. Barker, AIP Conf. Proc. 24, 505 (1974).
[16]G. Spronken (unpublished).
[17]G. T. Rado and J. R. Weertman, J. Phys. Chem. Solids 11, 315 (1959).
[18]C. H. Durney and C. C. Johnson, *Introduction to Modern Electromagnetics* (McGraw-Hill, New York, 1969).
[19]S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio* (Wiley, New York, 1945).
[20]G. Spronken, M. Sc. A. thesis (Ecole Polytechnique, Montreal, 1975) (unpublished).
[21]D. S. Rodbell, Physics 1, 279 (1965).
[22]Y. J. Liu, Ph.D. thesis (Yale University, 1974) (unpublished).
[23]A. Aharoni (private communication).