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Erratic behavior of superconducting loops with Josephson junctions

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An analysis of the transitions between quantum states of a superconducting loop is presented. On the basis of the simple equivalent circuit model, it is argued that the system's behavior should be erratic for very small damping. This corrects an oversight in the work of Smith and Blackburn, and brings it into agreement with previously published material.

INTRODUCTION

A superconducting loop that is interrupted by a Josephson tunnel junction is an example of a system with a set of macroscopic quantum states. When it settles into some quantum state, how does it choose which one to settle into? There have been conflicting statements in the literature about the predictability of this choice. This paper will attempt to clarify the situation, at least within the context of the simple equivalent circuit model that has been widely used.

Most work has actually involved loops with two Josephson junctions, but in many respects their behavior should be similar to single-junction loops. Many papers[1-5] have described erratic, apparently random behavior in situations where a loop was forced to select a quantum state. Several groups[4,5] have pointed out that this is to be expected, but detailed justifications were not presented.

On the other hand, Smith and Blackburn[6] (referred to in the following as SB) have described both experiments and computer simulations giving simple, predictable behavior. The resolution of these differences lies in the fact that they were interested primarily in the case of intermediate damping. For such loops, the equivalent-circuit model does predict simple behavior. However, they erred in supposing that their results could be extrapolated to the case of small damping, where, in fact, erratic behavior should occur.

The process we are interested in begins with the loop superconducting and with no magnetic field anywhere. The field, perpendicular to the plane of the loop, is then slowly increased from zero until the circulating current induced in the loop reaches the critical value for the Josephson junction. At this point a transient voltage will appear across the junction, and some amount of magnetic flux will enter the loop. Our problem is to predict this amount of flux, which is equivalent to predicting the final state.

THEORY

Since the theory of this system has been discussed repeatedly,[1-7] we will merely review it briefly. We treat the loop by an equivalent circuit consisting of the loop inductance L, the junction capacitance C, the junction quasiparticle tunneling resistance R, and an element obeying the Josephson equations with critical current i_c, all connected in parallel. Each of the above quantities is taken to be constant, although in fact R is known to be a function of voltage. It is hoped that neglecting the voltage dependence of R will not introduce any significant error, but eventually a more realistic model should be used.

To this equivalent circuit we add the requirement of fluxoid conservation and obtain, in Smith and Blackburn's notation,

$$\Phi = \Phi_x - \gamma \sin(2\pi\Phi) - \beta \frac{d\Phi}{dt_1} - \frac{d^2\Phi}{dt_1^2} \ . \quad (1)$$

Here Φ is the total magnetic flux in the loop, divided by the flux quantum 2×10^{-15} W, Φ_x is the flux that would be produced in the loop if the loop were open circuited, again divided by the flux quantum, and t_1 is the time divided by $\sqrt{LC}$. The damping parameter β is defined as $\sqrt{LC}/RC$, and γ is Li_c divided by the flux quantum. The junction critical current i_c is the largest current that can pass through the junction without a voltage drop. Thus γ is the maximum number of flux quanta that can be produced by a supercurrent circulating in the loop.

One way to gain insight into Eq. (1) is to note that it is also the equation of motion of a particle sliding in a potential well with a damping force that is proportional to velocity. The position coordinate is then Φ and the potential is

$$V(\Phi) = \tfrac{1}{2}(\Phi - \Phi_x)^2 - (\gamma/2\pi)\cos(2\pi\Phi) \ . \quad (2)$$

This represents a parabolic potential well, centered at $\Phi = \Phi_x$, with a sinusoidal modulation. The modulation will lead to a number of subsidiary wells, or metastable points. This potential is illustrated in Fig. 1. For simplicity, the figure is drawn for γ near unity, although our computations will be for larger γ. Note that for sufficiently large $|\Phi - \Phi_x|$, the modulation does not lead to local minima.

Zero applied field in the case of the superconducting loop means $\Phi_x = 0$, and so it corresponds to having the parabolic well centered at $\Phi = 0$. Applying a field increases Φ_x and moves the bottom of the well to larger Φ values. We must suppose that the particle remains near $\Phi = 0$, because this is what the superconducting loop does. That is, the particle stays in the same local minimum, which moves up the side of the well and grows shallower. When Φ_x gets large enough, the local minimum will disappear. The particle will then begin to slide down. The onset of sliding corresponds to the appearance of a voltage across the Josephson junction. Eventually the particle will come to rest in some minimum, and Φ will have changed from near zero to some new value. This corresponds to the loop changing to a new quantum state and to the entry of some flux. Following SB, the final value of Φ will be denoted Φ_{enter}. This is the quantity that we wish to compute.

The behavior of the system depends on the value of the damping parameter β. For very large β the particle will stop in the first minimum that it comes to. For smaller β, it will progress further down before being trapped. That is, as shown in SB, Φ_{enter} increases as β decreases.

At some value of β, which we will call β_0, the particle comes to rest in the lowest potential well, which is at $\Phi \cong \Phi_x$, so that $\Phi_{enter} \cong \Phi_x$. (The value of β_0 depends weakly on γ. We have used γ values from 30 to 1000, and we find $\beta_0 \cong 1.6$ in this range. In the limit of $\gamma = 0$, β_0 would be 2.0, corresponding to a critically damped oscillator.) It is the behavior at $\beta \lesssim \beta_0$ that SB failed to describe correctly.

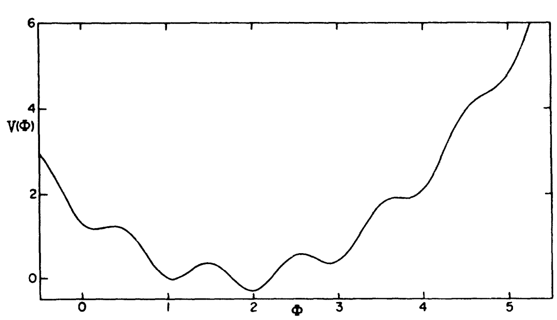

FIG. 1. Graph of the potential function $V(\Phi)$ for the case $\gamma = 2$, $\Phi_x = 1.8$. A particle that started at the lowest V at $\Phi_x = 0$ would now be in the shallow minimum at $\Phi = 0.15$. At $\Phi_x = 1.8$, the system has four stable or metastable states: $\Phi = 0.15$, 1.05, 2.00, 2.90, 3.78.

For $\beta < \beta_0$, the particle will pass the bottom and slide up the other side. If it is trapped in some minimum there, Φ_{enter} will be larger than Φ_x. For $\beta \ll \beta_0$, it will slide back and forth through $\Phi = \Phi_x$ many times before stopping. Clearly, just where it stops will be very sensitive to the initial conditions and the precise amount of damping.

At this point it is helpful in understanding the low-β case to consider an approximate analysis using $\gamma = 0$. This is the familiar damped-oscillator problem and can be solved exactly. Let Φ_x have some value and then release the particle from rest at $\Phi = \Phi_i$. The particle will slide back and forth with diminishing amplitude. The extrema of its motion, given by $d\Phi/dt_1 = 0$, are at

$$(\Phi - \Phi_x)/(\Phi_i - \Phi_x) = (-1)^N \exp[-\pi N/(4/\beta^2 - 1)^{1/2}]. \tag{3}$$

$N = 0$ gives the starting point $\Phi = \Phi_i$. The end of the first swing corresponds to $N = 1$, and Φ at the completion of the return swing, the end of the first cycle, is given by $N = 2$. As an example, take $\Phi_i = 0$ and $\beta = 0.1$. The extrema are at $\Phi = 0$, $1.85\Phi_x$, $0.27\Phi_x$, $1.62\Phi_x$, etc.

If we now return to our original problem with γ large, we can expect that Φ_{enter} should be near one of the members of the series calculated from Eq. (3). The reason is that Eq. (3) gives an estimate of the points where the particle will be moving slowly. These are just the places where the particle is likely to be trapped.

We call the values of Φ calculated from Eq. (3) with $N = 1$ and $N = 2$ the "approximate theoretical maximum" and "approximate theoretical minimum" values of Φ_{enter}. The actual Φ_{enter} should lie near or within these limiting values, and this is borne out by our numerical simulations, as we will see.

We expect Φ_{enter} to be very sensitive to the initial conditions when β is small. The normal initial conditions for our problem occur when the local minimum in which the particle started has been reduced to just an inflection point. The corresponding values of Φ_x and Φ_i are

$$\Phi_x = (\gamma^2 - 1/4\pi^2)^{1/2} + \Phi_i \tag{4}$$

and

$$\Phi_i = (1/2\pi)\cos^{-1}(-1/2\pi\gamma). \tag{5}$$

For $\gamma \gg 1$ these equations give $\Phi_x \cong \gamma + \frac{1}{4}$ and $\Phi_i \cong \frac{1}{4}$.

Now consider the effect of small changes in inital conditions, say by changing Φ_i slightly. Suppose that the normal initial conditions lead to a final state j. That is, suppose the particle comes to rest in the jth local minimum. Suppose further that it happens that the particle is almost able to escape from this state, but does not quite have enough energy to surmount the potential barrier. Then a slightly smaller Φ_i, corresponding to a larger initial potential energy, could mean that the particle is able to escape from the jth minimum. If it escapes, it will slide down and up the other side, perhaps to be trapped there. The result would then be a very large change in Φ_{enter} resulting from a small change in Φ_i.

A small noise pulse at any time could have a similar effect. If the particle almost escapes from its final state, the noise energy could lead to escape. The final state could then be quite different. Note that it is not the height of the final potential barrier that is important, but how close the particle would normally come to surmounting it. There is no general way to determine whether a particle will "almost escape" from its final state. This will depend sensitively on β and γ. Therefore, it is hard to make quantitative statements about how large a noise pulse or change in Φ_i will be important. Our approach will be to illustrate these effects by a few specific numerical simulations.

By a similar argument one can see that a small change in the damping constant β could lead to a large change in Φ_{enter}. Again, we will illustrate this with examples.

NUMERICAL SIMULATIONS

We have numerically integrated Eq. (1) using the fourth-order Runge-Kutta method. Most of the work was done using a program very kindly supplied to us by Smith and Blackburn. For some computations we used a program written here by Wilson, and modified by Chen, which was originally used for double-junction loops.[3] Initial conditions were chosen, and the integration was carried out until the system had clearly settled into some final minimum. Most computations were for $\gamma = 120$, but values from 30 to 1000 were also used.

To investigate the dependence on β, we carried out a series of computations with $\gamma = 120$, fixed initial conditions, and different values of β. The initial conditions were $d\Phi/dt_1 = 0$, $\Phi_i = 0.25$, and $\Phi_x = 120.251$. These are the normal initial conditions, mentioned before, except that 0.001 has been added to Φ_x to ensure that the particle would actually start sliding. Figure 2 shows the results.

Figure 2 supports the general remarks made in the previous section. For β less than about 1.6 the behavior becomes rather erratic. Φ_{enter} follows no discernible pattern other than that it lies between the approximate theoretical maximum and minimum.

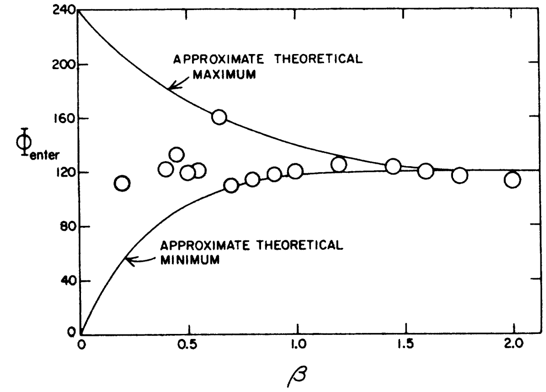

FIG. 2. Results of computer simulations for various values of the damping parameter β, with $\gamma = 120$. The initial conditions were $\Phi_x = 120.251$, and $\dot{\Phi} = 0$. These are the "normal" initial conditions, except that 0.001 was added to Φ_x to ensure that switching would occur. The solid lines were computed as described in the text. They give approximate limits for the value of Φ_{enter} for $\beta \lesssim 1.6$, where the behavior is erratic.

The sensitivity of Φ_{enter} to initial conditions is shown in Fig. 3. We see that it can change quite abruptly in a narrow range of values of Φ_i. Note in particular the spike that appears for the $\beta = 0.2$ case. Such spikes could easily be missed in numerical computations, since only a finite set of Φ_i values can be used.

CONCLUSION

The value of Φ_{enter} for a superconducting loop having small β is hard to predict for two reasons. First, it changes in an erratic way with β, so a small error in the choice of β could lead to a large error in the predicted Φ_{enter}. Second, the experi-

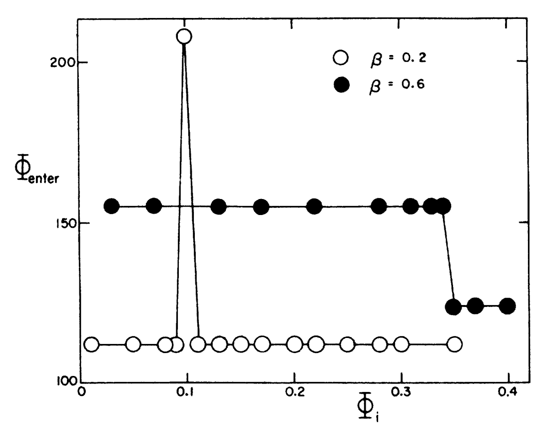

FIG. 3. Results of computer simulations for various initial values of $\dot{\Phi}$. The other parameters were $\gamma = 120$, $\Phi_x = 120.251$, $\dot{\Phi} = 0$. Note the abrupt changes in the value of Φ_{enter}.

mental results will be very sensitive to noise, and, to make matters worse, the degree of sensitivity to noise is liable to depend in an erratic way on the precise value of β. It is clear that devices made with superconducting loops which involve transitions between quantum states should avoid the low damping case. The simple, predictable behavior described by Smith and Blackburn applies only for $\beta > \beta_0$.

These conclusions are also relevant to superconducting loops having two Josephson junctions. The additional junction complicates the system, but should not alter the general erratic character that we have seen in single-junction loops.

ACKNOWLEDGMENTS

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