# Solitons in the continuous Heisenberg spin chain

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Solitons in the continuous Heisenberg spin system are studied in one dimension. We present results for soliton-soliton scattering in the isotropic case. For the anisotropic results we derive the functional form of the solitons. In both cases we investigated the linearized stability equations and found no evidence of instability.

## I. INTRODUCTION

In recent years there has been considerable interest in solitons and their relevance to some phenomena of classical and quantum physics.[1-3] We study in this paper the solitons in a continuum version of a classical linear Heisenberg chain of spins. In a subsequent paper we will discuss the relation between the quantum and classical solutions.[4] The classical system for the isotropic infinite chain was studied by two groups[5,6]; however, they did not present the most general single-soliton solutions. We will study in some detail the isotropic chain including the stability of single-soliton solutions and soliton-soliton scattering. We were unable to solve these problems analytically, but the solution of the equations on a computer presented no particular difficulty. The single-soliton solutions seem quite stable and the two-soliton scattering takes place with only a time delay to signify that any interaction occurred.

We also present single soliton solutions for the anisotropic chain and study their stability. For one sign of the anisotropy there is the possibility of two kinds of solitons, depending on the boundary conditions. The isotropic case corresponds to a background of spins pointing in the 3 direction and a localized traveling spin excitation. For the anisotropic case there is a corresponding soliton, but in addition there may be a background of spins in the 1-2 plane. In this paper we discuss only the solitons with spins pointing in the 3 direction at $x$ = infinity. We will discuss the other case in a separate publication.[7]

In Sec. II we present for completeness the equations and solutions for the isotropic case. Section III is devoted to a brief discussion of stability. Section IV is a discussion of the results of soliton-soliton scattering, and Sec. V presents the solutions for the anisotropic solitons analogous to the isotropic ones.

## II. EQUATIONS OF MOTION

The system we consider is an infinite linear chain with spin-density $S_i(x,t)$ $(i=1,2,3)$. The spin densities satisfy the Poisson bracket relations

$$[S_i(x,t), S_j(y,t)] = \epsilon_{ijk} S_k(y,t)\,\delta(x-y)\,, \qquad (2.1)$$

which give for arbitrary functionals $A$ and $B$,

$$[A,B] = \epsilon_{ijk} \int \frac{\delta A}{\delta S_i(x)} \frac{\delta B}{\delta S_j(x)} S_k(x)\,dx\,. \qquad (2.2)$$

The length of the spins is denoted $s$,

$$\sum_{i=1}^{3} S_i^2(x) = s^2\,.$$

The Hamiltonian for the system is given by $H = \int \mathcal{H}(x)\,dx$,

$$\mathcal{H}(x) = \frac{J}{2}\left(\frac{\partial \vec{S}}{\partial x}\right)^2 + \frac{J}{2}\gamma\left(\frac{\partial S_3}{\partial x}\right)^2 - \frac{J\tau}{2}(S_3^2 - s^2) - (S_3 - s)\Omega\,. \qquad (2.3)$$

In this section we restrict ourselves to the isotropic case, $\gamma=\tau=0$ and to boundary conditions at $x=\pm\infty$,

$$S_3 = s;\quad S_1, S_2 = 0\,. \qquad (2.4)$$

The constants in the density have been chosen to make the total energy finite. The equations of motion are given by

$$\frac{dS_k(x)}{dt} = [S_k(x), H]\,,$$
$$\frac{d\vec{S}}{dx} = J\vec{S}\times\frac{\partial^2\vec{S}}{\partial x^2} + \vec{S}\times\vec{\Omega}_0\,. \qquad (2.5)$$

The Hamiltonian is obviously invariant under space translations and under rotations in spin space. It is useful to use the generators of these operations in solving the equations of motion, however, the translation operator is not simply given in terms of $\vec{S}$. We therefore introduce angle densities $\theta(x), \phi(x)$,

$$S_3(x) = s\cos\theta(x) \equiv sU(x)\,,$$
$$S_2 = s\sin\theta(x)\sin\phi(x)\,, \qquad (2.6)$$
$$S_1 = s\sin\theta(x)\cos\phi(x)\,.$$

The variables $S_3$ and $\phi$ are canonically conjugate variables

$$[\phi(x), S_3(y)] = \delta(x-y) \qquad (2.7)$$

or

$$[\phi(x), U(y)] = (1/s)\delta(x-y)\,.$$

In terms of $U$ and $\phi$, $\mathcal{H}$ becomes

$$\mathcal{H}(x) = \frac{Js^2}{2}\left[\left(\frac{\partial U}{\partial x}\right)^2\frac{1}{1-U^2} + \left(\frac{\partial\phi}{\partial x}\right)^2(1-U^2)\right] + s\Omega_0(1-U)\,. \qquad (2.8)$$

The generator of translations (momentum operator) is easily verified to be given by

$$P = s\int_{-\infty}^{\infty} dx\,\frac{\partial\phi(x,t)}{\partial x}[1-U(x,t)]\,,$$
$$[\phi(x), P] = -\frac{\partial\phi}{\partial x}\,, \quad [U(x), P] = -\frac{\partial U}{\partial x}\,. \qquad (2.9)$$

The generators of rotations (magnetization) are given by

$$M_k = \int_{-\infty}^{\infty} S_k(x)\,dx\,,$$

and we are mainly interested in

$$M_3 \equiv s\int_{-\infty}^{\infty}(U-1)\,dx\,, \qquad (2.10)$$

where we subtract a constant to make $M_3$ finite.

The operators $P$ and $\vec{M}$ are constants of the motion and it is useful to take advantage of this in solving the equations of motion and investigating the stability. Of course just as in quantum mechanics we can only specify the value of one component of $\vec{M}$ which we have chosen to be $M_3$. We thus look for minima of the Hamiltonian subject to two constraints

$$P = p_0\,, \quad M_3 = M\,; \qquad (2.11)$$

which gives the form

$$I(\Omega_1, v, U, \phi) = \int \mathcal{H}(x)\,dx - \Omega_1(M_3 - M) - v(P - p_0)\,; \qquad (2.12)$$

the equations are

$$\frac{\delta I}{\delta\phi(x)} = 0 = -Js^2\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}(1-U^2)\right) - sv\frac{\partial U}{\partial x} \qquad (2.13)$$

and

$$\frac{\delta I}{\delta U(x)} = 0 = Js^2\left[-\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\frac{1}{1-U^2}\right) + \frac{U(\partial U/\partial x)^2}{(1-U^2)^2} - U\left(\frac{\partial\phi}{\partial x}\right)^2\right] - s\Omega_0 - s\Omega_1 + sv\frac{\partial\phi}{\partial x}\,, \qquad (2.14)$$

plus the equations of constraint. If we compare with the equations of motion

$$\dot\phi = \frac{1}{s}\frac{\delta H}{\delta U} \quad\text{and}\quad -\dot U = \frac{1}{s}\frac{\delta H}{\delta\phi}\,, \qquad (2.15)$$

we see that we are essentially finding solutions such that

$$U(x,t) = U(x-vt)\,, \quad \phi(x,t) = \Omega_1 t + \hat\phi(x-vt)\,, \qquad (2.16)$$

and so the Lagrange multipliers have a simple interpretation as the angular and linear velocities. It is convenient to define

$$\Omega = (\Omega_0 + \Omega_1)/Js$$

and

$$V = v/Js\,. \qquad (2.17)$$

Equation (2.13) can be directly integrated to give

$$\frac{\partial\phi}{\partial x} = \frac{V}{1+U}\,. \qquad (2.18)$$

Equation (2.14) then simplifies to

$$\left(\frac{\partial\theta}{\partial x}\right)^2 = 4\Omega\frac{1-U}{1+U}\left(\frac{1+U}{2} - \frac{V^2}{4\Omega}\right). \qquad (2.19)$$

Introducing $\beta = \frac{1}{2}\theta$, and

$$\cos^2\beta_0 = V^2/4\Omega = 1 - b^2\,, \qquad (2.20)$$

the equation becomes

$$\left(\frac{d\beta}{dx}\right)^2 = \Omega\frac{\sin^2\beta}{\cos^2\beta}(\cos^2\beta - \cos^2\beta_0)\,. \qquad (2.21)$$

At $x = \pm\infty$, $\beta = 0$ and $\beta$ obtains the maximum value $\beta_0$ at some arbitrary point $x_0$. Equation (2.21) is easily integrated to give

$$\sin\beta = b\,\mathrm{sech}[b\sqrt{\Omega}(x-x_0)]\,,$$
$$\cos\theta = 1 - 2b^2\,\mathrm{sech}^2[b\sqrt{\Omega}(x-x_0)]\,,$$
$$\phi = \phi_0 + \tfrac{1}{2}V(x-x_0) + \tan^{-1}\left[\left(\frac{b^2}{1-b^2}\right)^{1/2}\tanh[b\sqrt{\Omega}(x-x_0)]\right]. \qquad (2.22)$$

The constants of motion $E = \int\mathcal{H}(x)\,dx$, $P$ and $M_3$ are easily calculated to be

$$E = 4Js^2 b\sqrt{\Omega} + (4Jsb/\sqrt{\Omega})\Omega_0\,,$$
$$M = 4bs/\sqrt{\Omega}\,, \quad P = 4s\sin^{-1}(b) = 4s\beta_0\,. \tag{2.23}$$

The energy can be written

$$E = 8Js^3[1-\cos(P/2s)]/M + M\Omega_0\,. \tag{2.24}$$

Referring to Eq. (2.20) we see that as $v \to 0$ either $b \to 1$ or $\Omega \to 0$. If $\Omega \to 0$, the soliton disappears as $E, P \to 0$. The point $b = 1(\beta_0 = \frac{1}{2}\pi)$ has

$$P = 2s\pi \quad\text{and}\quad E = 16Js^3/M + M\Omega_0\,,$$

with $V = 0$. However, this point is a singular point in that there is no turning point in the differential equation and $d\theta/dx$ must develop a discontinuity of magnitude $4\sqrt{\Omega}$. The solutions as given in Eq. (2.22) are continuous at this point.

## III. STABILITY

The stability of the solutions to the constrained minimization problem [Eqs. (2.12)–(2.14)] was studied by expanding $I$ [Eq. (2.12)] about the soliton solution and calculating numerically the eigenvalues of the resulting quadratic form. Writing $U = U_0 + \eta$ and $\phi = \phi_0 + \omega$, the quadratic form is given by

$$\mathcal{H}_{\text{eff}}(x) = \tfrac{1}{2}(1-U_0^2)\left(\frac{d\omega}{dx}\right)^2 - 2U_0\frac{d\phi_0}{dx}\eta\frac{d\omega}{dx} - \frac{\eta^2}{2}\left(\frac{d\phi_0}{dx}\right)^2 + V\frac{d\omega}{dx}\eta + \frac{3U_0^2+1}{2(1-U_0^2)^3}\left(\frac{dU_0}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\eta}{dx}\right)^2\frac{1}{1-U_0^2} + 2\eta\frac{d\eta}{dx}U_0\frac{dU_0}{dx}\frac{1}{(1-U_0^2)^3}\,. \tag{3.1}$$

For simplicity we have chosen $J = 1$ and $s = 1$. The constraint that $M$ = constant requires

$$\int_{-\infty}^{\infty}\eta\,dx = 0 \tag{3.2}$$

and $P$ = constant requires

$$\int_{-\infty}^{\infty}\left(\eta\frac{d\phi_0}{dx} - \omega\frac{dU_0}{dx}\right)dx = 0\,. \tag{3.3}$$

There was one negative eigenvalue and it was therefore necessary to calculate the eigenvector. In all case the eigenvector violated the constraint that $M$ is constant. Therefore that eigenvalue and eigenvector can be ignored in a stability analysis, since if it were included $M$ would change as the system evolved in time. As a check we calculated directly the eigenvalues in the equation of motion and there were none which would cause any instabilities. The numerical problems were quite severe when using this more direct method and the sign of the small eigenvalues somewhat less certain. In either case we conclude that for infinitesimal perturbations in the solution, the motion is of an oscillatory nature.

We solved the eigenvalue problem by two methods. The first method was only satisfactory for the Hamiltonian form and involved evaluating the Hamiltonian at a sequence of mesh points resulting in a large matrix with a nonzero band along the diagonal. The second method was to use a set of trial functions of the form

$$x^n\,\text{sech}^2 b\sqrt{\Omega}\,x\,.$$

To test stability under large perturbations we introduced changes in the initial conditions and observed numerically the evolution in time. One cannot of course do this systematically, nevertheless we obtain a strong indication of stability. The collisions of solitons with different parameters also provides a test of stability. We conclude from our investigations that the solitons are very likely completely stable and behave as those in exactly soluble systems such as the nonlinear Schrödinger equation.

## IV. SOLITON-SOLITON SCATTERING

We were unable to find analytic solutions for scattering so we simulated the collision on the computer. Insofar as we could detect numerically the only effect of the collision is an effective increase in the velocity during the collision. After the collision the soliton's shape and velocity were the same as before the collision. We used periodic boundary conditions on a finite chain and hence introduced a very slight error as we only know analytically the answer for the infinite chain.

In Fig. 1 we show the initial configuration, Fig. 2 shows a typical time during the collision, and Fig. 3 a short time after the collision. Figures 4 and 5 show a three-dimensional plot as a function of both space and time. In the upper corner a piece appears which is just the continuation due to the periodic boundary conditions. The negative time delay can be seen quite clearly in these pictures. Figures 6 and 7 show more of the structure of the actual collision. Finally in Fig. 8 we show the 1 component of the spin. From these pictures we observe that the solitons pass through each other with no effect on their size, shape, or velocities.

## V. ANISOTROPIC MODEL

We consider in this section the effects of the terms proportional to $\tau$ and $\gamma$ in Eq. (2.3). If the continuum limit of an anisotropic chain with nearest-neighbor interactions is taken, these terms occur. We restrict ourselves to anisotropy in the 3 direction only. The analysis proceeds exactly as in Sec. II except that Eq. (2.19) becomes

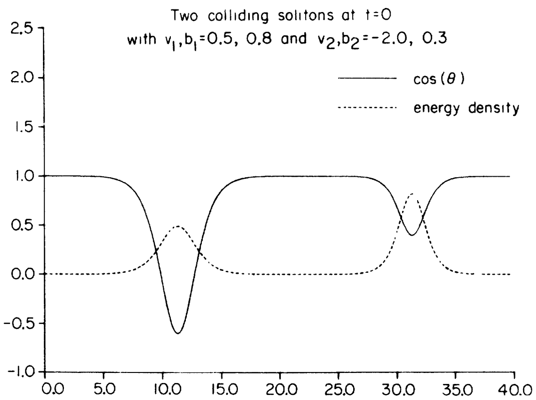


FIG. 1. Initial configuration for two solitons. Energy density and $\cos\theta = S_3/s$ are plotted. In Figs. 1–8 the soliton parameters are all the same. One soliton has velocity $V=0.5$ and amplitude $\sin\frac{1}{2}\theta = b = 0.8$. Second has velocity $V=-2$ and amplitude $b=0.3$.

$$\left(\frac{\partial\theta}{\partial x}\right)^2(1+\gamma\sin^2\theta)=4\Omega\frac{1-u}{1+u}\left(\frac{1+u}{2}+\frac{\tau(1+u)^2}{4\Omega}-\frac{V^2}{4\Omega}\right). \tag{5.1}$$

We will only consider $\gamma > -1$. We define the maximum deflection angle $\beta_0(=\frac{1}{2}\theta)$ by

$$\frac{V^2}{4\Omega}=\cos^2\beta_0+\frac{\tau}{\Omega}\cos^4\beta_0 . \tag{5.2}$$

Since the left-hand side of Eq. (5.1) is positive, we can infer some constraints on the parameters. We rewrite Eq. (5.1) as

$$\left(\frac{d\beta}{dx}\right)^2(1+\gamma\sin^2\theta)=\frac{\tau\sin^2\beta}{\cos^2\beta}(\cos^2\beta-\cos^2\beta_0)\times(\cos^2\beta+\cos^2\beta_0+\Omega/\tau). \tag{5.3}$$

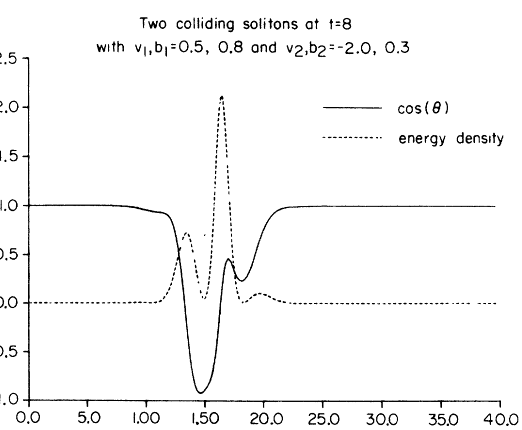


FIG. 2. Plot of $\cos\theta$ and energy density at a typical time during the collision.

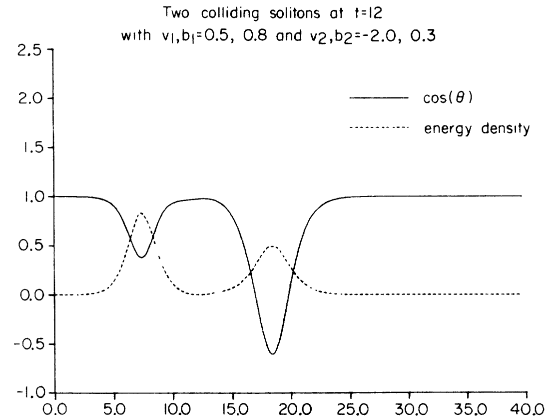


FIG. 3. Short time after the collision.

Eliminating $\Omega$ from the constraint we have

$$V^2/4\cos^2\beta_0 > -\tau . \tag{5.4}$$

Thus if $\tau<0$, there is a lower limit to the velocity for fixed maximum deflection angle unlike either $\tau>0$ or the isotropic case. Also if $\tau<0$, $\Omega$ must be positive. If $\tau>0$, it is possible to have $\Omega<0$ provided $\cos^2\beta_0+\Omega/\tau>-1$.

Although we can not write the solution to Eq. (5.1) in terms of elementary functions, we can easily integrate it numerically. By changing

$$\int dx \to \int d\beta\frac{dx}{d\beta},$$

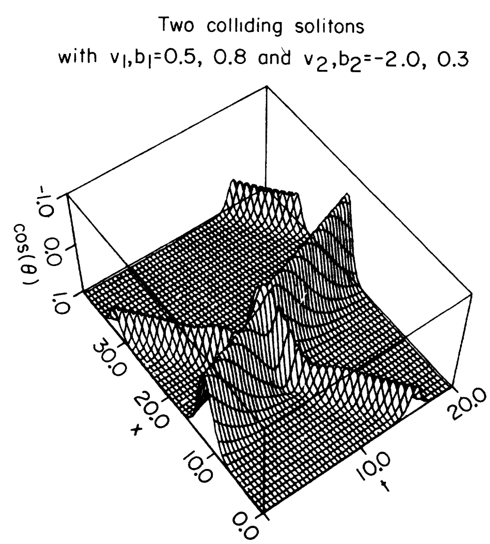


FIG. 4. Two-dimensional perspective plot showing $\cos\theta$ vs $x$ and $t$. By following the ridge line of one soliton one can easily see the time shift.

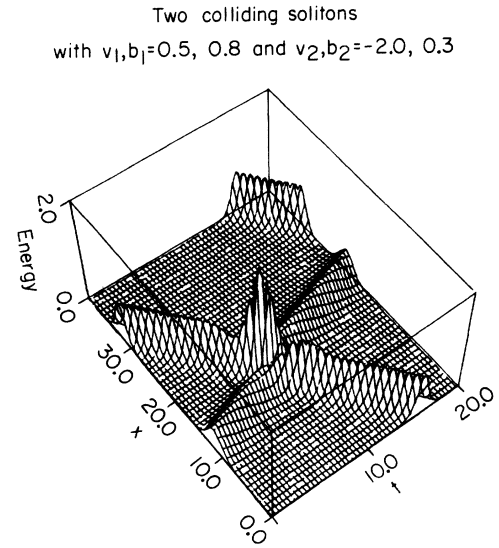


FIG. 5. Perspective plot of energy density.

we can easily express the energy, momentum, and magnetization in terms of elementary integrals.

$$E = Js^2\sqrt{\Omega}\int_0^{\beta_0} d\beta\cos\beta(\sin\beta)\left(1+\frac{2\tau}{\Omega}\cos^2\beta\right)\frac{A}{B}, \tag{5.5}$$

$$P = 4s\cos\beta_0\left(1+\frac{\tau}{\Omega}\cos^2\beta_0\right)^{1/2}\int_0^{\beta_0} d\beta\frac{\sin\beta}{\cos\beta}\frac{A}{B}, \tag{5.6}$$

$$M = \frac{4s}{\sqrt{\Omega}}\int_0^{\beta_0}\sin\beta(\cos\beta)\frac{A}{B}. \tag{5.7}$$

In the above

$$A = (1+\gamma\sin^2\theta)^{1/2}$$

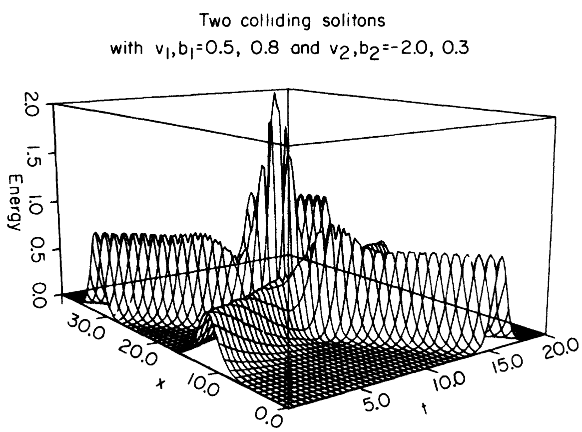


FIG. 6. Side view of energy density.

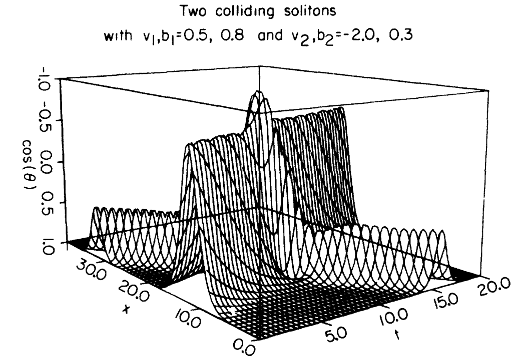


FIG. 7. Side view of $\cos\theta$.

and

$$B = [\cos^2\beta + (\tau/\Omega)\cos^4\beta - \cos^2\beta_0 - (\tau/\Omega)\cos^4\beta_0]^{1/2}.$$

The plot of $\cos[\theta(x)]$ for the anisotropic case cannot of course be much different from the isotropic case since there is only one zero of a derivative. We show in Fig. 9 some typical solitons. There are some significant differences in the energy-momentum relations with the magnetization fixed. In Fig. 10 we show energy versus momentum with $M=1$ and spin $s=\frac{1}{2}$ for the isotropic case and for one anisotropic case. We have plotted $E$ vs $\cos(P)$ since the isotropic answer is

$$E = (1/M)[1-\cos(P)]. \tag{2.24}$$

For $\tau > 0$ it is possible to have $v=0$ solitons, that is there is an energy gap $E(0)$. The gap can be

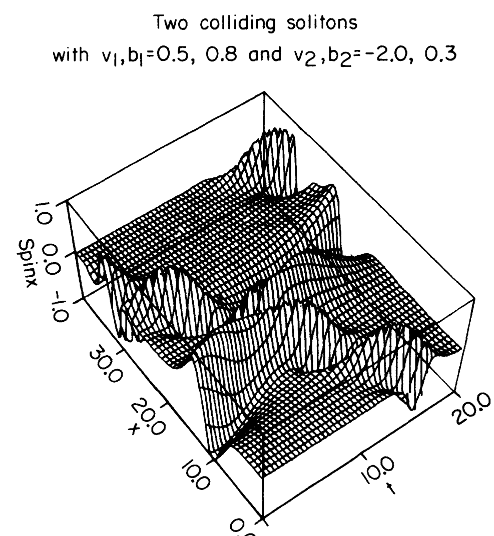


FIG. 8. 1-component of the spin showing the more complicated structure of the 1-2 spin components.

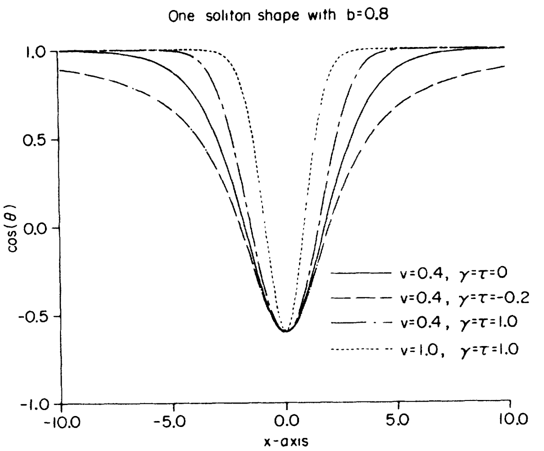


FIG. 9. Cos$\theta(x)$ for different anisotropy parameters.

easily calculated from Eqs. (5.5)–(5.7).

In Fig. 11 we show two anisotropic cases, and it is evident that for large anisotropy there are two branches of the curves. The linearized stability equations showed no indication of any instability for any of these situations, but we have not done the other numerical tests that were performed for the isotropic solitons. We have no explanation for the two branches of the curves. We did not examine the end point of the $\gamma=-1$ curve to see if it also has two branches, although the second branch is very short if it exists.

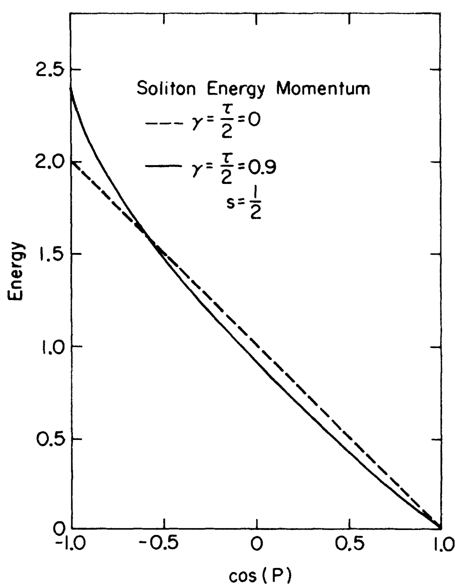


FIG. 10. Energy momentum spectrum. For the case of positive anisotropy, an energy gap appears. In the figure $E(0)$ has been subtracted.

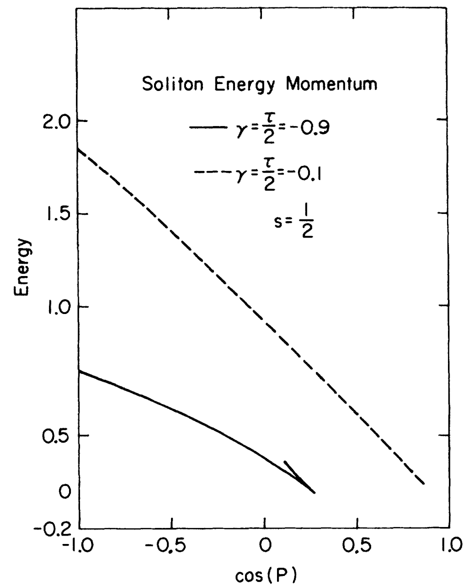


FIG. 11. Energy momentum spectrum.

## VI. DISCUSSION

We studied in some detail soliton solutions in the continuum Heisenberg chain. Unfortunately we were unable to find analytic solutions for multiple soliton solutions. Because of certain similarities of the quantum version of the discrete chain and the nonlinear Schrödinger equation we expect that analytic solutions do exist. The remarkable stability of the solitons in a collision lends some support to this conjecture. We also could not solve analytically the discrete chain, although using as initial conditions those for the continuum case leads to an apparently stable soliton. The numerical simulation leads to different amplitudes and velocities for the discrete chain, but the behavior was qualitatively the same as the continuum chain. Because of the exactly soluble nature of the quantum version for $s=\frac{1}{2}$, it is of some interest to have an exact solution for the discrete classical chain.

The anisotropic chain has some interesting features such as the existence of an energy gap ($v=0$ solitons) for one sign of the anisotropy and no gap but a peculiar energy momentum relationship for the other sign.

## ACKNOWLEDGMENTS

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[1]A. F. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973).
[2]R. Rajaraman, Phys. Rep. 21C, No. 5 (1975).
[3]Phys. Rep. 23C, No. 3 (1976).
[4]J. Tjon and Jon Wright (unpublished).
[5]K. Nakamura and T. Sadada, Phys. Lett. A 48, 321 (1974).
[6]M. Lakshmanan, Th. W. Ruijgrok, and C. J. Thompson, Physica (Utr.) 84A, 577 (1976).
[7]J. Tjon and Jon Wright (unpublished).