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Equations of state for bicritical points. II. Ising-like ordered phases

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Calculations of bicritical crossover scaling functions are extended into the Ising-like bicritical ordered phases. Closed-form expressions are derived to first order in $\epsilon = 4 - d$ for the specific heat, nonordering susceptibility, and longitudinal susceptibilities in this regime. Scaling functions in ordered and disordered regions are displayed graphically and compared with series-expansion results.

I. INTRODUCTION

In a previous paper,[1] henceforth referred to as I, the crossover scaling functions appropriate to the disordered phase of systems with bicritical points were calculated to first order in $\epsilon = 4 - d$. In particular, closed-form expressions were obtained for the free energy, specific heat, nonordering and longitudinal or parallel susceptibilities. The results for the parallel susceptibility, when compared with series-expansion work,[2] were found to give a qualitatively good description of bicritical crossover behavior, and represented a significant improvement over expressions derived from mean-field theory. Independent work by Kosterlitz and by Horner,[3] who have also calculated certain disordered-phase bicritical scaling functions to $O(\epsilon)$, should be noted.[4] Since it is expected that bicritical points will occur in uniaxial antiferromagnets such as $GdAlO_3$ and MnF_2 which exhibit spin-flop transitions,[5] there is hope that experimentally determined bicritical scaling functions may become available to compare with these calculations.

In this paper, we extend the calculations of I to regions of the phase plane for which series results are not available. The Hamiltonian

$$\bar{\mathcal{H}} = -\frac{\mathcal{H}}{k_B T} = -\int d\vec{R}\biggl(\frac{1}{2}\sum_{i=1}^{m} (\vec{\nabla} s_\parallel^i)^2 + \frac{1}{2}\sum_{j=1}^{n-m} (\vec{\nabla} s_\perp^j)^2 + \tfrac{1}{2} r_0 (|\vec{s}_\parallel|^2 + |\vec{s}_\perp|^2) - (g_0/2n)[(n-m)|\vec{s}_\parallel|^2 - m|\vec{s}_\perp|^2] + u_0(|\vec{s}_\parallel|^2 + |\vec{s}_\perp|^2)^2\biggr), \qquad (1.1)$$

depending on the m- and $(n-m)$-component spin fields $\vec{s}_\parallel(\vec{R})$ and $\vec{s}_\perp(\vec{R})$, is believed appropriate for the description of many real bicritical points,[6] and was discussed in I. The corresponding phase diagram as a function of the temperature variable $t_0 = (T - T_b)/T_b$ (which is related to the parameter r_0), and of the anisotropy g_0 is shown in Fig. 1 (for $n = 3$ and $m = 1$). Although both ordered phases shown in this figure are readily accessible in appropriate experiments, conventional series expansions at present offer little information about crossover behavior in this interesting region.

There are technical difficulties in calculations with the Hamiltonian (1.1) when $g_0 \gtrsim 0$ and $n > m \geq 2$. For example if $n = 3$ and $m = 2$ (which should be appropriate for MnF_2), then the upper ordered phase in Fig. 1 would have $n_\parallel = m = 2$ or XY-like symmetry. It is known,[7] however, that the parallel susceptibility, and certain other thermodynamic derivatives, are infinite everywhere in an ordered phase of continuous rotational symmetry; this fact considerably complicates the analysis. Although

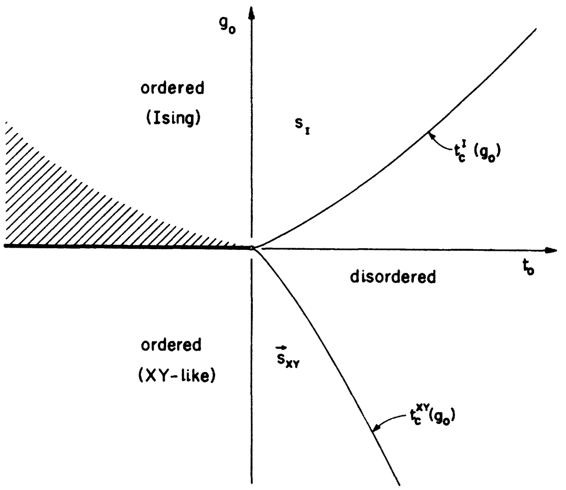

FIG. 1. Bicritical phase diagram (to scale for $n = 3$, $m = 1$, and $d = 3$) in the (t_0, g_0) plane. The present ordered-phase calculations are limited to the unshaded region of the Ising-like (or $m = 1$) ordered phase.

it has been possible to treat the analogous problems arising in a ferromagnet when $n \geq 2$,[8] we will, for simplicity, exclude them here by considering only $m = 1$, so that $s_{\parallel}(\vec{R})$ will represent a *scalar* or Ising-like field.

Real antiferromagnetic crystals will not display ordered phases with full rotational invariance. Symmetry breaking due to crystal fields destroys the rotational invariance and stabilizes the fluctuations and thermodynamic susceptibilities against incipient divergences within the ordered phase.[8] One way to allow for such effects when $m \geq 2$ is to add a symmetry-breaking perturbation of the form

$$\bar{\mathcal{H}}_v = v\left(\sum_{i=1}^{m} (s_{\parallel}^{i})^4 + \sum_{j=1}^{n-m} (s_{\perp}^{j})^4\right), \tag{1.2}$$

to (1.1). However, we will defer a detailed discussion of the effect of (1.2) on bicritical scaling functions to a future publication.[9] Bruce and Aharony[10] have shown that, although the perturbation (1.2) does not affect the bicritical phase diagram of Fig. 1 for $v \leq 0$, the bicritical point becomes *tetracritical* in nature when v is positive.

Unfortunately, there remain technical problems in approaching the bicritical first-order line (bold in Fig. 1) even under the restriction $m = 1$. Indeed, the first-order line is a boundary of coexistence of phases of higher symmetry than $m = 1$, and thermodynamic functions again diverge as this boundary is approached. For example, the nonordering susceptibility χ_g, discussed in I, should diverge as $g \to 0$ at fixed $T < T_b$ according to

$$\chi_g \sim g^{-\epsilon/2} \quad (\epsilon = 4 - d). \tag{1.3}$$

This divergence is reminiscent of the coexistence-curve behavior of the longitudinal susceptibility of Heisenberg ferromagnets as the magnetic field tends to zero.[7,8] For the present we will again bypass the difficulties connected with this behavior by excluding from consideration the shaded region in Fig. 1 (the explicit restriction will be specified more precisely in Sec. III). Similar limitations in fact apply to the original analysis of Brézin, Wallace, and Wilson[7] of the equation of state of a ferromagnet for $n \geq 2$ (although they were not explicitly emphasized; see, however, Ref. 8).

Even with the two restrictions discussed above, the results presented are of definite theoretical and experimental interest. The behavior of the bicritical thermodynamic functions will be determined [to $O(\epsilon)$] in crossover regions which have, thus far, eluded theoretical analysis.

In Sec. II we recapitulate the techniques developed in I for calculating bicritical scaling functions. The derivation of scaling functions for an isotropic system in the ordered phase[11] is reviewed in Appendix A. A detailed discussion of the "matching" analysis which leads to thermodynamic functions in the bicritical region is given in Sec. III and Appendix B. Closed-form expressions for the bicritical ordered-phase scaling functions are derived in Sec. IV. Finally, in Sec. V, we summarize the investigations reported here, exhibit the scaling functions graphically for $\epsilon = 1$, and compare with series-expansion calculations.[2]

Some readers may wish to proceed directly to the *results* reported here, bypassing a detailed account of the calculations. To this end, we suggest that the reader pass directly from Sec. II to Sec. IV. Explicit formulas for the bicritical magnetization, susceptibility, free energy, specific heat, and nonordering susceptibility are given in (4.5), (4.9), (4.29), (4.32), and (4.42), respectively. These formulas are expressed in terms of g_0, the anisotropy field, in terms of the deviation from the bicritical temperature, $t_0 = (T - T_b)/T_b$, and utilize the normalized crossover variable

$$y = Bt_0/g_0^{1/\phi}, \tag{1.4}$$

which takes the value unity on the critical line $t_c^I(g_0)$. Here ϕ is the standard anisotropy crossover exponent[2,12] with expansion

$$\phi = 1 + \tfrac{1}{2}\epsilon n/(n+8) + O(\epsilon^2). \tag{1.5}$$

In a real uniaxial antiferromagnet, t_0 and g_0 would correspond to linear combinations of the physical deviations from the bicritical point, $(T - T_b)/T_b$ and $(H - H_b)/H_b$.[5(b)] The function $R(x)$ entering in the final expressions, is defined in (4.2); in addition one has $K_4 = 1/8\pi^2$, and

$$\lambda_2 = 2 - 2\epsilon/(n+8) + O(\epsilon^2). \tag{1.6}$$

Ising critical exponents, to first order in ϵ, enter most of the expressions: explicitly, these are

$$\alpha_I \approx \tfrac{1}{6}\epsilon, \quad \beta_I \approx \tfrac{1}{2} - \tfrac{1}{6}\epsilon, \quad \gamma_I \approx 1 + \tfrac{1}{6}\epsilon, \tag{1.7}$$

although more accurate results for $d = 3$ ($\epsilon = 1$) are probably obtained if the series-expansion estimates $\alpha_I \simeq \frac{1}{8}$, $\beta_I \simeq \frac{5}{16}$, and $\gamma_I \simeq 1\frac{1}{4}$ are utilized in the scaling-function expressions. We hope the notational summary presented above will help readers not interested in the technical details of the calculations to appreciate and utilize the results.

II. THEORETICAL TECHNIQUE

The calculations reported here follow the procedures developed in I. Two distinct length scales enter the problem, namely, the parallel and perpendicular correlation lengths $\xi_{\parallel}$ and $\xi_{\perp}$. These lengths both decrease under the application of the renormalization-group transformation but, in general, at different rates. Our analysis relies on approximate solutions of differential renormalization-group recursion relations which are themselves exact to order $\epsilon = 4 - d$. The solutions remain sufficiently accurate to yield scaling functions exact to order ϵ, as long as the quadratic coefficients, $r_{\parallel}$ and $r_{\perp}$, in the renormalized Hamiltonian $\mathcal{H}(l)$, which are related to $\xi_{\parallel}$ and $\xi_{\perp}$, do not become too large. We first employ these approximate solutions to integrate the recursion relations up to a value $l = l^*$ chosen such that $\xi_{\perp}(l^*) = O(1)$. We must require, in addition, that the parallel quadratic coefficient $r_{\parallel}$ not exceed order unity at this stage. This requirement (which was satisfied automatically in the calculations reported in I) imposes a restriction on the region of the (t_0, g_0) plane over which our analysis is valid, as indicated in Fig. 1. This restriction is examined in detail in Sec. III D.

Having renormalized the full Hamiltonian up to $l = l^*$, we then integrate out the perpendicular spin components $\vec{s}_{\perp}(\vec{R})$. Since the perpendicular correlation length $\xi_{\perp}(l^*)$ is of order unity, this can be achieved to sufficient accuracy by using straightforward perturbation theory to yield a *reduced Hamiltonian* that depends only on the parallel spin components. This $\bar{\mathcal{H}}_{\mathrm{red}}$ is then written in a form which has been treated in detail by Rudnick and Nelson,[11] allowing their analysis to be taken over directly.

Working in a bicritical ordered phase gives rise to several difficulties not encountered in I. As usual in ordered-phase calculations,[7] we must shift the parallel spin component (we consider now only the case $m = 1$, $g \gtrsim 0$) by the exact magnetization M, namely we put

$$s_{\parallel} = \sigma + M , \tag{2.1}$$

which gives rise to three-spin interaction terms in σ spins in the shifted Hamiltonian. This procedure has been employed by Rudnick and Nelson[11] to obtain ordered-phase recursion relations; their approach for an Ising Hamiltonian is reviewed in Appendix A.

There is a second technical difficulty: upon integrating out the perpendicular spin components at $l = l^*$, we are left with a reduced Hamiltonian depending only on the parallel spin components. However, we must relate the various coefficients (or field parameters) in this reduced Hamiltonian directly to the *initial*, *unshifted* parameters which occur in the Ising Hamiltonian treated by Rudnick and Nelson.[11] This is done in Appendix B.

As was discussed in detail in I, renormalization-group theory relates thermodynamic quantities calculated close to the bicritical point to the corresponding quantities calculated with the Hamiltonian $\mathcal{H}_{\mathrm{red}}$. This "matching" procedure leads directly to the bicritical crossover scaling functions.

III. DERIVATION OF THE REDUCED HAMILTONIAN

A. Integration of the recursion relations

Let us return to the Hamiltonian of (1.1). Introducing a shift via (2.1) we obtain

$$\mathcal{H} = -\int d\vec{R}\left[\tfrac{1}{2} r_{\parallel} M^2 + uM^4 - hM + \tfrac{1}{2}(\nabla\sigma)^2 + \tfrac{1}{2}(\vec{\nabla}\vec{s}_{\perp})^2 + \tfrac{1}{2}\tilde{r}_{\parallel}\sigma^2 + \tfrac{1}{2}\tilde{r}_{\perp}|\vec{s}_{\perp}|^2 + w_{\perp}\sigma|\vec{s}_{\perp}|^2 + w_{\parallel}\sigma^3 + u(\sigma^2 + |\vec{s}|^2)^2 - \tilde{h}\sigma\right] , \tag{3.1}$$

where

$$r_{\parallel} = r_0 - g_0(n-m)/n , \quad r_{\perp} = r_0 + g_0 m/n , \quad \tilde{r}_{\parallel} = r_{\parallel} + 12uM^2 , \quad \tilde{r}_{\perp} = r_{\perp} + 4uM^2 , \quad w_{\parallel} = w_{\perp} = 4uM^2 , \quad \tilde{h} = h - r_{\parallel}M - 4uM^3 , \tag{3.2}$$

and we have allowed for a magnetic field term $hs_{\parallel}$ in (1.1). In general, we must also allow for quadratic couplings of the arbitrary form $\sum_{ij} u_{ij} s_i^2 s_j^2$. The differential recursion relations for these s^4 coupling coefficients u_{ij} take the form

$$\begin{aligned}\frac{du_{ij}}{dl} &= \epsilon u_{ij} - 4K_4\left(2u_{ij}u_{ii}q_i^2 + 4u_{ij}^2 q_i q_j + 2u_{ij}u_{jj}q_j^2 + \sum_m u_{im}u_{mj}q_m^2\right) + O(uw^2, w^4) ,\\ q_i &= q_{\parallel} = 1/(1+\tilde{r}_{\parallel}) , \quad i = 1\\ &= q_{\perp} = 1/(1+\tilde{r}_{\perp}) , \quad i = 2, \ldots, n\end{aligned} \tag{3.3}$$

where $K_4 = 1/8\pi^2$. This complicated set of equations simplifies considerably, provided we take the initial coupling constants[1,11] all equal, namely $u_{ij} \equiv u_0$ (all i,j), where, as usual, u_0 is supposed of order ϵ. In this case, the solutions can be written[1,11] to leading order in ϵ as

$$u_{ij}(l) = u(l) + O(\epsilon u(l), u^2(l)) , \tag{3.4}$$

where

$$u(l) = u_0 e^{\epsilon l}/Q(l) , \quad Q(l) = 1 + 4K_4(n+8)u_0(e^{\epsilon l} - 1)/\epsilon , \tag{3.5}$$

provided $\tilde{r}_{\parallel}(l)$ and $\tilde{r}_{\perp}$ do not become too large. When the $\tilde{r}$ do become large, the isotropy of the solutions $u_{ij}(l)$ implied by (3.4) will be broken. Within the domain of validity implied by the condition $\tilde{r}_{\parallel}(l), \tilde{r}_{\perp}(l) \lesssim O(1)$, $u(l)$ remains bounded and of order ϵ.

Within the single-coupling-constant approximation based on (3.4), the remaining recursion relations are found to be

$$\frac{d\tilde{r}_{\parallel}}{dl} = 2\tilde{r}_{\parallel} + 12K_4\, uq_{\parallel} + 4(n-1)K_4 uq_{\perp} - 18K_4 w_{\parallel}^2 q_{\parallel}^2 - 2(n-1)K_4 w_{\perp}^2 q_{\perp}^2, \tag{3.6}$$

$$\frac{d\tilde{r}_{\perp}}{dl} = 2\tilde{r}_{\perp} + 4(n+1)K_4 uq_{\perp} + 4K_4 uq_{\parallel} - 4K_4 w_{\perp}^2 q_{\perp} q_{\parallel}, \tag{3.7}$$

$$\frac{dw_{\perp}}{dl} = (1+\tfrac{1}{2}\epsilon)w_{\perp} - 12K_4 uw_{\parallel}q_{\parallel}^2 - 4(n+1)K_4 uw_{\perp}q_{\perp}^2 - 16K_4 uw_{\perp}q_{\parallel}q_{\perp} + O(w_{\parallel}w_{\perp}^2), \tag{3.8}$$

$$\frac{dw_{\parallel}}{dl} = (1+\tfrac{1}{2}\epsilon)w_{\parallel} - 36K_4 uw_{\parallel}q_{\parallel}^2 - 4(n-1)K_4 uw_{\perp}q_{\perp}^2 + O(w_{\parallel}^3, w_{\perp}^3), \tag{3.9}$$

$$\frac{d\tilde{h}}{dl} = (3-\tfrac{1}{2}\epsilon)\tilde{h} - 3K_4 w_{\parallel}q_{\parallel} - (n-1)K_4\, w_{\perp}q_{\perp}, \tag{3.10}$$

where $q_{\parallel}$ and $q_{\perp}$ were defined in (3.3). Following the approach reviewed in Appendix A, we look for solutions of these equations of the form

$$\tilde{r}_{\parallel}(l) = r_{\parallel}(l) + 12u(l)M^2(l), \tag{3.11}$$

$$\tilde{r}_{\perp}(l) = r_{\perp}(l) + 4u(l)M^2(l), \tag{3.12}$$

$$w_{\parallel}(l) = w_{\perp}(l) = 4u(l)M(l) = w(l), \tag{3.13}$$

$$\tilde{h}(l) = h(l) - r_{\parallel}(l)M(l) - 4u(l)M_{\parallel}^3(l), \tag{3.14}$$

where we have set

$$M(l) = M_0 \exp[(1-\tfrac{1}{2}\epsilon)l]. \tag{3.15}$$

Note that the equality between $w_{\parallel}(l)$ and $w_{\perp}(l)$ is accurate only to first order in ϵ. On substituting (3.11)–(3.15) into (3.6)–(3.10) one obtains equations for $r_{\parallel}(l)$ and $r_{\perp}(l)$ quite similar to those encountered in the disordered phase,[1] namely

$$\frac{dr_{\parallel}}{dl} = 2r_{\parallel} + 4(n+2)K_4 u - 12K_4\, ur_{\parallel} - 4(n-1)K_4\, ur_{\perp} + \frac{12K_4\, u\tilde{r}_{\parallel}^2}{1+\tilde{r}_{\parallel}} + \frac{4(n-1)K_4 u\tilde{r}_{\perp}^2}{1+\tilde{r}_{\perp}} + K_4 w^2\left(\frac{18(\tilde{r}_{\parallel}^2 + 2\tilde{r}_{\parallel})}{(1+\tilde{r}_{\parallel})^2} + 2(n-1)\frac{\tilde{r}_{\perp}^2 + 2\tilde{r}_{\perp}}{(1+\tilde{r}_{\perp})^2}\right), \tag{3.16}$$

$$\frac{dr_{\perp}}{dl} = 2r_{\perp} + 4(n+2)K_4 u - 4(n+1)K_4 ur_{\perp} - 4K_4 ur_{\parallel} + 4(n+1)K_4 u\frac{\tilde{r}_{\perp}^2}{1+\tilde{r}_{\perp}} + \frac{4K_4 u\tilde{r}_{\parallel}^2}{1+\tilde{r}_{\parallel}} + 4K_4\, w^2 \frac{\tilde{r}_{\parallel}\tilde{r}_{\perp} + \tilde{r}_{\parallel} + \tilde{r}_{\perp}}{(1+\tilde{r}_{\parallel})(1+\tilde{r}_{\perp})}. \tag{3.17}$$

Following I, it is now convenient to define new variables

$$r_s = [r_{\parallel} + (n-1)r_{\perp}]/n, \quad r_d = r_{\perp} - r_{\parallel}; \tag{3.18}$$

which decouple the recursion relations in leading order. For these we obtain

$$\frac{dr_s}{dl} = 2r_s + 4(n+2)K_4 u - 4(n+8)K_4 ur_s + \frac{4(n+2)K_4 u}{n}\left(\frac{\tilde{r}_{\parallel}^2}{1+\tilde{r}_{\parallel}} + \frac{(n-1)\tilde{r}_{\perp}^2}{1+\tilde{r}_{\perp}}\right) + K_4\, w^2\left(\frac{18(\tilde{r}_{\parallel}^2 + 2\tilde{r}_{\parallel})}{(1+\tilde{r}_{\parallel})^2} + 2(n-1)\frac{\tilde{r}_{\perp}^2 + 2\tilde{r}_{\perp}}{(1+\tilde{r}_{\perp})^2} + 4(n-1)\frac{\tilde{r}_{\parallel}\tilde{r}_{\perp} + \tilde{r}_{\parallel} + \tilde{r}_{\perp}}{(1+\tilde{r}_{\parallel})(1+\tilde{r}_{\perp})}\right), \tag{3.19}$$

and

$$\frac{dr_d}{dl} = 2r_d - 8K_4 ur_d + 8K_4 u\left(\frac{\tilde{r}_{\perp}^2}{1+\tilde{r}_{\perp}} - \frac{\tilde{r}_{\parallel}^2}{1+\tilde{r}_{\parallel}}\right) + K_4 w^2\left(4\frac{\tilde{r}_{\parallel}\tilde{r}_{\perp} + \tilde{r}_{\parallel} + \tilde{r}_{\perp}}{(1+\tilde{r}_{\parallel})(1+\tilde{r}_{\perp})} - 18\frac{\tilde{r}_{\parallel}^2 + 2\tilde{r}_{\parallel}}{(1+\tilde{r}_{\parallel})^2} - 2(n-1)\frac{\tilde{r}_{\perp}^2 + 2\tilde{r}_{\perp}}{(1+\tilde{r}_{\perp})^2}\right). \tag{3.20}$$

As in I, the solutions of these equations are conveniently expressed in terms of special combinations of the temperature and nonordering scaling fields, namely

$$t_0 = r_s(0) + 2(n+2)K_4\, u_0, \quad g_0 = r_d(0). \tag{3.21}$$

The solutions of (3.19) and (3.20) may be expressed in implicit form by defining

$$\begin{aligned} t(l) \equiv{} & r_s(l) + 2K_4(n+2)u(l)\left(1 - \frac{1}{n}\{\tilde{r}_\parallel(l)\ln[1+\tilde{r}_\parallel(l)] + (n-1)\tilde{r}_\perp(l)\ln[1+\tilde{r}_\perp(l)]\}\right) \\ & - \frac{16K_4}{n}u^2(l)M^2(l)\left((n+8)\ln[1+\tilde{r}_\parallel(l)] + \frac{9\tilde{r}_\parallel(l)}{1+\tilde{r}_\parallel(l)} + 2(n-1)\ln[1+\tilde{r}_\perp(l)]\right. \\ & \left. + (n-1)\frac{\tilde{r}_\perp(l)}{1+\tilde{r}_\perp(l)} + (n-1)\frac{\tilde{r}_\parallel(l)+\tilde{r}_\perp(l)}{\tilde{r}_\parallel(l)-\tilde{r}_\perp(l)}\ln\frac{1+\tilde{r}_\parallel(l)}{1+\tilde{r}_\perp(l)}\right) \\ \equiv{} & t_0 e^{2l}/[Q(l)]^{(n+2)/(n+8)} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} g(l) \equiv{} & r_d(l) - 4K_4 u(l)\{\tilde{r}_\perp(l)\ln[1+\tilde{r}_\perp(l)] - \tilde{r}_\parallel(l)\ln[1+\tilde{r}_\parallel(l)]\} \\ & + 16K_4 u^2(l)M^2(l)\left(8\ln[1+\tilde{r}_\parallel(l)] + 9\frac{\tilde{r}_\parallel(l)}{1+\tilde{r}_\parallel(l)} + (n-2)\ln[1+\tilde{r}_\perp(l)]\right. \\ & \left. + (n-1)\frac{\tilde{r}_\perp(l)}{1+\tilde{r}_\perp(l)} + \frac{\tilde{r}_\parallel(l)+\tilde{r}_\perp(l)}{\tilde{r}_\parallel(l)-\tilde{r}_\perp(l)}\ln\frac{1+\tilde{r}_\parallel(l)}{1+\tilde{r}_\perp(l)}\right) \\ ={} & g_0 e^{2l}/[Q(l)]^{2/(n+8)}. \end{aligned} \tag{3.23}$$

The final step is to solve these implicit expressions for $\tilde{r}_\parallel(l)$ and $\tilde{r}_\perp(l)$ expressed in terms of the simple functions $t(l)$ and $g(l)$. On setting

$$T_\parallel(l) = t(l) - (n-1)g(l)/n + 12u(l)M^2(l), \tag{3.24}$$

$$T_\perp(l) = t(l) + g(l)/n + 4u(l)M^2(l), \tag{3.25}$$

we finally obtain

$$\begin{aligned} \tilde{r}_\parallel(l) ={} & T_\parallel(l) - 2K_4(n+2)u(l) + 6K_4 u(l)T_\parallel(l)\ln[1+T_\parallel(l)] + 2K_4(n-1)u(l)T_\perp(l)\ln[1+T_\perp(l)] \\ & + 144K_4 u^2(l)M^2(l)\left(\ln[1+T_\parallel(l)] + \frac{T_\parallel(l)}{1+T_\parallel(l)}\right) + 16K_4(n-1)u^2(l)M^2(l)\left(\ln[1+T_\perp(l)] + \frac{T_\perp(l)}{1+T_\perp(l)}\right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} \tilde{r}_\perp(l) ={} & T_\perp(l) - 2K_4(n+2)u(l) + 2K_4 u(l)T_\parallel(l)\ln[1+T_\parallel(l)] + 2(n+1)K_4 u(l)T_\perp(l)\ln[1+T_\perp(l)] \\ & + 32K_4 u^2(l)M^2(l)\frac{T_\parallel(l)\ln[1+T_\parallel(l)] - T_\perp(l)\ln[1+T_\perp(l)]}{T_\parallel(l) - T_\perp(l)}. \end{aligned} \tag{3.27}$$

Note that $\tilde{r}_\parallel(l) = T_\parallel(l) + O(\epsilon, u(l))$ and $\tilde{r}_\perp(l) = T_\perp(l) + O(\epsilon, u)$; thus, the correlation lengths are determined, to a leading approximation, by knowledge of $T_\parallel(l)$ and $T_\perp(l)$.

The solution to the recursion relation (3.10) for $\tilde{h}(l)$ may now be found straightforwardly. To $O(\epsilon^{1/2})$ it is

$$\begin{aligned} \tilde{h}(l) ={} & h_0\exp[(3-\tfrac{1}{2}\epsilon)l] - [t(l) - (n-1)g(l)/n]M(l) - 4u(l)M^3(l) + 2(n+2)K_4 u(l)M(l) \\ & - 6K_4 u(l)M(l)T_\parallel(l)\ln[1+T_\parallel(l)] - 2(n-1)K_4 u(l)M(l)T_\perp(l)\ln[1+T_\perp(l)]. \end{aligned} \tag{3.28}$$

B. Reduced Ising-like Hamiltonian

The leading-order solutions of the recursion relations just obtained are valid, as explained after (3.5), provided both $\tilde{r}_\parallel$ and $\tilde{r}_\perp$ remain bounded. Thus we will use the solutions up to a value $l = l^*(t_0, g_0, M_0)$ which may, in view of the discussion following (3.27), be defined explicitly by

$$T_\perp(l^*) = t(l^*) + n^{-1}g(l^*) + 4u(l^*)M^2(l^*) = 1. \tag{3.29}$$

Of course, it would suffice to set $T_\perp(l^*)$ equal to any number of order unity in order to integrate out the perpendicular spin field: the number 1 is chosen here purely for convenience. In addition, we must require that $\tilde{r}_\parallel(l^*)$, and, hence, $T_\parallel(l^*)$, is not too large. The region of the (t_0, g_0) plane where this requirement is met will be discussed in Sec. III D.

At the value $l = l^*$ (which is determined explicitly in Sec. III C), we may perform the trace over the perpendicular spin components, treating the nonquadratic terms by perturbation theory. Taking the partial trace, we generate terms in the new, reduced Hamiltonian which involve only the parallel spin components. The first of these is a constant, which will enter only the free energy. Contributions corresponding to the diagrams of Fig. 2(a) modify the quadratic coefficient $\tilde{r}_{\parallel}(l^*)$, while the field $\tilde{h}(l^*)$ is changed via terms arising from the diagrams in Fig. 2(b). Further contributions which modify $h(l^*)$ and $w(l^*)$ are of higher order in ϵ and $u(l^*)$ and will be neglected. In this way, we obtain a reduced Hamiltonian of Ising type, namely

$$\overline{\mathcal{H}}_{\mathrm{red}} = -\int d\vec{\mathrm{R}}\left[\tfrac{1}{2}(\nabla\sigma)^2 + \tfrac{1}{2}\tilde{r}_{\mathrm{red}}\sigma^2 + w_{\mathrm{red}}\sigma^3 + u_{\mathrm{red}}\sigma^4 - \tilde{h}_{\mathrm{red}}\sigma + \mathrm{const}\right], \tag{3.30}$$

where

$$\begin{aligned}\tilde{r}_{\mathrm{red}} &= \tilde{r}_{\parallel}(l^*) - 32(n-1)K_4u^2(l^*)M^2(l^*)\int_0^1 \frac{q^3dq}{[T_{\perp}(l^*)+q^2]^2} + 4K_4(n-1)u(l^*)\int_0^1\frac{q^3dq}{T_{\perp}(l^*)+q^2}\,,\\ &= T_{\parallel}(l^*) - 6K_4u(l^*)\{1 - T_{\parallel}(l^*)\ln[1+T_{\parallel}(l^*)]\}\\ &\quad + 144K_4u^2(l^*)M^2(l^*)\{\ln[1+T_{\parallel}(l^*)] + T_{\parallel}(l^*)/[1+T_{\parallel}(l^*)]\} + 16K_4(n-1)u^2(l^*)M^2(l^*)\,,\end{aligned} \tag{3.31}$$

$$w_{\mathrm{red}} = 4u(l^*)M(l^*) + O(\epsilon^{3/2})\,,\quad u_{\mathrm{red}} = u(l^*)\,, \tag{3.32}$$

and

$$\begin{aligned}\tilde{h}_{\mathrm{red}} &= \tilde{h}(l^*) - 4K_4(n-1)u(l^*)M(l^*)\int_0^1\frac{q^3dq}{T_{\perp}(l^*)+q^2}\\ &= h_0\exp[(3-\tfrac{1}{2}\epsilon)l^*] - [t(l^*) - (n-1)g(l^*)/n]M(l^*) - 4u(l^*)M^3(l^*) + 6K_4\,u(l^*)M(l^*)\{1 - T_{\parallel}(l^*)\ln[1+T_{\parallel}(l^*)]\}.\end{aligned} \tag{3.33}$$

We shall focus attention on the situation where the magnetic field h_0 is zero. (Note that h_0 being the ordering field, would correspond to a *staggered* uniaxial field in a bicritical antiferromagnetic system.)

The reduced Hamiltonian described above can now, in turn, serve as the initial Hamiltonian in a renormalization-group analysis. The problem of determining the thermodynamic functions for an Ising system below T_c has been treated by Rudnick and Nelson.[11] However, their solutions are expressed in terms of parameters r_I (or t_I), u_I, and M_I defined with respect to an original *unshifted* Hamiltonian [with no cubic interaction term such as appears in (3.30)]. In order to utilize their results, we must thus identify r_I, u_I, and M_I in such a way that $\tilde{r}_{\mathrm{red}}$, $\tilde{h}_{\mathrm{red}}$, w_{red}, and u_{red} do indeed result from shifting the spin of an appropriate zero-field Ising Hamiltonian by an amount M_I. If we try a representation of the form

$$\begin{aligned}\tilde{r}_{\mathrm{red}} &= r_I + 12uM_I^2\,,\\ w_{\mathrm{red}} &= 4u_IM_I\,,\\ \tilde{h}_{\mathrm{red}} &= h_I - r_IM_I - 4u_IM_I^3 = -r_IM_I - 4u_IM_I^3\end{aligned} \tag{3.34}$$

(since we take $h_0=0$, it follows that $h_I=0$), it is straightforward to make the leading-order identifications

$$\begin{aligned} t_I &\approx t(l^*) - (n-1)g(l^*)/n \equiv \bar{t}\,,\\ u_I &\approx u(l^*)\,,\quad M_I \approx M(l^*)\,.\end{aligned} \tag{3.35}$$

In order to determine the reduced variables to first order in ϵ (as we require) we postulate the forms

$$\begin{aligned} t_I &= \bar{t}(1+x\epsilon)\,,\\ u_I &= u(l^*)(1+a\epsilon)\,,\quad M_I = M(l^*)(1+b\epsilon)\,,\end{aligned} \tag{3.36}$$

where x, a, and b are numerical coefficients. In Appendix B we determine these coefficients and find

$$x = \frac{1}{2}\left(\frac{n-1}{n+8}\right)\epsilon\,,\quad a = \left(\frac{n-1}{n+8}\right)\epsilon\,,\quad b = -\frac{1}{4}\left(\frac{n-1}{n+8}\right)\epsilon\,. \tag{3.37}$$

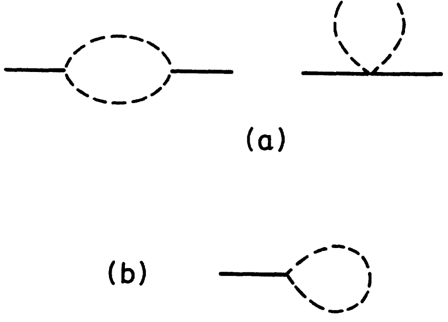

FIG. 2. Feynmann graphs needed in the calculation (a) of $\tilde{r}_{\mathrm{red}}$ and (b) of $\tilde{h}_{\mathrm{red}}$. The broken lines represent transverse propagators $[r_{\perp}(l^*)+q^2]^{-1}$.

C. Determination of l^*

For given t_0, g_0, and M_0, we can determine $l^*(t_0, g_0, M_0)$ from the definition $T_{\perp}(l^*)=1$ by methods developed in I. However, M_0 is not a known initial parameter: hence we must determine M_0 as a function of t_0, g_0, and u_0. We need only do this approximately here; the full expression for M_0 is given in Sec. IV.

We will solve (3.29) by approximating the term $4u(l^*)M^2(l^*)$. It follows from (3.38) that, to leading order in ϵ, we can make the replacement,

$$4u(l^*)M^2(l^*) \approx 4u_I M_I^2 . \tag{3.38}$$

Then to zeroth order in ϵ, we can simply replace M_I by its mean-field value, namely

$$M_I \approx (-t_I/4u_I)^{1/2} , \tag{3.39}$$

which gives, finally,

$$4u(l^*)M^2(l^*) \approx -t_I . \tag{3.40}$$

Substitution of the expression (3.35) for t_I into (3.29) gives a very simple condition determining l^*, namely,

$$g(l^*) = 1 . \tag{3.41}$$

We have, of course, neglected $O(\epsilon)$ corrections to this result throughout. However, as discussed following (3.29), the precise choice of the matching condition is immaterial, so these terms at $l = l^*$ cannot alter our scaling-function results in order ϵ.

So far we have allowed a general initial (isotropic) quartic coupling constant u_0 of order ϵ; the analysis, however, simplifies considerably if we take

$$u_0 = u^*(n) = \epsilon/[4K_4(n+8)] ,$$

namely the fixed point value of u_0 for the isotropic case ($g_0 \equiv 0$). With this special choice, (3.41) simplifies further to

$$g_0 e^{\lambda_2 l^*} = 1 , \quad \lambda_2 = 2 - 2\epsilon/(n+8) + O(\epsilon^2) . \tag{3.42}$$

In addition, the function $t(l)$ now behaves simply as

$$t(l^*) = t_0 e^{\lambda_1 l^*} , \quad \lambda_1 = 2 - (n+2)\epsilon/(n+8) + O(\epsilon^2) . \tag{3.43}$$

The results in I were presented in terms of the variable

$$z \equiv g_0/t_0^{\phi} , \quad \phi = 1 + n\epsilon/2(n+8) + O(\epsilon^2) . \tag{3.44}$$

However, in the bicritical ordered phase there is a line on which $t_0 \rightarrow 0$ for finite g_0 (see Fig. 1), and $z \rightarrow \infty$. Accordingly it will be convenient to exhibit our new results in terms of the complementary variable which remains finite in the region under discussion, namely,

$$y = t_0 g_0^{-1/\phi} / \dot{z}^{-1/\phi} = (z/\dot{z})^{-1/\phi} . \tag{3.45}$$

The relation

$$z = g_0/t_0^{\phi} = \dot{z} , \tag{3.46}$$

where

$$\dot{z} \approx n^{\phi}/(n-1)^{\phi} , \tag{3.47}$$

determines the upper critical line.[1,5] The variable y ranges from $+1$ on the boundary between the ordered and disordered phases, i.e., the critical line $t_c^{\parallel}(g)$, to $-\infty$ on the first-order flop line ($g_0 = 0$, $t_0 < 0$). The region of large negative y, however, will be excluded from the analysis by the restriction discussed in the following section.

The condition (3.42) determining l^* can, of course, be solved trivially to give

$$e^{l^*} = g_0^{-1/\lambda_2} . \tag{3.48}$$

With this expression for l^*, the temperature-like variable entering the effective Ising problem follows from (3.36) and (3.37), as

$$\begin{aligned} t_I &= \bar{t}(l^*)\left(1 + \frac{1}{2}\frac{n-1}{n+8}\epsilon\right) \\ &= [t_0 e^{\lambda_1 l^*} - (n-1) g_0 e^{\lambda_2 l^*}/n]\left[1 + \left(\frac{n-1}{n+8}\right)\frac{\epsilon}{2}\right] \\ &= \frac{n-1}{n}(y-1)\left(1 + \frac{n-1}{n+8}\,\frac{\epsilon}{2}\right) . \end{aligned} \tag{3.49}$$

D. Restriction $T_{\parallel}(l^*) \lesssim O(1)$

We now determine analytically the region of the (t_0, g_0) plane excluded from our analysis by the restriction $T_{\parallel}(l^*) \lesssim O(1)$. At $l = l^*$, we have

$$g(l^*) = 1 , \quad t(l^*) = t_0 g_0^{-1/\phi} = (n-1)y/n , \tag{3.50}$$

while $T_{\parallel}(l^*)$ was defined in (3.24). Using (3.40), we find that $T_{\parallel}(l^*)$ may be represented to leading order by

$$T_{\parallel}(l^*) = 2(n-1)(1-y)/n . \tag{3.51}$$

Provided that $y > -|y_{\min}|$, where $|y_{\min}|$ is a constant of order unity, $T_{\parallel}(l^*)$ is bounded above and of order unity. This constraint excludes the shaded region shown in Fig. 1. Our calculations break down asymptotically as this region is penetrated.

It is interesting to note that precisely in this region Bruce and Aharony[10] found distinct new behavior when they introduced the cubic interaction term (1.2). For the appropriate sign of this term, a new, "intermediate" ordered phase appears, *within* the Ising-like ordered phase shown in Fig. 1, in which the perpendicular components of the spin also ordered and the bicritical point takes on a tetracritical aspect.[5,9]

Difficulties similar to the large negative y restriction were encountered by Brézin, Wallace and Wilson,[7] in their original Feynman-graph calculation of the isotropic equation of state for $n \geq 2$. Indeed their analysis also breaks down asymptotically upon approaching the ferromagnetic coexistence curve. The difficulties they encountered were later resolved by a parquet-graph resummation procedure,[8] and it is likely that the same sort of analysis could be utilized here. However, we will defer such an analysis to the future,[9] and proceed to a description of the ordered-phase results obtained thus far.

IV. THERMODYNAMIC FUNCTIONS

A. Spontaneous magnetization

In order to determine the spontaneous magnetization, we recall from (A22) in Appendix A that the magnetization of the reduced Ising-like system is

$$M_I = \frac{(-t_I)^{\beta_I}}{2\sqrt{u_I}} R^{1/3}(-t_I)[1 - 6K_4 u_I \ln 2/R(-t_I)], \qquad (4.1)$$

with (choosing $u_0 = u_n^* = u_I$)

$$R(x) = \frac{n-1}{n+8} x^{\epsilon/2} + \frac{9}{n+8}, \quad \beta_I = \tfrac{1}{2} - \tfrac{1}{6}\epsilon + O(\epsilon^2). \quad (4.2)$$

Using the relations (3.36) and (3.37) for t_I, u_I, and M_I then yields

$$M(l^*) = \frac{1}{2\sqrt{u_0}} [-\bar{t}(l^*)]^{\beta_I} R^{1/3}[-\bar{t}(l^*)] \times \left(1 - \frac{3\epsilon \ln 2}{2(n+8) R(-\bar{t}(l^*))}\right). \qquad (4.3)$$

The spontaneous magnetization M_0 can now be found using (A11) in the form

$$M_0(t_0, g_0) = M(l^*) \exp[-(1 - \tfrac{1}{2}\epsilon) l^*], \qquad (4.4)$$

where $M(l^*)$ is given by (4.3) and by (3.48). In terms of the normalized scaling variable $y = Bt_0/g_0^{1/\phi}$ defined in (3.45) (which takes the value 1 on the critical line) we finally have

$$M_0(t_0, g_0) \approx g_0^{\beta_H/\phi} \tilde{W}(y), \qquad (4.5)$$

where β_H is the isotropic (Heisenberg-like) magnetization exponent appearing here via

$$\frac{\beta_H}{\phi} = \frac{2 - \epsilon + O(\epsilon^2)}{2\lambda_2}, \quad \lambda_2 = 2 - \frac{2\epsilon}{n+8} + O(\epsilon^2) \qquad (4.6)$$

while the scaling function is

$$\tilde{W}(y) = \tfrac{1}{2} u_0^{-1/2} \left(\frac{n-1}{n}\right)^{1/2} (1-y)^{\beta_I} R^{1/3}(1-y) \times \left(1 - \frac{3\epsilon \ln[2(1-n^{-1})]}{2(n+8) R(1-y)}\right). \qquad (4.7)$$

Note the factor $(1-y)^{\beta_I}$, where β_I is the Ising-like magnetization exponent, which ensures correct behavior as the critical line is approached from below at fixed $g_0 > 0$.

B. Susceptibility

Upon using the simple multiplicative recursion relation[1,11]

$$\chi = e^{2l^*} \chi(l^*), \qquad (4.8)$$

for the parallel susceptibility χ, in conjunction with (A23), and substituting for t_I from (3.49), and for e^{l^*} from (3.48) we obtain the result

$$\chi(t_0, g_0) \approx g_0^{-\gamma_H/\phi} \tilde{X}_{\parallel}^{-}(y), \qquad (4.9)$$

where γ_H is the isotropic (Heisenberg-like) susceptibility exponent entering here in the form

$$\frac{\gamma_H}{\phi} = \frac{2-\eta}{\lambda_2} = \frac{2}{\lambda_2} + O(\epsilon^2), \qquad (4.10)$$

while the crossover scaling function is given by

$$\tilde{X}_{\parallel}^{-}(y) = \frac{1}{2}\left(\frac{n}{n-1}\right)(1-y)^{-\gamma_I} R^{1/3}(1-y) \times \left(1 - \frac{(n-1)}{2(n+8)}\epsilon - \frac{3[3 + \ln 2(1-n^{-1})]}{2(n+8)R(1-y)}\epsilon\right), \qquad (4.11)$$

where $R(x)$ is defined in (4.2). (The tilde on the scaling function indicates use of the scaling variable y is place of x: see below.) The factor $(1-y)^{-\gamma_I}$ with Ising-like susceptibility exponent

$$\gamma_I = 1 + \tfrac{1}{6}\epsilon + O(\epsilon^2), \qquad (4.12)$$

yields correct behavior on the approach to the critical line.

Now we may check this result for the crossover scaling function, $\tilde{X}_{\parallel}^{-}(y)$, describing the ordered-phase susceptibility by computing the amplitude ratio of susceptibilities calculated above and below the transition.[13] This ratio should be a universal constant independent of the details of the model, and, in particular, independent of n. The result derived in I for the disordered-phase susceptibility may[1] be written

$$\chi(t_0, g_0) \approx t_0^{-\gamma_H} X_{\parallel}^{+}(x), \qquad (4.13)$$

where the scaling function is

$$X_{\parallel}^{+}(x) = (1-x)^{-\gamma_I} R^{1/3}(x) \times \left(1 - \epsilon\Delta_1(x) + \frac{\epsilon}{2}\,\frac{m+2}{n+8} \ln[1 + mx/(n-m)] \Big/ R(x)\right), \qquad (4.14)$$

with

$$\Delta_1(x) = -\frac{\epsilon}{2(1-x)}\left[\left(\frac{2x-n-2}{n+8}\right)\ln\left(1+\frac{m}{n-m}x\right) + \frac{n}{n+8}x\ln\left(\frac{n}{n-m}\right)\right], \quad (4.15)$$

while the normalized scaling variable is now

$$x = \frac{z}{\dot{z}} \approx \frac{g_0}{t_0^{\phi}} \Big/ \left(\frac{n}{n-m}\right)^{\phi}. \quad (4.16)$$

The algebra involved in taking the limits $y \to 1$ and $x \to 1$ in (4.11) and (4.14) is straightforward but tedious. In the ordered phase we find for this limit,

$$\chi \approx C_\infty^- |t_0|^{-1-\epsilon(n+2)/2(n+8)} (x-1)^{-\gamma_I}, \quad (4.17)$$

with

$$C_\infty^- = \frac{1}{2}\left(\frac{9}{n+8}\right)^{1/3}\left\{1 - \tfrac{1}{6}\epsilon\ln 2 - \tfrac{1}{2}\epsilon - \frac{\epsilon}{2}\left[\frac{n+2}{n+8}\ln\left(\frac{n}{n-1}\right) - \frac{1}{n+8} + \tfrac{1}{3}\ln\left(\frac{n}{n-1}\right)\right]\right\}, \quad (4.18)$$

while in the disordered phase we have

$$\chi \approx C_\infty^+ t_0^{-1-\epsilon(n+2)/2(n+8)} (1-x)^{-\gamma_I}, \quad (4.19)$$

with

$$C_\infty^+ = \left(\frac{9}{n+8}\right)^{1/3}\left\{1 - \frac{\epsilon}{2}\left[\frac{n+2}{n+8}\ln\left(\frac{n}{n-1}\right) - \frac{1}{n+8} - \tfrac{1}{3}\ln\left(\frac{n}{n-1}\right)\right]\right\}. \quad (4.20)$$

The amplitude ratio on the critical line is hence given by

$$C^+/C^- = C_\infty^+/C_\infty^- = 2[1 + \tfrac{1}{6}\epsilon\ln 2 + \tfrac{1}{2}\epsilon + O(\epsilon^2)], \quad (4.21)$$

in precise agreement with the results of Brézin *et al.*[13] for Ising-like systems. It is especially gratifying that this agreement is obtained from the very elaborate functional forms for the crossover scaling functions displayed in (4.11) and (4.14). The appearance of the scaling functions will be discussed in Sec. V.

C. Free energy and specific heat

The free energy is given in terms of a trajectory integral by[11]

$$F = \tfrac{1}{2}r_\parallel M_0^2 + u_0 M_0^4 + \int_0^{l^*} dl\, e^{-dl}G(l) + e^{-dl}F(l^*), \quad (4.22)$$

where

$$G(l) = \tfrac{1}{2}K_4(\ln[1+\tilde{r}_\parallel(l)] - \tfrac{1}{2} + (n-1)\{\ln[1+\tilde{r}_\perp(l)] - \tfrac{1}{2}\}). \quad (4.23)$$

This trajectory integral can be evaluated along lines devised by Nelson and Rudnick.[11] It consists of pieces that are regular or cancel against contributions from $e^{-dl^*}F(l^*)$, and of a piece that gives the leading singular behavior, namely

$$I = -\frac{K_4}{d}\int_0^{l^*} [\tilde{r}_\parallel^2(l) + (n-1)\tilde{r}_\perp^2(l)]\, e^{-dl}\, dl. \quad (4.24)$$

This integral can be evaluated to yield various regular terms, a term that cancels the terms $\frac{1}{2}r_\parallel M_0^2 + u_0 M_0^4$ in (4.22), and the contribution

$$\begin{aligned} I^{(1)} &= \frac{1}{4du_0}\left(\frac{n}{4-n}t_0^2 e^{(4-n)\epsilon l^*/(n+8)} + \frac{n-1}{n(n+4)} g_0^2 e^{(n+4)\epsilon l^*/(n+8)}\right) \\ &\quad - e^{dl^*}[u_0 M^4(l^*) + \tfrac{1}{2}\bar{t}(l^*) M^2(l^*)], \\ &\equiv I_1^{(1)} + I_2^{(1)}. \end{aligned} \quad (4.25)$$

We have also dropped certain regular terms needed to give a finite result here in the limit $n \to 4$. The factor $F(l^*)$ in (4.22) may be calculated by first integrating over the perpendicular components, which yields a fluctuation term that serves to cancel parts of the trajectory integral not included in (4.24). This leaves the free energy of the reduced Ising-like Hamiltonian, which contains the terms $-\frac{1}{2}t_I M_I^2 - u_I M_I^4$. These must completely cancel $I_2^{(1)}$ in (4.25). (Otherwise the free energy would contain spurious terms like $tM^2 \sim t^{1+2\beta}$, which are more singular than $t^{2-\alpha}$.) Specifically, we must have

$$\bar{t}(l^*) M^2(l^*) = t_I M_I^2, \quad (4.26)$$

$$u_0 M^4(l^*) = u_I M_I^4. \quad (4.27)$$

These equations can, in turn, be used to find a, b, and x in (3.36). As shown in Appendix B, one obtains *four* equations for these three parameters, thus providing a consistency check on this approach. The remaining part of $F(l^*)$ is the Ising-like free energy first derived by Nelson and Rudnick,[11] namely,

$$F_I = -(1/3d)u_I^{-1}(-t_I)^{2-\epsilon/6}R^{1/3}(-t_I) \times [1 - 6u_I K_4 \ln 2/R(-t_I)]. \quad (4.28)$$

Again substituting for u_I and t_I we obtain finally

$$
\begin{aligned}
F &= I_1^{(1)} + e^{-dl^*}F_I \\
&= -\frac{K_4(n+8)}{4\epsilon}\, g^{2-\epsilon(n+4)/2(n+8)} \left(\frac{n-1}{n}\right)^2 \\
&\quad \times [Q_1(y) + (1-y)^{2-\epsilon/6} Q_2(y)]
\end{aligned}
\tag{4.29}
$$

where

$$
Q_1(y) = \frac{n}{4-n}\, y^2 + \frac{n}{(n-1)(n+4)} - \tfrac{1}{3}(y-1)^2 ,
\tag{4.30}
$$

$$
Q_2(y) = \tfrac{4}{3}[R(1-y)]^{1/3} \left(1 - \frac{3\epsilon \ln[2(1-n^{-1})]}{2(n+8)R(1-y)}\right) .
\tag{4.31}
$$

The specific heat can be obtained by calculating $-(\partial^2 F/\partial t_0^2)$ using (4.29). We find that the leading singular part may be written

$$
\Delta C(t_0, g_0) \approx g_0^{-\alpha_H/\phi} \tilde{Z}^-(y) ,
\tag{4.32}
$$

where the isotropic specific-heat exponent is given by

$$
\alpha_H = (4-n)\epsilon/2(n+8) + O(\epsilon^2) ,
\tag{4.33}
$$

while the crossover scaling function is

$$
\tilde{Z}^-(y) = \frac{K_4(n+8)}{2\epsilon} \left(\frac{n}{4-n} - \tfrac{1}{3} + (1-y)^{-\alpha_I} Q_2(y)\right)
\tag{4.34}
$$

where the Ising-like specific-heat exponent is

$$
\alpha_I = \tfrac{1}{6}\epsilon + O(\epsilon^2) ,
\tag{4.35}
$$

while $Q_2(y)$ is defined in (4.31).

For $t_0 > 0$ we can express the specific heat in terms of the scaling variable x, defined in (4.16) as

$$
\Delta C(t_0, g_0) \approx t_0^{-\alpha_H} Z^-(x) ,
\tag{4.36}
$$

$$
\begin{aligned}
Z^-(x) = \frac{K_4(n+8)}{2\epsilon} \Bigg[& \frac{n}{4-n} - \frac{1}{3} - \frac{2\epsilon}{3}\, \frac{n-1}{n+8} \ln\left(\frac{nx}{n-1}\right) \\
& + (x-1)^{-\epsilon/6} Q_3(x) \Bigg] ,
\end{aligned}
\tag{4.37}
$$

where

$$
\begin{aligned}
Q_3(x) = \tfrac{4}{3}[R(x-1)]^{1/3} & \\
\times \Bigg[1 &+ \frac{3\epsilon}{2(n+8)R(x-1)} \ln\left(\frac{nx}{2(n-1)}\right) \\
&- \frac{4-n}{n+8}\, \frac{\epsilon}{2} \ln\left(\frac{nx}{n-1}\right) \Bigg] .
\end{aligned}
\tag{4.38}
$$

It is now straightforward to compute the specific-heat amplitude ratio from the disordered-phase expression for the specific heat given in I. Specifically, we verify that

$$
A^+/A^- = \tfrac{1}{4}[1 + O(\epsilon)] ,
\tag{4.39}
$$

which is independent of n along the critical line. This result agrees with work by Brézin *et al.*[13] to lowest order in ϵ. Because the specific-heat amplitude ratio is undefined to zeroth order in ϵ (mean-field theory gives a jump discontinuity), A^+/A^- is determined correctly to one less order in ϵ than is C^+/C^-. This peculiarity was, indeed, noted by Brézin and co-workers.[13]

D. Nonordering susceptibility

In order to calculate the nonordering susceptibility,

$$
\chi_g = -\left(\frac{\partial^2 F}{\partial g_0^2}\right) ,
\tag{4.40}
$$

it is convenient to use (4.28) and (4.24) in the form

$$
\begin{aligned}
F = -\frac{K_4(n+8)}{\epsilon} \Bigg(& \frac{n}{n-4}\, t_0^2 e^{(4-n)\epsilon l^*/(n+8)} \\
& + \frac{n-1}{n(n+4)}\, g_0^2 e^{(4+n)\epsilon l^*/(n+8)} \Bigg) \\
& + e^{-dl^*} F_I(-t_I) .
\end{aligned}
\tag{4.41}
$$

On differentiating this expression, ignoring the g dependence of l^* (since F must be independent of the precise choice of l^*), we find

$$
\chi_g(t_0, g_0) \approx g_0^{-\tilde{\gamma}/\phi} \tilde{Y}^-(y) ,
\tag{4.42}
$$

where the isotropic exponent is

$$
\tilde{\gamma} = -(2 - \alpha_H - 2\phi) = \frac{1}{2}\left(\frac{n+4}{n+8}\right)\epsilon + O(\epsilon^2) ,
\tag{4.43}
$$

while the scaling function is given by

$$
\begin{aligned}
\tilde{Y}^-(y) = \frac{K_4(n+8)}{2\epsilon} \Bigg[& \frac{n-1}{n(n+4)} - \frac{(n-1)^2}{3n^2}\, y^2 \\
& + \frac{(n-1)^2}{n^2}\left(1 - \frac{n\epsilon}{n+8}\right) y^2 Q_2(y) \Bigg] ,
\end{aligned}
\tag{4.44}
$$

with $Q_2(y)$ again defined as in (4.31). For $t_0 > 0$ this may be written in terms of the alternate scaling variable introduced in (4.16), as

$$
\chi_g \approx t_0^{-\tilde{\gamma}} Y^-(x) ,
\tag{4.45}
$$

where

$$
Y^-(x) = \frac{K_4(n+8)}{2\epsilon} \left(\frac{n-1}{n(n+4)} - Q_4(x) + (x-1)^{-\alpha_I} Q_5(x)\right) ,
\tag{4.46}
$$

where

$$Q_4(x)=\frac{1}{3}\left(\frac{n-1}{nx}\right)^2\left(1+\frac{n\epsilon}{n+8}\ln x\right)+\frac{n-1}{n(n+8)}\,\frac{\epsilon}{2}\ln\left(\frac{nx}{n-1}\right)-\frac{1}{3}\left(\frac{n-1}{nx}\right)^2\frac{n+4}{n+8}\,\frac{\epsilon}{2}\ln\left(\frac{nx}{n-1}\right), \tag{4.47}$$

$$Q_5(x)=\frac{4}{3}\left(\frac{n-1}{nx}\right)^2[R(x-1)]^{1/3}\left[1+\frac{3\epsilon}{2(n+8)R(x-1)}\ln\left(\frac{nx}{2(n-1)}\right)-\frac{n\epsilon}{n+8}+\frac{n\epsilon}{n+8}\ln x-\frac{n+4}{n+8}\,\frac{\epsilon}{2}\ln\left(\frac{nx}{n-1}\right)\right]. \tag{4.48}$$

Note that the nonordering susceptibility diverges like the specific heat on the critical line $t_c(g_0)$, i.e., the scaling function diverges at $x=1$ with exponent α_I. By comparison with I we may again check that the amplitude ratios above and below $t_c(g_0)$ obey (4.39). The graphical appearance of the χ_g scaling functions will also be discussed in the next section.

E. Disordered-phase transverse susceptibility

We take the opportunity to present here the scaling functions for the *transverse* susceptibility in the *disordered* phase. (Horner[3] has also calculated $\chi_\perp^{(m)}$ to order ϵ for the special case $m=1, n=2$.) The derivation of this function was sketched in Appendix B of I. The transverse susceptibility is defined by

$$\chi_\perp^{(m)}=\frac{1}{n-m}\langle\vec{s}_\perp(\vec{q})\vec{s}_\perp(\vec{q}\,')\rangle\Big/\delta(\vec{q}+\vec{q}\,')\Big|_{\vec{q}=0}, \tag{4.49}$$

and can be represented[1] as

$$\chi_\perp^{(m)}=e^{2l^*}[1-4u(l^*)E_m(r_{\rm red},u(l^*))] \tag{4.50}$$

where

$$E_m(r_{\rm red},u)=2\frac{\partial F_m^{(s)}(r_{\rm red},u)}{\partial r_{\rm red}}. \tag{4.51}$$

After some tedious but straightforward algebra, we obtain

$$\chi_\perp^{(m)}(t_0,g_0)\approx t_0^{-\gamma_H}X_\perp^{(m)}(x), \tag{4.52}$$

with scaling function

$$X_\perp^{(m)}(x)=Q_6(x)+\frac{m}{4-m}(1-x)^{1-\alpha_m}Q_7(x), \tag{4.53}$$

to first order in ϵ, where x is still defined as in (4.16), α_m is the specific-heat exponent for an m-component system, and

$$Q_6(x)=[\psi_0(x)]^2[1+\epsilon\tilde{\psi}(x)], \tag{4.54}$$

$$Q_7(x)=[\psi_0(x)]^4\left[[R(1-x)]^{2\alpha_m}\left(1-\epsilon\frac{4-m}{n+8}\,\frac{\ln\psi_0(x)}{R(1-x)}+\epsilon\Delta_1(x)+2\epsilon\tilde{\psi}(x)\right)-(1-x)^{\alpha_m}[1+\epsilon\Delta_1(x)+2\tilde{\psi}(x)]\right], \tag{4.55}$$

with

$$\Delta_1(x)=-\frac{1}{2}\left[\frac{2x-n-2}{n+8}\ln\left(1+\frac{m}{n-m}x\right)+\frac{nx}{n+8}\ln\left(\frac{n}{n-m}\right)\right]\Big/(1-x), \tag{4.56}$$

$$\tilde{\psi}(x)=\frac{-[\psi_0(x)]^2}{2(n-m)(n+8)}\left[nmx\ln\left(\frac{n}{n-m}\right)+[(n-m)(n+2)+2mx]\ln\left(1+\frac{m}{n-m}x\right)\right], \tag{4.57}$$

$$\psi_0(x)=\left(1+\frac{m}{n-m}x\right)^{-1/2}. \tag{4.58}$$

Now the susceptibility $\chi_\perp^{(m)}(t_0,g_0)$ in the disordered phase for $g_0>0$ should be just the analytic continuation of $\chi_\parallel^{(n-m)}(t_0,g_0)$ [obtained when the "perpendicular" $(n-m)$-component field is dominant] to $g_0<0$; the same should apply to the scaling functions $X_\perp^{(m)}(x)$ and $X_\parallel^{(n-m)}(x)$. However, the two functions are obtained in our calculations by using different procedures in different regions of the (t_0, g_0) plane. Nevertheless, since all our calculations are correct to first order in ϵ, we expect the two functions to be identical when *completely* expanded in powers of ϵ for fixed x (not equal to $\dot{x}$ or $\ddot{x}$) and when terms higher than first order in ϵ are neglected. This is indeed the case; in terms of the variable $z=\dot{z}x=g_0/t_0^{\phi}$ we find

$$X_{\perp}^{(m)}(z)=\frac{1}{1+mz/n}\left\{1-\frac{\frac{1}{2}\epsilon}{1+mz/n}\left[\frac{n-m+2}{n+8}\left(1+\frac{m}{n}z\right)\ln\left(1+\frac{m}{n}z\right)+\frac{m}{n+8}\left(1-\frac{n-m}{n}z\right)\ln\left(1-\frac{n-m}{n}z\right)\right]\right\}$$
$$+O(\epsilon^2),$$
$$=X_{\parallel}^{(n-m)}(-z)+O(\epsilon^2), \tag{4.59}$$

as expected. It is interesting to note that although the two functions do differ at $O(\epsilon^2)$ the differences are quite small numerically even at $\epsilon=1$. Thus graphically (see Fig. 3 below) the two plots join quite smoothly through $x=0$. Specifically we find

$$X_{\parallel}^{(m)}(z)-X_{\perp}^{(n-m)}(-z)=K_1(n,m)\epsilon^2 z-K_2(n,m)\epsilon^2 z^2+O(\epsilon^2 z^2,\epsilon^3 z) \tag{4.60}$$

where

$$K_1(n,m)=\left[(n-m)\ln\left(\frac{n}{m}\right)-n\ln\left(\frac{n}{n-m}\right)\right]\Big/2(n+8)^2, \tag{4.61}$$

while $K_2(n,m)$ is given by a similar but much longer expression. For $n=3$, $m=1$ we have the surprisingly small numerical values

$$K_1(3,1)=\tfrac{1}{242}\ln\tfrac{8}{3}\simeq 0.004\,05, \tag{4.62}$$

$$K_2(3,1)=-\tfrac{31}{2376}-\tfrac{7}{363}\ln 3-\tfrac{3}{121}(\ln 3)^2+\tfrac{45}{968}\ln\tfrac{3}{2}+\tfrac{27}{484}(\ln\tfrac{3}{2})^2\simeq 0.0542. \tag{4.63}$$

V. SUMMARY AND DISCUSSION

We have demonstrated that thermodynamic functions in the ordered phase of relatively complicated multicritical systems can be determined from renormalization-group recursion relations, trajectory integrals, and a "matching" approach. Although the calculations presented here are restricted to Ising-like ordered phases, they nevertheless have produced detailed information about a region of the bicritical phase diagram not previously explored by other theoretical techniques. Explicit expressions, correct to order $\epsilon=4-d$, have been obtained for the crossover scaling functions of spontaneous magnetization, ordering susceptibility, free energy, specific heat, and for the subdominant or nonordering susceptibility. [These results are contained in Sec. IV, Eqs. (4.5), (4.9), (4.29), (4.32), and (4.42).] We have also reported calculations of the transverse susceptibility in the disordered region.

Our results are essentially complete in the ordered region *above* the bicritical point ($t_0>0$; see Fig. 1) but they must be expressed in terms of the modified scaling variable $y=Bt_0/g_0^{1/\phi}$ if their analyticity across the line $t_0=0$ (or $T=T_b$) for $g_0>0$ is to be made evident. However, as the first-order bicritical phase boundary is approached below T_b (shaded region in Fig. 1) the scaling variable y approaches $-\infty$ and our results asymptotically lose validity. This experimentally interesting region has to be handled separately since the appearance of "massless Goldstone bosons" considerably complicates the analysis; however, the appropriate calculations will be postponed for later publication.[9] As explained in Sec. I, small symmetry-breaking terms of cubic or similar character may play a crucial role close to the phase boundary and, in particular, tetracritical behavior may be realized.

It is interesting to combine the present calculations for the Ising-like ordered region with the previous work in I for the disordered phases, and to examine the related scaling functions. In Fig. 3 the crossover scaling function, $X_{\parallel}(x)\approx t_0^{\gamma_H}\chi(t_0,g_0)$, for the longitudinal or parallel susceptibility is ex-

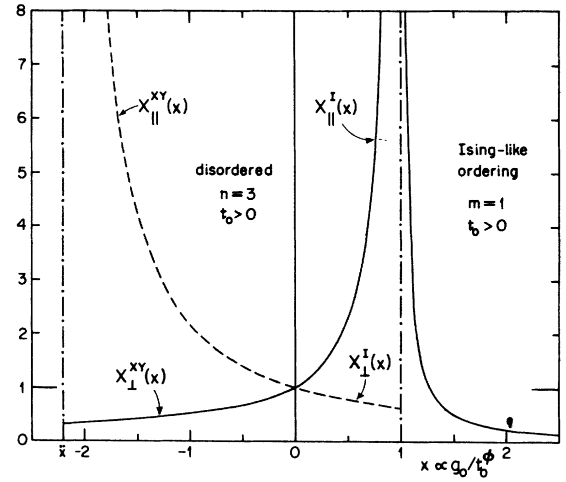

FIG. 3. Scaling functions, $X_{\parallel}(x)$ and $X_{\perp}(x)$, for the ordering and transverse susceptibilities, as a function of $x=(g_0/t_0^{\phi})/(\frac{3}{2})^{\phi}$ for crossover from Heisenberg to Ising and XY behavior ($n=3$, $m=1,2$) evaluated at $\epsilon=1$ ($d=3$). The dot-dashed lines at $\dot{x}=1$ and $\ddot{x}\simeq-2.20$ correspond to the Ising-like and XY-like critical lines (see Fig. 1).

hibited for $n=3$, $m=1$, and $\epsilon=1$, as a function of the variable $x \propto g_0/t_0^\phi$, which has been normalized to 1 on the Ising-like critical line, $t_c^I(g_0)$ [see (4.9) and I]. Note that by using the results of Sec. IV E the plot for $X_\parallel^I(x)$ may be extended smoothly to negative x ($g_0<0, t_0>0$) where it represents the *transverse susceptibility* in an XY-like system, which does not diverge. In this region it is, of course $\chi^{(m=2)} \propto X_\parallel^{XY}$ which diverges as the XY-like critical line $t_c^{XY}(g_0)$ is approached. In terms of the scaled variable x (normalized by $\dot{x}=1$) this critical line occurs at $x=\ddot{x} \approx -(n-1)^\phi \simeq -2.20$ [using the results (3.47)]. This should be contrasted with the best series estimates,[2] namely, $|\ddot{x}_3| \simeq 2.51$. In a complementary fashion the longitudinal susceptibility scaling function $X_\parallel^{XY}(x)$ for $x<0$ extends into the transverse Ising susceptibility $X_\perp^I(x)$ as displayed in Fig. 3. It should be noted that although they do not diverge, the transverse susceptibilities display energylike singularities, proportional to $(t-t_c)^{1-\alpha_I}$ as $t \to t_c$. At fixed g_0 these terms will give rise to maxima in $\chi_\perp$ above t_c and a vertical tangent *at* $t=t_c$ for $\alpha_m>0$. These singularities appear in the expressions in Sec. IV E for the crossover scaling functions. However, owing to the numerically small values of α_I and α_{XY} and the small amplitudes, they are not easily visible. This may be checked by inspection of Fig. 4 which exhibits the appropriate region of the scaling function $X_\perp^I(x)$ on a large scale.

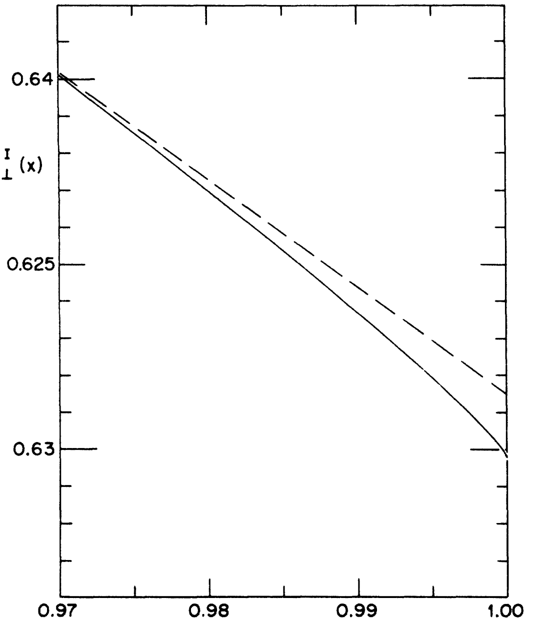

FIG. 4. Detail of the scaling function for the transverse susceptibility near the Ising-like critical point at $x=\dot{x}=1$, displaying an energy-like singularity (with a vertical tangent at the critical point). The straight dashed line is purely for comparison purposes.

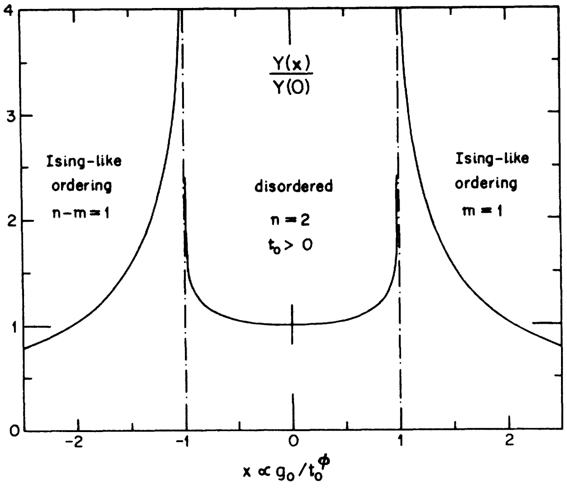

FIG. 5. Normalized scaling function $Y(x)/Y(0)$ for the nonordering susceptibility for XY to Ising crossover ($n=2$, $m=1$) evaluated at $\epsilon=1$. Note the symmetry about $x=0$.

The scaling functions for the subdominant or nonordering susceptibility $\chi_g = \partial^2 F/\partial g_0^2$ may be examined in the same way. In Fig. 5 the corresponding scaling function $Y(x)$, normalized by the value at $x=0$, is plotted at $\epsilon=1$ for the case $n=2, m=1$ where crossover occurs from XY to Ising-like behavior symmetrically for positive or negative g_0.

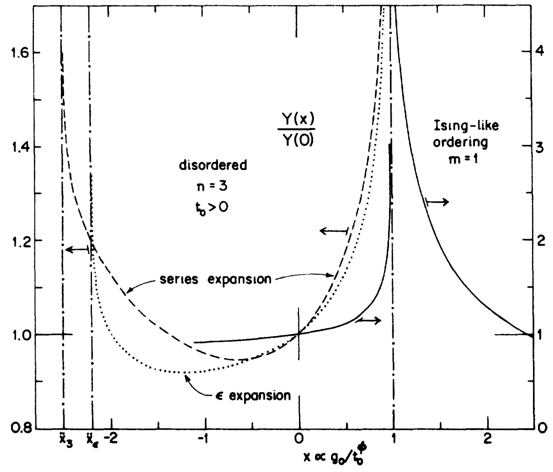

FIG. 6. Normalized scaling function for the nonordering susceptibility for Heisenberg to Ising and XY crossover ($n=3$, $m=1,2$), evaluated at $\epsilon=1$ (solid curve, right-hand scale). In the disordered region the dashed curve (and left-hand scale) shows the series-extrapolation results of Gerber and Fisher (Ref. 15) while the dotted curve shows the ϵ-expansion results on the same enlarged scale (left-hand axis). The XY critical line corresponds to $x=\ddot{x}_3$ for the series-expansion results but to $x=\ddot{x}_\epsilon$ to first order in ϵ.

In considering the size of χ_g in the ordered region it must be recalled that the ratio A^-/A^+ has the value 4 in leading order in ϵ [see (4.39)], whereas in reality the ratio for $d=3$ should be[14] about 1.6.

A complementary plot is displayed in Fig. 6 for crossover from Heisenberg ($n=3$) to Ising-like and XY-like behavior for positive and negative g_0, respectively, as in Fig. 3. Our calculations at $\epsilon=1$ (solid and dotted curves) in the disordered region may be compared with the recent calculations of Gerber and Fisher[15] (dashed curve) who extrapolated the high-temperature series expansions for the anisotropic classical Heisenberg model. For $0\leq x<1$ the agreement is fairly good — better than 6 or 7% for $x\leq 0.9$ — although the critical exponent α_I is given as $\frac{1}{6}$ in the truncated ϵ expansion in place of $\alpha_I\simeq 0.125$. On the other hand for $x<0$ the main error is in the value of $\ddot{x}$ which determines the critical line. (For $x>0$, of course, this problem is obviated by the normalization chosen.) In addition the series results, which are probably reliable to within 2%, exhibit a less pronounced minimum which lies correspondingly closer to the origin. Over-all, however, the agreement is quite encouraging considering the first-order nature of our calculations.

Finally in Fig. 7 we plot the crossover scaling function for the specific heats. Because the value of α_H at $d=3$ is negative it is convenient to write the specific heat in the modified scaling form

$$\Delta C(t_0,g_0)\approx A^+t_0^{-\alpha_H}+t_0^{-\alpha_H}Z^*(x), \tag{5.1}$$

and to plot the "subtracted scaling function" $Z^*(x)$ normalized by $(d^2Z^*/dx^2)_{x=0}$ so that the leading behavior at small x is $\frac{1}{2}x^2+O(x^3)$. The dashed curve in the disordered region represents the series-extrapolations of Gerber and Fisher.[15] Significant numerical discrepancies arise in the region $x<-1$. In comparing the ordered and disordered regions, the low accuracy of the leading-order ϵ-expansion estimate $A^-/A^+=4$ should again be borne in mind.

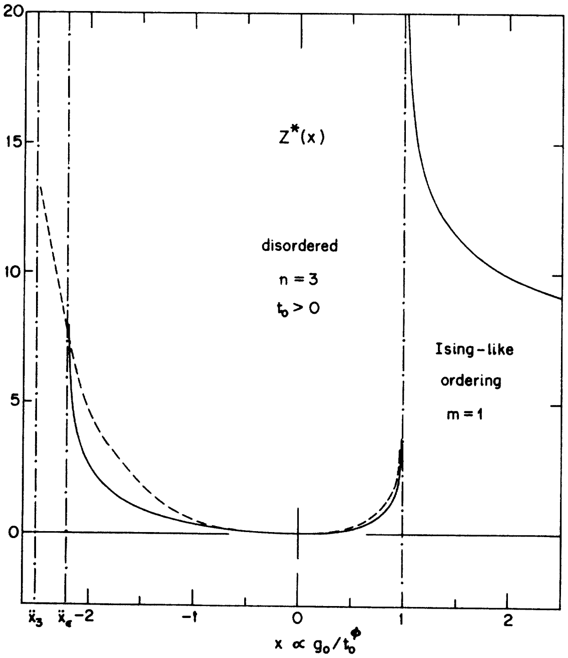

FIG. 7. Plot of the "subtracted" specific-heat crossover scaling function $Z^*(x)$ [see Eq. (5.1)] evaluated at $\epsilon=1$ (solid curve). The dashed curve represents the Gerber-Fisher (Ref. 15) numerical-series-extrapolation results for $d=3$.

When sufficiently extensive data from experiments such as Rohrer's striking work[16] on $\mathrm{GdAlO_3}$ become available, it should be possible to test our various calculations directly against real antiferromagnetic behavior. We may also hope that other physical systems which display accessible bicritical points will be studied in detail and compared with our theoretical predictions.

ACKNOWLEDGMENTS

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APPENDIX A: ORDERED-PHASE SCALING FUNCTIONS FROM RECURSION RELATIONS

Consider the Hamiltonian

$$\overline{\mathcal{H}}[s]=-\int d\vec{R}[\tfrac{1}{2}r_0s^2+\tfrac{1}{2}(\nabla s)^2+us^4-h_0s]. \tag{A1}$$

On setting $s=\sigma+M_0$, where M_0 is the *exact* magnetization, we get

$$\overline{\mathcal{H}}[\sigma]=-\int d\vec{R}[\tfrac{1}{2}r_0M_0^2+u_0M_0^4-h_0M_0+\tfrac{1}{2}\tilde{r}_0\sigma^2 + \tfrac{1}{2}(\nabla\sigma)^2+w_0\sigma^3+u\sigma^4-\tilde{h}_0\sigma], \tag{A2a}$$

where

$$\begin{aligned}\tilde{r}_0&=r_0+12uM_0^2, \qquad w_0=4u_0M_0,\\ \tilde{h}_0&=h_0-r_0M_0-4u_0M_0^3.\end{aligned} \tag{A2b}$$

It is important to note that at this point M_0 is considered a known parameter; it will be calculated at the end of the procedure (and turn out to be of order $\epsilon^{-1/2}$). The recursion relations for the shifted Hamiltonian are[11]

$$\frac{d\tilde{r}}{dl}=2\tilde{r}+\frac{12K_4u}{1+\tilde{r}}-\frac{18K_4w^2}{(1+\tilde{r})^2}, \tag{A3}$$

$$\frac{dw}{dl}=(1+\tfrac{1}{2}\epsilon)w-\frac{36K_4wu}{(1+\tilde{r})^2}+\frac{36K_4w^3}{(1+\tilde{r})^3}, \tag{A4}$$

$$\frac{du}{dl}=\epsilon u-\frac{36K_4u^2}{(1+\tilde{r})^2}+\frac{216K_4w^2u}{(1+\tilde{r})^3}-\frac{162K_4w^4}{(1+\tilde{r})^4}, \tag{A5}$$

$$\frac{d\tilde{h}}{dl}=(3-\tfrac{1}{2}\epsilon)\tilde{h}-\frac{3K_4w}{1+\tilde{r}}. \tag{A6}$$

The terms retained in these recursion relations are such as to include, for all functions, the leading power of ϵ and one higher power. [Note that $M_0=O(\epsilon^{-1/2}), u_0=O(\epsilon), w=O(\epsilon^{1/2})$.]

These equations are now solved as follows. Consider first the equation

$$\frac{d\bar{u}}{dl}=\epsilon\bar{u}-36K_4\bar{u}^2, \tag{A7}$$

with initial condition $\bar{u}(0)=u_0$; the solution is easily found to be

$$\bar{u}(l)=u_0e^{\epsilon l}/Q(l), \tag{A8}$$

$$Q(l)=1+36K_4u_0(e^{\epsilon l}-1)/\epsilon. \tag{A9}$$

Note that $\bar{u}(l)$ remains of order u_0 for all l. Then it is not hard to show that $u(l)$, the solution of Eq. (A5), is given by

$$u(l)=\bar{u}(l)+O(\epsilon^2,\epsilon\bar{u}(l)), \tag{A10}$$

provided $r(l)$ remains bounded above. Now *define* functions $M(l), \bar{w}(l), r(l), h(l)$ by

$$M(l)=M_0e^{[1-(\epsilon/2)]l}, \tag{A11}$$

$$\bar{w}(l)=4\bar{u}(l)M(l), \tag{A12}$$

$$r(l)=\tilde{r}(l)-12\bar{u}(l)M(l)^2, \tag{A13}$$

$$h(l)=\tilde{h}(l)+r(l)M(l)+4\bar{u}(l)M(l)^3. \tag{A14}$$

Substitution of (A12) into (A4) yields

$$w(l)=\bar{w}(l)[1+O(\epsilon,u)]. \tag{A15}$$

Substituting $\tilde{r}(l)$ from (A13), with (A15) and (A10) for $w(l)$ and $u(l)$ into (A3), we get an equation for $r(l)$ *correct* to $O(\epsilon)$, namely,

$$\frac{dr}{dl}=2r+12K_4\bar{u}(1-r)+\frac{12K_4\bar{u}\tilde{r}^2}{1+\tilde{r}}$$
$$+\frac{288\bar{u}^2M(l)^2(\tilde{r}^2+2\tilde{r})}{(1+\tilde{r})^2}. \tag{A16}$$

The solution, correct to $O(\epsilon)$, is

$$\tilde{r}(l)=T(l)-6K_4\bar{u}(l)\{1-T(l)\ln[1+T(l)]\}$$
$$+144K_4\bar{u}(l)^2M(l)^2\left(\ln[1+T(l)]+\frac{T(l)}{1+T(l)}\right). \tag{A17}$$

In a similar manner we get

$$\tilde{h}(l)=h_0e^{[3-(\epsilon/2)]l}-t(l)M(l)-4\bar{u}(l)M(l)^3$$
$$+6K_4\bar{u}(l)M(l)\{1-T(l)\ln[1+T(l)]\}, \tag{A18}$$

where

$$T(l)=t(l)+12\bar{u}(l)M(l)^2,$$
$$t(l)=t_0e^{2l}[Q(l)]^{1/3}, \tag{A19}$$
$$t_0=r_0-6K_4u_0.$$

These solutions are used until $\tilde{r}(l^*)=O(1)$. Since, by (A17), $\tilde{r}(l)=T(l)+O(\epsilon)$, this means $T(l^*)=K=O(1)$. Since the results are independent of the exact value[11] of K we choose for convenience $K=1$, yielding the condition

$$T(l^*)=1. \tag{A20}$$

When this is used in conjunction with

$$\langle\sigma\rangle_{\bar{\mathcal{H}}(l^*)}=0, \tag{A21}$$

we get, for $h_0=0$, the spontaneous magnetization *and* e^{l^*} as functions of t_0 and u_0; then the various thermodynamic functions are obtained. The explicit results to first order in ϵ for magnetization, susceptibility, free energy, and specific heat are

$$M_0=\frac{(-t_0)^{\beta}{}_IR_0^{1/3}}{2u_0^{1/2}}\left(1-\frac{6K_4u_0\ln2}{R_0}\right), \tag{A22}$$

$$\chi=\tfrac{1}{2}(-t_0)^{-\gamma}{}_IR_0^{1/3}\left(1-\frac{6K_4u_0\ln2}{R_0}-\frac{18K_4u_0}{R_0}\right), \tag{A23}$$

$$F_s=\frac{(-t_0)^{2-\alpha}{}_IR_0^{1/3}}{12u_0}\left(1-\frac{6u_0K_4\ln2}{R_0}-\frac{t_0^2}{4}\right), \tag{A24}$$

$$C=-2F_s/(-t_0)^2, \tag{A25}$$

with

$$R_0=R(-t_0)=[1-36K_4(u_0/\epsilon)](-t_0)^{\epsilon/2}+36K_4(u_0/\epsilon), \tag{A26}$$

and

$$\gamma_I\approx1+\tfrac{1}{6}\epsilon,\quad \beta_I\approx\tfrac{1}{2}(1-\tfrac{1}{3}\epsilon),\quad \alpha_I\approx\tfrac{1}{6}\epsilon. \tag{A27}$$

APPENDIX B: THE REDUCED HAMILTONIAN

After integrating the recursion relations to l^*, and taking a partial trace over the perpendicular spin components one is left with the reduced, Ising-like Hamiltonian

$$\bar{\mathcal{H}}_{\mathrm{red}}=-\int d\vec{\mathrm{R}}[\tfrac{1}{2}\tilde{r}_{\mathrm{red}}\sigma(\vec{\mathrm{R}})^2+\tfrac{1}{2}(\nabla\sigma)^2+u_{\mathrm{red}}\sigma(\vec{\mathrm{R}})^4$$
$$+w_{\mathrm{red}}\sigma(\vec{\mathrm{R}})^3-\tilde{h}_{\mathrm{red}}\sigma(\vec{\mathrm{R}})], \tag{B1}$$

with, as follows from (3.31)–(3.33),

$$\begin{aligned}\tilde{r}_{\text{red}} = T_{\parallel}(l^*) &- 6K_4 u(l^*)\{1 - T_{\parallel}(l^*)\ln[1 + T_{\parallel}(l^*)]\} \\ &+ 144K_4 u(l^*)^2 M(l^*)^2\{\ln[1 + T_{\parallel}(l^*)] \\ &\qquad + T_{\parallel}(l^*)/[1 + T_{\parallel}(l^*)]\} \\ &+ 16K_4(n-1)u(l^*)^2 M(l^*)^2,\end{aligned} \tag{B2}$$

$$T_{\parallel}(l^*) = \bar{t} + 12u(l^*)M(l^*)^2, \tag{B3}$$

$$u_{\text{red}} = u(l^*) = u, \quad w_{\text{red}} = 4u(l^*)M(l^*), \tag{B4}$$

and, in zero external field,

$$\begin{aligned}\tilde{h}(l^*) = -\bar{t}M(l^*) &- 4u(l^*)M^3(l^*) \\ &+ 6K_4 u(l^*)M(l^*)\{1 - T_{\parallel}(l^*)\ln[1 + T_{\parallel}(l^*)]\}.\end{aligned} \tag{B5}$$

These parameters resemble closely the initial parameters of an Ising Hamiltonian, whose spin variable has been shifted by $M(l^*)$, and whose temperaturelike variable is $\bar{t}$.

To make this idea precise we thus look for parameters t_I, u_I, and M_I such that an Ising Hamiltonian defined initially by t_I and u_I and then shifted by M_I will yield a Hamiltonian whose parameters agree with (B2)–(B5) to appropriate order (that is to order ϵ for $\tilde{r}_{\text{red}}$, to order $\epsilon^{1/2}$ for w_{red}, and order ϵ for u_{red}, and order $\epsilon^{-1/2}$ for $\tilde{h}_{\text{red}}$). On using (A17) we thus find the conditions

$$\begin{aligned}\tilde{r}_{\text{red}} = T_I &- 6K_4 u_I[1 - T_I\ln(1 + T_I)] \\ &+ 144K_4 u_I^2 M_I^2[\ln(1 + T_I) + T_I/(1 + T_I)],\end{aligned} \tag{B6}$$

$$\tilde{h}_{\text{red}} = -t_I M_I - 4u_I M_I^3 + 6K_4 u_I M_I[1 - T_I\ln(1 + T_I)], \tag{B7}$$

$$T_I = t_I + 12u_I M_I^2. \tag{B8}$$

If we now equate the right-hand sides of (B2) and (B6), and of (B1) and (B7), we see that the equalities are satisfied in leading order if

$$\begin{aligned}&t_I = \bar{t}(1 + \epsilon x), \\ &u_I = u(1 + \epsilon a), \quad M_I = M(l^*)(1 + \epsilon b).\end{aligned} \tag{B9}$$

On equating terms of next highest order in ϵ or u_0 in the $\tilde{r}$ and $\tilde{h}$ equations one finds

$$16K_4(n-1)u^2 M(l^*)^2 = [\bar{t}x + 12uM(l^*)^2(a + 2b)]\epsilon, \tag{B10}$$

$$0 = [-\bar{t}x - \bar{t}b - 4uM(l^*)^2(a + 3b)]\epsilon. \tag{B11}$$

We now invoke the same argument that led to equation (3.39), namely, the fact that in (B10) and (B11) we may replace $4uM(l^*)^2$ by its value to zero order in ϵ, which is

$$4uM(l^*)^2 \approx 4u_I M_I^2 \approx -t_I \simeq -\bar{t}. \tag{B12}$$

Substituting this in (B10) and (B11) yields again two equations for a, b, and x; but since a and b enter only through the combination $a + 2b$, we can solve to find

$$x = \frac{1}{2}\left(\frac{n-1}{n+8}\right)\epsilon = a + 2b. \tag{B13}$$

This yields t_I and is sufficient to calculate the susceptibility.

In the calculation of the free energy in Sec. IV C we obtained two further conditions that should be satisfied, namely (4.26) and (4.27), which state

$$\bar{t}M(l^*)^2 = t_I M_I^2, \tag{B14}$$

$$uM(l^*)^4 = u_I M_I^4. \tag{B15}$$

These reduce simply to

$$0 = x + 2b, \tag{B16}$$

$$0 = a + 4b, \tag{B17}$$

the first of which with (B13) yields

$$a = \frac{n-1}{n+8}, \tag{B18}$$

$$b = -\frac{1}{4}\frac{n-1}{n+8}. \tag{B19}$$

Note that (B17) is then also satisfied identically. The importance of calculating a can be seen in the free energy,

$$F \approx (-t_I)^{2-\epsilon/6}/u_I, \tag{B20}$$

which upon substitution of t_I and u_I from (B9) (with the calculated values of x and a) reduces to

$$F \approx (-\bar{t})^{2-\epsilon/6}/u. \tag{B21}$$

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†To whom reprint requests should be addressed.

[1] D. R. Nelson and E. Domany, Phys. Rev. B 13, 236 (1976). This paper contains a number of typographical errors specifically in Eqs. (3.43b), (3.44a) and (3.44b), etc. A list will be sent on request to either of the authors. In addition the plots of $\gamma^{\text{eff}}(\tilde{t})$ in Figs. 2 and 4 should have much less pronounced minima. We are grateful to D. A. D. Bruce and Professor H. Horner for bringing these latter errors to our attention.

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