PHYSICAL REVIEW B VOLUME 15, NUMBER 7 1 APRIL 1977

Transport processes in superfluid ^{3}He-B at low temperatures*

C. J. Pethick†
Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801 and Nordita, DK-2100 Copenhagen Ø, Denmark

H. Smith
Physics Laboratory I, H.C. Ørsted Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

P. Bhattacharyya‡
Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801
(Received 21 October 1976)

We derive and solve the Boltzmann equation for viscosity and diffusive thermal conductivity at low temperatures in the B phase of superfluid ^{3}He. The viscosity η is shown to tend towards a constant value as the temperature tends to zero, with the constant being inversely proportional to an angular average of the collision probability. A general expression for the collision probability valid at any temperature is given in terms of the singlet and triplet components of the normal-state scattering amplitude. If one takes for the normal-state amplitude the s- and p-wave approximation, the constant viscosity is found to equal about one third of its value at the transition temperature. The diffusive thermal conductivity κ_D is found to vary as T^{-1}, as in the normal state, and with roughly the same coefficient of proportionality. We calculate as a function of pressure the viscosity and diffusive thermal conductivity in the normal state and in the superfluid at $T = 0$, and the normal-state quasiparticle relaxation time at the Fermi energy. The results are compared with experimental data, and the adequacy of the s- and p-wave approximation for the normal-state scattering amplitude is discussed. Finite temperature corrections to η and $\kappa_D T$ are obtained for a particularly simple normal-state scattering amplitude, showing that η initially decreases with increasing temperature while $\kappa_D T$ increases.

I. INTRODUCTION

In the preceding paper[1] (referred to as I) transport and relaxation processes in superfluid liquid ^{3}He close to the transition temperature T_c were considered. The approach involved a systematic expansion in terms of the small parameter Δ/k_BT, where Δ is a typical value of the gap in the excitation spectrum. In the present paper we treat the opposite limit, in which the thermal energy k_BT is small compared to the gap energy, or, equivalently $T \ll T_c$. (For reference to earlier work on this subject we refer the reader to I.) We consider the B phase of liquid ^{3}He, which is the only phase presently accessible to experiment at $T \ll T_c$, and we shall assume this phase to be the Balian-Werthamer (BW) state.[2] The experimental evidence for this identification, particularly from magnetic resonance experiments, is now quite considerable.[3]

There are a number of important differences between the two limits $k_BT \gg \Delta$ and $k_BT \ll \Delta$. In the region close to T_c we worked in terms of quasiparticles having a positive energy above the Fermi surface, and a negative energy below, so that in the limit $\Delta \to 0$ one recovers the normal-state dispersion relation. At low temperatures ($k_BT \ll \Delta$) it is more convenient to use quasiparticle states which all have positive energy, both above and below the Fermi surface. The methods for solving the Boltzmann equation are also rather different. Close to T_c it is solved by perturbation theory starting from the normal-state solution. In the low-temperature limit the equation may be solved exactly rather simply; in particular we show that the simple relaxation-time approximation for the collision term gives the exact results for the viscosity and diffusive thermal conductivity. The characteristic relaxation time is inversely proportional to the number density of excitations, and as a consequence the shear viscosity tends to a constant, while the diffusive thermal conductivity has the same temperature dependence as in the normal state ($\sim T^{-1}$).[4]

The temperature dependence of the transport coefficients does not depend on detailed properties of the quasiparticle scattering amplitude, but the numerical values of quantities of course do. Given the interaction between normal-state quasiparticles one can find the interaction between superfluid quasiparticles by performing a Bogoliubov transformation. The scattering amplitude for quasiparticles in the superfluid is a linear combination of normal-state amplitudes. As a result of the anisotropy of the superfluid state, the scattering amplitude for superfluid quasiparticles generally has a much more complicated angular dependence than the corresponding amplitude for normal state quasiparticles, and also depends on the quasiparticle energy. We give detailed expressions for the transition probability for superfluid quasiparticles. This is used to obtain expressions for the characteristic relaxation time when the scattering amplitude for normal-state quasiparticles is given by the s- and p-wave approximation in terms of Landau parameters.[5] The leading finite-temperature corrections to the low-temperature behavior of the transport coefficients are of order k_BT/Δ, and we evaluate the coefficients for the particular case of a constant normal quasiparticle scattering amplitude acting only in singlet spin states.

The paper is organized as follows. In Sec. II we derive and solve the low-temperature Boltzmann equation for viscosity and diffusive thermal conductivity. Section III and the Appendix contain the evaluation of the collision probability in terms of the singlet and triplet components of the normal-state scattering amplitude, in addition to explicit expressions for the transport coefficients in terms of the Landau parameters occurring in the s- and p-wave approximation. In Sec. IV we calculate the finite temperature corrections for the simplified normal-state scattering amplitude. Section V contains a discussion of the experimentally important question of the magnitude of the mean free path, and also a consideration of the adequacy of the particular approximate scattering amplitude used to calculate the magnitude of the transport coefficients.

II. BOLTZMANN EQUATION

Low-frequency long-wavelength transport and relaxation processes in the superfluid may be dealt with in the framework of a Boltzmann equation, provided ω and qv_F are very much less than Δ. Here ω and q are the frequency and wave number of the disturbance, v_F is the Fermi velocity, and Δ is the superfluid gap, which is isotropic for the BW state. The streaming terms in the Boltzmann equation have the standard form, and in the hydrodynamic limit, when ω and qv_F are small compared with a typical quasiparticle collision rate, all distribution functions in the streaming terms may be replaced by local equilibrium distribution functions. Throughout this paper we shall confine ourselves to the hydrodynamic limit. Thus for the case of the shear viscosity the streaming terms reduce to

$$\vec{v}_{\vec{p}}\cdot\vec{\nabla}n_{\vec{p}}=p_xv_y\frac{\partial u_x}{\partial y}\left(-\frac{\partial n_p^0}{\partial E_p}\right). \tag{2.1}$$

On the left-hand side of (2.1) the gradient operator acts only on the velocity field. Terms involving the gradient of the quasiparticle energy cancel in the usual fashion. For definiteness we assume the velocity field $\vec{u}$ to have only an x component, whose magnitude varies only in the y direction. $\vec{p}$ is the quasiparticle momentum. $\vec{v}$, which depends on $\vec{p}$, is the quasiparticle velocity, and n_p^0 is the equilibrium distribution function

$$n_p^0=(e^{E_p/k_BT}+1)^{-1}, \tag{2.2}$$

where E_p is the quasiparticle energy. In (2.1) we have retained only terms of first order in the velocity gradient, which is assumed to be small. We also suppress spin indices for the moment.

The quasiparticle energy in the state of global equilibrium is given by

$$E_p=(\Delta^2+\xi_p^2)^{1/2}, \tag{2.3}$$

where $\xi_p=(p-p_F)v_F$ is the normal-state quasiparticle energy, measured with respect to the chemical potential and p_F is the Fermi momentum. The quasiparticle velocity is

$$v=\frac{\partial E_p}{\partial p}=v_F\frac{\xi_p}{E_p}, \tag{2.4}$$

which is an odd function of ξ_p, in contrast to the case for a normal Fermi liquid, where it is an even function of ξ_p, if one works in terms of the usual quasiparticle states. The driving term for the case of viscosity is therefore odd in ξ_p, and has $l=2$ angular symmetry.

In the case of thermal conductivity the driving term is

$$\vec{v}\cdot\vec{\nabla}n_p=E_p\vec{v}\cdot\frac{\vec{\nabla}T}{T}\left(-\frac{\partial n_p^0}{\partial E_p}\right), \tag{2.5}$$

which again is odd in ξ_p.

Let us now turn to the collision term. In I we found that close to T_c coalescence and decay processes played an important role. However at low temperatures and for superfluid states with a finite gap everywhere, coalescence and decay processes are less important by a factor $\sim e^{-\Delta/k_BT}$ than the two-excitation scattering processes. The total rate of coalescence processes is proportional to the probability of three quasiparticles colliding, $\sim(e^{-\Delta/k_BT})^3$; the rate for decay processes is also proportional to $e^{-3\Delta/k_BT}$ since a quasiparticle with an energy of at least 3Δ is required for the decay to be kinematically allowed. On the other hand, the rate for two-quasiparticle scattering processes is proportional to $e^{-2\Delta/k_BT}$. In this paper we concern ourselves only with the low-temperature limit, and corrections to the low-temperature limit involving powers of T. Thus we may neglect the coalescence and decay processes, and the collision term may be written in the standard form for two-quasiparticle scattering. It is convenient to work with the deviation function $\psi_{p\sigma}$, which is defined in terms of the deviation δn_i of the distribution function from the local equilibrium distribution, by

$$\delta n_i = n_i^0(1-n_i^0)\psi_i \,, \tag{2.6}$$

where i denotes both momentum and spin variables $\vec{p}_i$ and σ_i. The linearized collision integral may then be written

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_s(1,2;3,4)\, n_1^0 n_2^0 \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \times \delta(E_1+E_2-E_3-E_4) \times(\psi_1+\psi_2-\psi_3-\psi_4)\,. \tag{2.7}$$

The usual factor $(1-n_3^0)(1-n_4^0)$ which takes into account blocking of transitions due to occupation of the final states has been neglected in (2.7) since it differs from unity only by amounts of order $e^{-\Delta/k_BT}$. $W_s(1,2;3,4)$ is the collision probability for a transition in which quasiparticles in the states 1 and 2 are scattered to the states 3 and 4.

At low temperatures the Boltzmann equation may be further simplified. First, typical values of ξ_p are small compared with Δ, and therefore we may expand E_p in powers of ξ_p:

$$E_p = (\Delta^2+\xi_p^2)^{1/2} \simeq \Delta + \xi_p^2/2\Delta\,. \tag{2.8}$$

That this expansion is valid at low temperatures is apparent from the corresponding approximate form of the distribution function n^0:

$$n^0(E) \simeq e^{-\Delta/k_BT} e^{-\xi^2/2\Delta k_BT}\,, \tag{2.9}$$

which shows that the important values of ξ are of order $(\Delta k_B T)^{1/2}$ at low temperatures. The quasiparticle group velocity corresponding to the quasiparticle energy (2.8) is

$$v(\xi_p) = (\xi_p/\Delta) v_F\,, \tag{2.10}$$

so (2.8) has the form

$$E = \Delta + \tfrac{1}{2} m_{\text{eff}} v^2\,, \tag{2.11}$$

with the "effective mass" given by

$$m_{\text{eff}} = \Delta / v_F^2\,. \tag{2.12}$$

The collision probability in (2.7), in general, depends both on the angles between the momenta $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, and $\vec{p}_4$ and also on the quasiparticle energies. As one can see from the detailed expression for the collision probability given in Sec. III, the energy dependence comes from the ξ_p dependence of the superfluid coherence factors. The latter is on an energy scale of order Δ, and therefore in the low-temperature limit all values of ξ_p of interest are much less than Δ, which means that the collision probability may be evaluated for $\xi_1=\xi_2=\xi_3=\xi_4=0$. Note, however, that this is not true for evaluating the finite temperature corrections.

To reduce the collision integral (2.7) to manageable form we convert the nine summations over momentum variables into three integrals over the normal-state energies and two angular integrals over θ and ϕ in precisely the same way as is done in normal Fermi liquid calculations.[6] (To do this we have used momentum conservation to remove three of the nine momentum sums and performed one integration over an angular variable which does not enter the collision probability.) Also in the low-temperature limit we may use the approximate forms (2.8) and (2.10). It is then easy to show that the Boltzmann equation for the case of the shear viscosity, given by equating the streaming term (2.1) to the collision term (2.7), reduces to

$$\frac{1}{k_BT} p_x v_y \frac{\partial u_x}{\partial y} = -H\psi\,, \tag{2.13}$$

where the operator H is defined by

$$(H\psi)_1 = -[n_1^0(1-n_1^0)]^{-1}\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} \tag{2.14}$$

$$= \frac{m^{*3}}{8\pi^4\hbar^6}\int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_4\, \langle W_s^0\rangle n_2^0\, \delta\!\left(\frac{1}{2\Delta}(\xi_1^2+\xi_2^2-\xi_3^2-\xi_4^2)\right)(\psi_1+\psi_2-\psi_3-\psi_4)\,. \tag{2.15}$$

Here

$$\langle W_s^0\rangle = \int_{-1}^{1}\frac{d(\cos\theta)}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{W_s^0(\theta,\phi)}{\cos(\frac{1}{2}\theta)} \tag{2.16}$$

is the collision probability averaged over spins and angles of the initial and final quasiparticle states. θ is the angle between $\vec{p}_1$ and $\vec{p}_2$, and ϕ is the angle between the plane of $\vec{p}_1$ and $\vec{p}_2$ and the plane of $\vec{p}_3$ and $\vec{p}_4$. The spin-averaged transition probability is defined by

$$W_s^0(\theta,\phi) = \frac{1}{4}\times\frac{1}{2}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} W_s^0(1,2;3,4)\,. \tag{2.17}$$

In (2.17) the factor of $\frac{1}{4}$ averages over initial spins, and the factor of $\frac{1}{2}$ avoids double counting final states. The superscript 0 indicates that the probability is to be evaluated for $\xi_1=\xi_2=\xi_3=\xi_4=0$.

It is simple to solve the Boltzmann equation since if χ is an odd function of ξ, then

$$(H\chi)_1 = \chi_1/\tau \,, \tag{2.18}$$

where the relaxation time τ, which, as we shall see, is independent of ξ_1, is given by

$$\frac{1}{\tau} = \frac{m^{*3}}{8\pi^4\hbar^6}\int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_4 \,\langle W_s^0\rangle\, n_2^0 \times \delta\left(\frac{\xi_1^2+\xi_2^2-\xi_3^2-\xi_4^2}{2\Delta}\right). \tag{2.19}$$

This follows from the fact that the χ_2, χ_3, and χ_4 terms in (2.15) vanish identically if χ is an odd function of ξ, irrespective of the angular dependence of χ, because the integral kernel is an even function of the ξ_i.

The integral over ξ_3 and ξ_4 in (2.19) is easily performed if one works in terms of the variable $\xi_3^2+\xi_4^2$, and one finds

$$\frac{1}{\tau} = \frac{\pi^2}{6}\,\frac{\Delta}{\hbar}\,\frac{n_{\text{ex}}}{n}\left(\frac{\hbar}{2\pi}\,\nu^2(0)\langle W_s^0\rangle\right), \tag{2.20}$$

where

$$n_{\text{ex}} = \sum_{\vec{p}_2\sigma_2} n_2 = \nu(0)(2\pi\Delta k_BT)^{1/2} e^{-\Delta/k_BT} \tag{2.21}$$

is the number density of excitations, which tends to zero exponentially at low temperatures, and $\nu(0)$ is the density of states for both spins at the Fermi surface in the normal state. n is the number density of ^{3}He atoms, and m^* is the effective mass. The fact that the relaxation time is independent of energy is a direct consequence of the fact that the density of states of a pair of excitations in the superfluid, moving in definite directions, is independent of energy for energies small compared with Δ. The quantity in large parentheses in (2.20) is a dimensionless number which turns out to be rather large for liquid ^{3}He (~12 at the melting pressure). Clearly by symmetry the solution to (2.13) must be an odd function of ξ, and therefore using (2.18) one finds

$$\psi = -\frac{1}{k_BT}\,p_x v_y \frac{\partial u_x}{\partial y}\,\tau \,. \tag{2.22}$$

Having obtained ψ, it is straightforward to find the viscosity from the expression for the shear stress Π_{xy},

$$\Pi_{xy} = \sum_{\vec{p}\sigma} p_x v_y \,\delta n_{\vec{p}\sigma} \,, \tag{2.23}$$

where δn, given in terms of ψ by (2.6), is

$$\delta n_{\vec{p}\sigma} = \tau \frac{\partial n_p^0}{\partial E_p}\, p_x v_y \frac{\partial u_x}{\partial y} \,. \tag{2.24}$$

One finds by combining (2.23) and (2.24) with the definition of the viscosity η,

$$\Pi_{ij} = -\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\vec{\nabla}\cdot\vec{u}\,\delta_{ij}\right), \tag{2.25}$$

that

$$\eta = \tfrac{1}{5}\rho_n \langle v^2\rangle_{\text{th}}\tau \tag{2.26}$$

$$= \tfrac{1}{15}\,(v_F^2 p_F^2/\Delta) n_{\text{ex}}\tau \,, \tag{2.27}$$

where

$$\rho_n = \sum_{\vec{p}\sigma} \frac{1}{k_BT}\, p_x^2 n_p^0 = \frac{p_F^2}{3k_BT}\, n_{\text{ex}} \tag{2.28}$$

is the density of the normal fluid, and

$$\langle v^2\rangle_{\text{th}} = (k_BT/\Delta)\, v_F^2 \tag{2.29}$$

is the mean-square thermal quasiparticle velocity. Since $\tau \propto n_{\text{ex}}^{-1}$ it follows that η is independent of temperature.

An alternative way of writing the result (2.27) is

$$\eta = \tfrac{1}{5} n m^* v_F^2 \tau_s \,, \tag{2.30}$$

where

$$\tau_s^{-1} = \frac{\pi^2}{4}\,\frac{\Delta^2}{\hbar E_F}\left(\frac{\hbar}{2\pi}\,\nu^2(0)\langle W_s^0\rangle\right). \tag{2.31}$$

Here we have introduced the "Fermi energy" $E_F = p_F^2/2m^*$. The results (2.30) and (2.31) indicate that the zero-temperature viscosity is not too different from the viscosity at T_c, since $\Delta \sim k_BT_c$. The detailed calculations below give $\eta(0) \simeq \frac{1}{3}\eta(T_c)$ for the particular approximate normal-state scattering amplitude used.

The diffusive thermal conductivity κ_D may be calculated by precisely similar methods to those used for the viscosity. The driving term (2.5), proportional to the heat current carried by a quasiparticle $E_p v_p = \xi_p v_F$, is again an odd function of ξ_p and the Boltzmann equation may be solved exactly. The deviation function is found to be

$$\psi_{\vec{p}} = -(1/k_BT^2) E_p \vec{v}\cdot\vec{\nabla}T\tau \,, \tag{2.32}$$

and the deviation from local equilibrium is

$$\delta n_{\vec{p}} = \frac{E_p}{T}\,\vec{v}\cdot\vec{\nabla}T\tau \frac{\partial n_p^0}{\partial E_p} \,. \tag{2.33}$$

From this one can calculate the diffusive heat current $\vec{j}_D$, given by

$$\vec{j}_D = \sum_{\vec{p}\sigma} E\vec{v}\,\delta n_{\vec{p}\sigma} \,, \tag{2.34}$$

whence the diffusive thermal conductivity κ_D, defined by

$$\vec{j}_D = -\kappa_D \vec{\nabla} T , \quad (2.35)$$

is given by

$$\kappa_D = \tfrac{1}{3} c_V \langle v^2 \rangle_{th} \tau \quad (2.36)$$

$$= \tfrac{1}{3} (\Delta/T) v_F^2 n_{ex} \tau \quad (2.37)$$

$$= (n/m^*)(\Delta^2/T)\tau_s \quad (2.38)$$

or

$$\kappa_D T = \frac{2}{\pi^2} n v_F^2 \hbar \left(\frac{\hbar}{2\pi} \nu^2(0) \langle W_s^0 \rangle \right)^{-1} . \quad (2.39)$$

Here

$$c_V = (\Delta^2 / k_B T^2) n_{ex} \quad (2.40)$$

is the heat capacity per unit volume. As one can see from (2.39), κ_D has the same temperature dependence as in the normal state, and $\kappa_D T$ does not depend explicitly on the gap. As we shall see below $\kappa_D T$ is not very different from the normal-state value.

In contrast to the Boltzmann equation for the normal Fermi liquid, that for the superfluid in the low-temperature limit is very simple to solve, and the reasons for this are perhaps worth remarking on. One is that in the low-temperature limit the collision probability can be replaced by its value for $\xi_i = 0$, since its variation with the ξ_i occurs only on scales of order Δ. If one were to do transport theory for the normal state using only positive energy excitations one could not neglect the ξ_i dependence, because the nature of the excitations in the normal state changes discontinuously from being holelike to particlelike at $\xi_i = 0$. The collision probability in the normal state is therefore not invariant under the replacement of ξ_i by $-\xi_i$. As we shall see in Sec. IV, this symmetry also ceases to exist in the superfluid when the temperature is finite.

Another quantity of direct physical importance is the quasiparticle mean free path l, since for the concept of a transport coefficient to make sense spatial variation must occur over length scales large compared with l. l is given by the quasiparticle relaxation time τ multiplied by a characteristic quasiparticle velocity, which we take to be the root-mean-square thermal velocity (2.29). Thus

$$l = (\langle v^2 \rangle_{th}^{1/2}) \tau = (2\pi)^{-1/2} v_F \tau_s e^{\Delta/k_B T} . \quad (2.41)$$

We return to a discussion of the magnitude of l in Sec. V.

It is a simple consequence of the existence of the energy independent relaxation time τ common to both the viscosity and heat-conductivity equation that the ratio of these two transport coefficients is independent of the collision probability. One finds from (2.27) and (2.37) the simple relation

$$\kappa_D T / \eta = 5\Delta^2 / p_F^2 . \quad (2.42)$$

III. COLLISION PROBABILITY

To estimate the collision probability in the superfluid state we wish to make use of the experimental information one has about scattering in the normal state. We shall make the standard weak-coupling assumption that the residual interaction in the superfluid is the same as in the normal liquid. One might question this assumption in view of the importance strong-coupling effects have in stabilizing the A phase.[7] However, in the case of free-energy calculations the strong-coupling corrections are particularly important because they have to be compared with the rather small difference between the free energies of two phases. In the case of transport calculations it seems unlikely that strong coupling effects play such an important role. A second reason for neglecting strong-coupling effects on the residual interaction is that these are almost certainly small compared with uncertainties in our knowledge of the normal-state residual interaction. We note, however, that estimates of the strong-coupling effects may be obtained quite straightforwardly within the framework of, for example, the spin-fluctuation model.

Once the weak-coupling assumption has been made, the scattering amplitude for superfluid quasiparticles may be found by performing a Bogoliubov transformation on the corresponding normal-state amplitude. As discussed in Sec. II we take into account only the two quasiparticle scattering processes. The result for the spin-averaged transition probability is

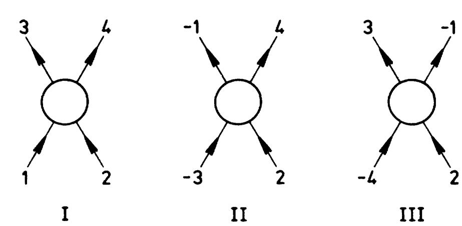

FIG. 1. Diagrammatic illustration of the scattering amplitudes entering the collision probability.

$$
\begin{aligned}
W_s(\vec{p}_1,\vec{p}_2;\vec{p}_3,\vec{p}_4) &\equiv \frac{2\pi}{\hbar}\frac{1}{4}\frac{1}{2}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} W_s(1,2;3,4) \\
&= \frac{2\pi}{\hbar}\frac{1}{8}\{(u_1^2u_2^2u_3^2u_4^2+v_1^2v_2^2v_3^2v_4^2)(S_I^2+3T_I^2)+2u_1v_1u_2v_2u_3v_3u_4v_4 \\
&\quad\times[\cos^2\theta_{12}S_I^2+(2\cos^2\theta_{13}+2\cos^2\theta_{14}-\cos^2\theta_{12})T_I^2+(\cos^2\theta_{13}-\cos^2\theta_{14})2T_IS_I] \\
&\quad-[u_1v_1u_2v_2(u_3^2v_4^2+v_3^2u_4^2)+u_3v_3u_4v_4(v_1^2u_2^2+u_1^2v_2^2)]\cos\theta_{12}(S_{II}S_{III}-5T_{II}T_{III}+S_{II}T_{III}-T_{II}S_{III}) \\
&\quad+(v_1^2u_2^2v_3^2u_4^2+u_1^2v_2^2u_3^2v_4^2)(S_{II}^2+3T_{II}^2)+2u_1v_1u_2v_2u_3v_3u_4v_4 \\
&\quad\times[\cos^2\theta_{14}S_{II}^2+(2\cos^2\theta_{13}+2\cos^2\theta_{12}-\cos^2\theta_{14})T_{II}^2+(\cos^2\theta_{13}-\cos^2\theta_{12})2S_{II}T_{II}] \\
&\quad-[u_2v_2u_3v_3(u_1^2u_4^2+v_1^2v_4^2)+u_1v_1u_4v_4(u_2^2u_3^2+v_2^2v_3^2)]\cos\theta_{14}(S_IS_{III}+5T_IT_{III}-S_IT_{III}-T_IS_{III}) \\
&\quad+(v_1^2u_2^2u_3^2v_4^2+u_1^2v_2^2v_3^2u_4^2)(S_{III}^2+3T_{III}^2)+2u_1v_1u_2v_2u_3v_3u_4v_4 \\
&\quad\times[\cos^2\theta_{13}S_{III}^2+(2\cos^2\theta_{12}+2\cos^2\theta_{14}-\cos^2\theta_{13})T_{III}^2+(\cos^2\theta_{12}-\cos^2\theta_{14})2S_{III}T_{III}] \\
&\quad-[u_2v_2u_4v_4(u_1^2u_3^2+v_1^2v_3^2)+u_1v_1u_3v_3(u_2^2u_4^2+v_2^2v_4^2)]\cos\theta_{13}(S_IS_{II}+5T_IT_{II}+S_IT_{II}+T_IS_{II})\}.
\end{aligned} \tag{3.1}
$$

For brevity we have omitted the modulus signs on the $|u_i|^2$ and $|v_i|^2$ in this equation.

The details of the calculation are given in the Appendix, in which the corresponding results for the decay and coalescence processes are also included. Here S and T are the singlet and triplet quasiparticle scattering amplitudes for quasiparticles in the normal state. The Roman numeral subscripts label the various normal-state scattering amplitudes which contribute to the superfluid amplitude. S_I and T_I give the amplitudes for a process in which the incoming quasiparticles have momenta $\vec{p}_1$ and $\vec{p}_2$ and the outgoing quasiparticles have momenta $\vec{p}_3$ and $\vec{p}_4$. S_{II} and T_{II} are the amplitudes for quasiparticles of momenta $-\vec{p}_3$ and $\vec{p}_2$ to be scattered to states $-\vec{p}_1$ and $\vec{p}_4$. Finally S_{III} and T_{III} are the amplitudes for quasiparticles of momenta $-\vec{p}_4$ and $\vec{p}_2$ to be scattered to states $\vec{p}_3$ and $-\vec{p}_1$. The processes are illustrated diagrammatically in Fig. 1. θ_{ij} is the angle between $\vec{p}_i$ and $\vec{p}_j$, and the coherence factors u and v are given explicitly in Eqs. (A8) and (A9).

We observe that the second and third group of terms in (3.1) are obtained from the first group by the replacements $u_1 \leftrightarrow v_3$, $u_3 \leftrightarrow v_1$, $\cos\theta_{12} \leftrightarrow \cos\theta_{14}$ and $u_1 \leftrightarrow v_4$, $u_4 \leftrightarrow v_1$, $\cos\theta_{12} \leftrightarrow \cos\theta_{13}$, respectively, provided one makes the obvious replacements of S_I^2 by S_{II}^2 and S_{III}^2 and of $S_{II}S_{III}$ by S_IS_{III} and S_IS_{II}, etc., in addition to making the appropriate changes of sign in the cross terms.

First, let us consider the simplest possible case, in which the singlet amplitude is a constant $S_I=S_{II}=S_{III}=S$, and the triplet amplitude vanishes $T_I=T_{II}=T_{III}=0$. As we shall see, this is a rather good first approximation for liquid ^{3}He. The corresponding collision probability is

$$W_s^0(\theta,\phi)=(2\pi/\hbar)\tfrac{1}{32}S^2\sin^4(\tfrac{1}{2}\theta)(3+\cos^2\phi). \tag{3.2}$$

To obtain this result the coherence factors u and v in (3.1) were put equal to $1/\sqrt{2}$, their value for $\xi_i=0$, and the angles θ_{ij} were expressed in terms of the usual angles $\theta(\equiv\theta_{12})$ and ϕ defined below Eq. (2.16). The relations are

$$\cos\theta_{12}=\cos\theta, \tag{3.3}$$

$$\cos\theta_{13}=\cos\theta_{24}=\cos^2(\tfrac{1}{2}\theta)+\sin^2(\tfrac{1}{2}\theta)\cos\phi, \tag{3.4}$$

and

$$\cos\theta_{14}=\cos\theta_{23}=\cos^2(\tfrac{1}{2}\theta)-\sin^2(\tfrac{1}{2}\theta)\cos\phi. \tag{3.5}$$

Note that W_s^0 vanishes in the forward direction ($\theta=0$). This is due to two effects. First, the scattering amplitude for initial or final states having quasiparticles in the same state must vanish by virtue of the Pauli principle; this is the argument which shows that the triplet amplitude for $\theta=0$ must vanish in the normal state. The fact that in the superfluid the other amplitudes also vanish in the forward direction is a consequence of a cancellation due to interference between S_I, S_{II}, and S_{III} which occurs only when the singlet amplitude is a constant. We remark that the angular dependence of $W_s^0(\theta,\phi)$ in this simple case is due entirely to the anisotropy of the superfluid state, since the normal-state scattering amplitude is constant.

The angular average $\langle W_s^0\rangle$ is given by

$$\langle W_s^0\rangle=\int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-1}^{1}\frac{d(\cos\theta)}{2}\,\frac{W_s^0(\theta,\phi)}{\cos(\frac{1}{2}\theta)} \tag{3.6}$$

$$=(2\pi/\hbar)\tfrac{7}{60}S^2. \tag{3.7}$$

The corresponding quantity for the normal state is

$$\langle W_N\rangle=(2\pi/\hbar)\tfrac{1}{4}S^2, \tag{3.8}$$

which shows that the effect of the superfluid correlations is to reduce the amount of scattering.

We have also calculated the transition probability in the BCS state of an s-wave superfluid for the constant singlet amplitude and find the same angle-independent value as in the normal state

$$W_s^0\big|_{\mathrm{BCS}} = W_N = (2\pi/\hbar)\tfrac{1}{8}S^2 \tag{3.9}$$

the angular average of which is of course the same as for the normal state, Eq. (3.8). For a given normal-state interaction, one therefore expects less scattering in the BW state of a p-wave superfluid than in the BCS s-wave state.

To make more-realistic estimates of the scattering amplitude we need to take into account the angular dependence of the normal-state scattering amplitudes. S_{I} and T_{I} are functions of the variables θ and ϕ. S_{II} and T_{II} are the same functions of the angles θ_{II} and ϕ_{II}, related to process II in the same way as θ and ϕ are related to process I. Similar remarks apply for S_{III} and T_{III}, and the corresponding angles are θ_{III} and ϕ_{III}. For the normal-state scattering amplitude we use the s- and p-wave approximation, as in Paper I:

$$\nu(0)S_I = C_1 + C_2\cos\theta \tag{3.10}$$

and

$$\nu(0)T_I = (C_3 + C_4\cos\theta)\cos\phi . \tag{3.11}$$

The dimensionless coefficients C_i are related to the Landau parameters by the relations given in Eq. (E13) of I.

Substituting (3.10) and (3.11) into (3.1), and putting all coherence factors equal to $1/\sqrt{2}$, one obtains a rather lengthy expression for $\langle W_s^0\rangle$. The number of distinct integrals may be reduced by using the fact that

$$\int d(\cos\theta)\, d\phi/\cos(\tfrac{1}{2}\theta)\cdots$$

may be replaced by

$$\int d(\cos\theta_{\mathrm{II}})\, d\phi_{\mathrm{II}}/\cos(\tfrac{1}{2}\theta_{\mathrm{II}})\cdots$$

or

$$\int d(\cos\theta_{\mathrm{III}})\, d\phi_{\mathrm{III}}/\cos(\tfrac{1}{2}\theta_{\mathrm{III}})\cdots,$$

as is obvious from the fact that the phase space for the scattering process may be described equally well in terms of the angles for any of the processes I, II, or III. To perform the final integrals it is convenient to use the variable

$$x = \cos(\tfrac{1}{2}\theta), \tag{3.12}$$

since $d(\cos\theta)/\cos(\frac{1}{2}\theta) = 4\,dx$.

The angles θ_{II}, θ_{III}, ϕ_{II}, and ϕ_{III} are expressed in terms of x and ϕ by

$$\cos\theta_{\mathrm{II}} = -\cos\theta_{23} = -x^2 + (1-x^2)\cos\phi , \tag{3.13}$$

$$\cos\theta_{\mathrm{III}} = -\cos\theta_{24} = -x^2 - (1-x^2)\cos\phi , \tag{3.14}$$

$$\cos\phi_{\mathrm{II}} = \frac{(1-x^2)\cos\phi + 3x^2 - 1}{(-1+x^2)\cos\phi + 1 + x^2}, \tag{3.15}$$

and

$$\cos\phi_{\mathrm{III}} = \frac{(1-x^2)\cos\phi - 3x^2 + 1}{(1-x^2)\cos\phi + 1 + x^2}. \tag{3.16}$$

We find then for the entire collision probability $\langle W_s^0\rangle$ within the s- and p-wave approximation [(3.10) and (3.11)]

$$\nu^2(0)\langle W_s^0\rangle = \frac{2\pi}{\hbar}\sum_{ij} D_{ij} C_i C_j , \tag{3.17}$$

where the symmetric matrix D_{ij} is given by

$$D = \begin{bmatrix} \frac{7}{60} & -\frac{5}{84} & \frac{1}{240} & \frac{1}{112} \\ -\frac{5}{84} & \frac{13}{140} & -\frac{1}{280} & \frac{1}{168} \\ \frac{1}{240} & -\frac{1}{280} & -\frac{1139}{480} + \frac{15}{4}\ln 2 & -\frac{3021}{1120} + \frac{15}{4}\ln 2 \\ \frac{1}{112} & \frac{1}{168} & -\frac{3021}{1120} + \frac{15}{4}\ln 2 & -\frac{1183}{480} + \frac{15}{4}\ln 2 \end{bmatrix}, \tag{3.18}$$

or approximately as

$$D = 10^{-3} \times \begin{bmatrix} 117 & -60 & 4 & 9 \\ -60 & 93 & -4 & 6 \\ 4 & -4 & 226 & -98 \\ 9 & 6 & -98 & 135 \end{bmatrix} . \tag{3.19}$$

The numbers in (3.19) with the exception of D_{11} differ somewhat from those given in Ref. 4 due to a computational error in the previous work, where the integrals were done numerically rather than analytically.

To calculate the zero-temperature viscosity $\eta(0)$ and the thermal conductivity we must finally combine (2.30), (2.31), and (2.39) with (3.17)–(3.19). The values of the coefficients C_i are obtained from the available experimental information on the Landau parameters F_0^s, F_0^a, and F_1^s with F_1^a fixed by the forward-scattering sum rule. The results of such a calculation using Wheatley's[3] tabulated values of the Landau parameters are shown in Figs. 2 and 3 and in Table I. We have used the weak-coupling gap $\Delta = (\pi/\gamma)k_B T_c \simeq 1.76 k_B T_c$. A discussion of these results will be given in Sec. V. Here we shall only point out that the calculated $\eta(0)$ is between 0.27 and 0.34 times the value of the viscosity at T_c, the ratio changing slightly with pressure between 0 and 34 bar. For $\kappa_D T$ the corresponding variation is between 0.63 and 0.72 of $\kappa_D(T_c)T_c$.

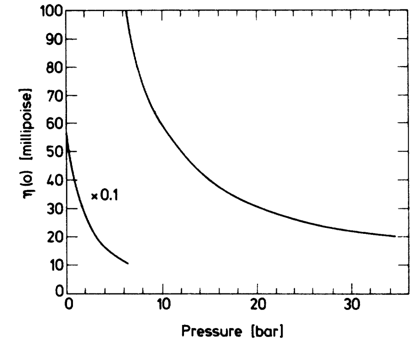

FIG. 2. Calculated zero-temperature viscosity $\eta(0)$ as a function of pressure. The rapid increase with decreasing pressure is due primarily to the decrease in the magnitude of the zero-temperature gap.

The magnitudes of the matrix elements of D, Eq. (3.19), cause the singlet part of the amplitude to dominate the angular average of the collision probability because of the large value of C_1 in particular. Note that there is little mixing between the singlet and the triplet parts of the normal-state scattering amplitude due to the smallness of the relevant matrix elements. In the constant singlet approximation ($C_1 \neq 0$, $C_2 = C_3 = C_4 = 0$) the value of $\langle W_s^0 \rangle$ differs by approximately 10% from the value obtained from the s- and p-wave approximation. However the ratio $\eta(0)/\eta(T_c)$ in the constant singlet approximation is 0.53 which is significantly different because of the considerable change in the value of $\eta(T_c)$ in the two approximations. For thermal conductivity the corresponding number is 0.87.

IV. FINITE-TEMPERATURE CORRECTIONS

At finite temperatures the exact solution of the Boltzmann equation is not given by the relaxation-time expression. There are two classes of corrections that must be taken into account. The first arises because one must retain higher-order terms in ξ in the expressions for the quasiparticle energy and related quantities. We shall refer to these corrections as *kinematical* corrections. The quasiparticle energy (2.3) is thus approximated by

$$E \simeq \Delta + \xi^2/2\Delta - \xi^4/8\Delta^3 \,. \tag{4.1}$$

Previously we kept only the quadratic term in the expansion (4.1). The inclusion of the quartic term modifies the group velocity of the excitations according to

$$v = v_F(\xi/\Delta)[1 - (k_B T/\Delta)x^2] \,, \tag{4.2}$$

where

$$x = \xi/(2\Delta k_B T)^{1/2} \tag{4.3}$$

is a convenient dimensionless variable. The occupation factors $n^0(E)$ change as well, and for the purposes of calculating the leading finite-temperature corrections may be replaced by

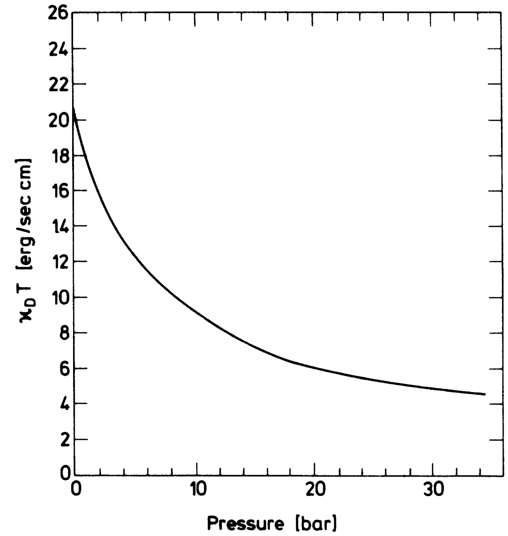

FIG. 3. Calculated zero-temperature limiting value of $\kappa_D T$. Note that $\kappa_D T$ is given in terms of η by the relation $\kappa_D T/\eta = 5\Delta^2/p_F^2$.

$$n^0(E) \simeq e^{-\Delta/k_B T} e^{-x^2}[1 + \tfrac{1}{2}(k_B T/\Delta)x^4] \,. \tag{4.4}$$

The second class of corrections comes from changes in the collision probability due to the ξ dependence of the coherence factors. We shall refer to these corrections as *dynamical* ones. The coherence factors are given by

$$u = [\tfrac{1}{2}(1 + \xi/E)]^{1/2}$$

and (4.5)

$$v = [\tfrac{1}{2}(1 - \xi/E)]^{1/2} \,.$$

When expanded to second order in ξ the quantities u^2, v^2, and uv, which are the only ones that enter the expression (3.1) for the collision probability, are

$$u^2 = \tfrac{1}{2}(1 + \xi/\Delta) \,,$$
$$v^2 = \tfrac{1}{2}(1 - \xi/\Delta) \,, \tag{4.6}$$

and

$$uv = \Delta/2E \simeq \tfrac{1}{2}(1 - \xi^2/2\Delta^2) \,. \tag{4.7}$$

The corrections to the viscosity and diffusive thermal conductivity due to the above sources are of order $k_B T/\Delta$ relative to the low-temperature limiting value. A further source of corrections comes from decay and coalescence processes, and the $1-n$ factors in the two-quasiparticle scattering process. These give contributions of relative order $e^{-\Delta/k_B T}$, and thus at low temperatures

TABLE I. Transport properties of liquid ^{3}He as a function of pressure, calculated in the s- and p-wave approximation for the normal-state scattering amplitude. The values of T_c are taken from Ref. 3. Those at the lowest pressures were read off from the phase diagram, Fig. 1 of Ref. 3.

Pressure (bar)	0	3	6	9	12	15	18	21	24	27	30	33	34.36
ηT^2 (PmK^2)	1.88	1.19	0.97	0.79	0.70	0.61	0.54	0.53	0.48	0.47	0.46	0.42	0.44
κT (erg/sec cm)	35	22	18	14	12	11	9.4	8.9	8.1	7.7	7.4	6.8	6.9
DT^2 (cm^2 mK^2/sec)	1.66	0.81	0.56	0.39	0.31	0.24	0.19	0.18	0.15	0.14	0.13	0.11	0.11
α_η	1.25	1.13	1.10	1.06	1.05	1.03	1.01	1.02	1.01	1.02	1.03	1.03	1.05
α_κ	2.48	2.13	2.06	1.93	1.90	1.84	1.78	1.82	1.79	1.82	1.85	1.83	1.89
α_D	0.028	−0.198	−0.246	−0.328	−0.350	−0.382	−0.422	−0.396	−0.418	−0.396	−0.378	−0.388	−0.350
$\tau(0)T^2$ (μsec mK^2)	0.61	0.44	0.37	0.32	0.29	0.26	0.24	0.23	0.22	0.21	0.21	0.20	0.20
$\eta(0)$ (mP)	589	216	116	69	53	42	35	31	27	25	23	21	21
T_c (mK)	0.92	1.30	1.62	1.94	2.09	2.21	2.31	2.39	2.46	2.51	2.55	2.59	2.60

vanish more rapidly than the contributions discussed above. Finally, we remark that the temperature dependence of the gap may also be neglected, since the leading term is proportional to $e^{-\Delta/k_BT}$.

The simplest way to calculate the leading finite temperature corrections to the transport coefficients is to use the variational principle which gives a lower bound on the viscosity and the thermal conductivity. For the viscosity one has

$$\eta \geq k_BT[(U,X)^2/(U,HU)] , \tag{4.8}$$

where the scalar product is defined by

$$(A,B)=\sum_{\vec{p}_1\sigma_1} A_{\vec{p}_1\sigma_1} B_{\vec{p}_1\sigma_1} n_1^0(1-n_1^0) , \tag{4.9}$$

and the linear operator H is given by Eqs. (2.14) and (2.15). Also

$$X_{\vec{p}_1\sigma_1} = -v_{1_x} p_{1_y}/k_BT,$$

which equals the driving term in the Boltzmann equation (2.13) with $-\partial u_x/\partial y$ replaced by unity. The trial function U in (4.8) is arbitrary, but for U equal to ψ, the exact solution of the Boltzmann equation, the bound (4.8) is equal to the true viscosity, given by

$$\eta = k_BT(\psi, X) . \tag{4.10}$$

One may calculate the leading finite-temperature corrections by evaluating the variational expression using the zero-temperature solution as the trial function. Because of the variational property, deviations of ψ from the zero-temperature solution do not affect the leading corrections, and therefore one can avoid solving the finite-temperature Boltzmann equation. The trial function U is therefore taken to be proportional to the zero-temperature solution (2.22),

$$U_{\vec{p}_1\sigma_1} = \xi_1 p_{1_x} p_{1_y} . \tag{4.11}$$

Let us consider the numerator and the denominator of (4.8) in turn, leaving out for the moment the dynamical corrections, which will be added later. For the term in the numerator in (4.8) one finds

$$(X,U)=(X,U)_0(1+\tfrac{3}{8}k_BT/\Delta) , \tag{4.12}$$

where $(\ ,\ )_0$ is to be understood as the $T\to 0$ limit of the quantity. This result is easily established by noting that (X,U) is proportional to the integral

$$\int_{-\infty}^{\infty} d\xi\, \xi v(\xi) n^0(\xi) .$$

If one now expands v and n^0 in powers of x, using (4.2) and (4.4), the integral is proportional to

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2}\left(1-\frac{k_BT}{\Delta}x^2\right)\left(1+\frac{1}{2}\frac{k_BT}{\Delta}x^4\right) = \frac{\sqrt{\pi}}{2}\left(1+\frac{3}{8}\frac{k_BT}{\Delta}\right) \tag{4.13}$$

to linear order in k_BT/Δ.

When the changes in the collision probability are ignored, the relevant integral in the denominator is

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_4\, e^{-(x_1^2+x_2^2)} x_1^2 \left(1+\frac{1}{2}\frac{k_BT}{\Delta}x_1^2\right)\left(1+\frac{1}{2}\frac{k_BT}{\Delta}x_2^2\right) \times \delta\left(x_1^2+x_2^2-x_3^2-x_4^2-\frac{1}{2}\frac{k_BT}{\Delta}(x_1^4+x_2^4-x_3^4-x_4^4)\right) = \frac{\pi^2}{2}\left(1+\frac{15}{4}\frac{k_BT}{\Delta}\right) \tag{4.14}$$

to linear order in k_BT/Δ, since the delta function when integrated with respect to x_3 and x_4 gives

$$\int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_4\, \delta\left(x_1^2+x_2^2-x_3^2-x_4^2-\frac{1}{2}\frac{k_BT}{\Delta}(x_1^4+x_2^4-x_3^4-x_4^4)\right) = \pi\left(1+\frac{3}{4}(x_1^2+x_2^2)\frac{k_BT}{\Delta}\right) . \tag{4.15}$$

Consequently, the kinematical corrections combine to reduce the viscosity by the factor $(1-3k_BT/\Delta)$.

Now let us turn to the dynamical corrections. After performing integrals over all angular variables except θ and ϕ one finds for the denominator in (4.8), neglecting kinematical corrections,

$$(U,HU) \propto \int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-1}^{1}\frac{d(\cos\theta)}{\cos(\frac{1}{2}\theta)}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}dx_3\int_{-\infty}^{\infty}dx_4\, e^{-(x_1^2+x_2^2)}\delta(x_1^2+x_2^2-x_3^2-x_4^2)W_s(\theta,\phi,x_1,x_2,x_3,x_4) \times x_1[x_1+x_2P_2(\cos\theta)-x_3P_2(\cos\theta_{13})-x_4P_2(\cos\theta_{14})] , \tag{4.16}$$

where θ_{13} and θ_{14} are the angles between $\vec{p}_1$ and $\vec{p}_3$, and $\vec{p}_1$ and $\vec{p}_4$. The P_2 functions are $l=2$ Legendre polynomials, which enter the viscosity by virtue of the $l=2$ symmetry of the driving term.

We now evaluate (4.16) assuming that the normal state scattering amplitude is a constant S acting only in the singlet spin state. This approximation gives results for the low-temperature viscosity which differ by less than 10% from those of the full s- and p-wave approximation. The collision probability expanded to second order in the x_i is

$$W_s(\theta,\phi,x_1,x_2,x_3,x_4)\propto 6-\frac{k_BT}{\Delta}(x_1x_2+x_3x_4-x_1x_3-x_1x_4-x_2x_3-x_2x_4)$$
$$+2\left(1-\frac{k_BT}{\Delta}\sum_{i=1}^{4}x_i^2\right)(\cos^2\theta+\cos^2\theta_{\rm II}+\cos^2\theta_{\rm III})$$
$$-\left(4-2\frac{k_BT}{\Delta}\sum_{i=1}^{4}x_i^2\right)(\cos\theta-\cos\theta_{\rm II}-\cos\theta_{\rm III})$$
$$+(x_1x_2+x_3x_4)\cos\theta+(x_1x_4+x_2x_3)\cos\theta_{\rm II}+(x_1x_3+x_2x_4)\cos\theta_{\rm III}\ . \qquad (4.17)$$

When this is substituted into (4.16), the only terms proportional to k_BT/Δ which survive are ones involving integrals of x_1^4, $x_1^2x_2^2$, $x_1^2x_3^2$, or $x_1^2x_4^2$: the rest vanish by symmetry. The final angular integral to be performed is

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-1}^{1}\frac{d(\cos\theta)}{\cos(\frac{1}{2}\theta)}\Big(2[(1-\cos\theta)^2+(1+\cos\theta_{\rm II})^2+(1+\cos\theta_{\rm III})^2]$$
$$-\frac{8k_BT}{\Delta}(\cos^2\theta+\cos^2\theta_{\rm II}+\cos^2\theta_{\rm III}-\cos\theta+\cos\theta_{\rm II}+\cos\theta_{\rm III})-\frac{2k_BT}{\Delta}(1-\cos\theta)P_2(\cos\theta)$$
$$-\frac{4k_BT}{\Delta}(1+\cos\theta_{\rm II})P_2(\cos\theta_{13})-\frac{4k_BT}{\Delta}(1+\cos\theta_{\rm III})P_2(\cos\theta_{14})\Big)\propto 1-\frac{121}{98}\frac{k_BT}{\Delta}\ . \qquad (4.18)$$

Note that this integral depends on the angular part of the trial function, through the P_2, and will therefore not be the same for other transport coefficients. It is therefore not possible at finite temperatures to use a common relaxation time to describe both the viscosity and the diffusive thermal conductivity. Combining the kinematical and dynamical corrections one finds

$$\eta=\eta(0)(1-\tfrac{173}{98}k_BT/\Delta)\ , \qquad (4.19)$$

where $\eta(0)$ denotes the zero-temperature viscosity calculated in Sec. III, in this case corresponding to the result (3.7) for the collision probability. In weak-coupling theory

$$\Delta(T=0)=(\pi/\gamma)k_BT_c\simeq 1.76\ k_BT_c,$$

and therefore the finite-temperature correction term is close to $-\eta(0)T/T_c$. We have not made estimates of the correction term for more general interactions, but it seems likely that for ones appropriate for liquid ^{3}He, η will decrease with increasing temperature.

Analogous calculations can be carried out for the finite temperature contribution to the diffusive thermal conductivity. The variational expression for the diffusive thermal conductivity is similar to (4.8) but with $X_{\vec{p}_1\sigma_1}$ proportional to the heat current carried by a quasiparticle

$$E_1v_1=\xi_1v_F\ . \qquad (4.20)$$

The appropriate trial function is again proportional to the zero-temperature solution ψ, given by

$$\psi_{\vec{p}_1}\propto E_1\vec{v}_1\cdot\vec{\nabla}T=\xi_1v_F\hat{p}_1\cdot\vec{\nabla}T\ . \qquad (4.21)$$

Note that ψ is linear in ξ_1, as in the case of viscosity, but has $l=1$ angular dependence. After calculations similar to those described above for the viscosity, one finds for the kinematical corrections

$$(X,U)=(X,U)_0(1+\tfrac{15}{8}k_BT/\Delta)\ , \qquad (4.22)$$

$$(U,HU)=(U,HU)_0(1+\tfrac{15}{4}k_BT/\Delta)\ . \qquad (4.23)$$

Thus the kinematical corrections to κ_DT vanish.

For a constant singlet effective interaction in the normal state the dynamical corrections are given by an expression identical with (4.16), apart from the replacement of the P_2 that occur there by P_1. One finds

$$(U,HU)=(U,HU)_0(1-\tfrac{11}{14}k_BT/\Delta)\ , \qquad (4.24)$$

and hence

$$\kappa_DT=\kappa_DT|_{T=0}(1+\tfrac{11}{14}k_BT/\Delta)\ . \qquad (4.25)$$

Thus κ_DT, in contrast to η, increases with increasing temperature. The main reason for the different behavior is that the current X is rigorously proportional to ξ for *all* ξ in the case of the heat current, but not in the case of the momentum current.

V. DISCUSSION

In this concluding section we first discuss the experimentally important question of the magnitude of the mean free path at low temperatures.

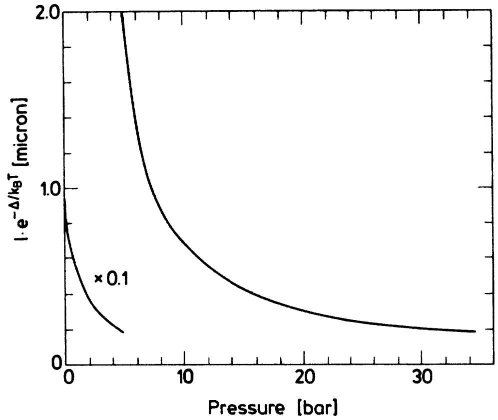

FIG. 4. Calculated values of the low-temperature mean free path l. $le^{-\Delta/k_BT}$ is shown as a function of pressure [cf. Eq. (2.41)].

Subsequently we shall deal with the adequacy of the s- and p-wave approximation to the scattering amplitude used in the calculation of the magnitude of the transport coefficients.

To perform an experiment which measures the low-temperature viscosity characteristic of the bulk liquid the mean free path must be less than the dimensions of the sample and other characteristic lengths such as the viscous penetration depth. We have defined the mean free path l by Eq. (2.41) in terms of the thermally averaged group velocity of an excitation and the relaxation

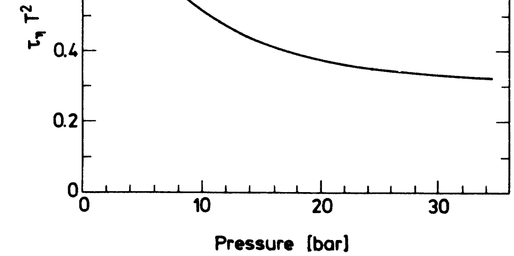

FIG. 5. Pressure dependence of the normal-state viscous relaxation time τ_η times T^2, where the viscosity is related to τ_η by $\eta = \frac{1}{5}m^* n v_F^2 \tau_\eta$. The circles are experimental values from Ref. 3, and the full line is the theoretical value calculated using the s- and p-wave approximation to the scattering amplitude.

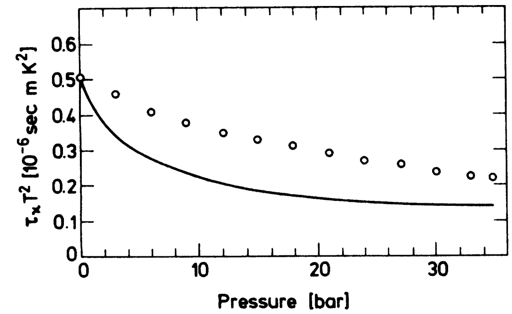

FIG. 6. Pressure dependence of the normal-state thermal-conductivity relaxation time τ_κ times T^2, where the thermal conductivity κ is related to τ_κ by $\kappa = \frac{1}{3}c_V v_F^2 \tau_\kappa$. The circles are experimental values taken from Ref. 3, and the full line is the theoretical value calculated using the s- and p-wave approximation.

time τ. Once the angularly averaged collision probability is calculated, one can from it derive τ_s, Eq. (2.31), and hence the magnitude of the temperature-independent quantity $le^{-\Delta/k_BT}$. The latter is shown in Fig. 4 as a function of pressure. The rapid increase at lower pressures reflects the increase in the normal-state mean free path at the transition temperature T_c with decreasing T_c. To obtain the low-temperature value of l at a given temperature and pressure, one may use the plot in Fig. 4 together with the value of the exponential e^{Δ/k_BT} at the temperature in question. Note that on the melting curve l is no greater than about 6μ at $T=\frac{1}{2}T_c$. It is therefore possible to have l smaller than any characteristic length in an experiment, even when $k_BT<\Delta$. In this connection one should note that the viscous penetration depth δ also increases as the temperature is lowered, since $\delta=(\eta/\rho_n\omega)^{1/2}\propto T^{1/4}e^{\Delta/(2k_BT)}$. The magnitude at a given frequency ω is readily calculated from the value of the zero temperature viscosity together with the expression for ρ_n given by Eqs. (2.28) and (2.21).

In Figs. 2 and 3 we displayed results for the shear viscosity and diffusive thermal conductivity calculated using the s- and p-wave approximation for the normal state scattering amplitude. To give some idea of how good the s- and p-wave approximation is we give in Figs. 5 and 6 and in Tables I and II results for various normal-state properties and $\eta(0)$ calculated using this approximation. The theoretical expressions for the normal-state quantities are all given in the preceding paper.[1] The first point to be made is that the s- and p-wave approximation accounts rather well for the magnitudes of the measured normal-state relaxation times, considering the limited amount of experimental information the approximation employs. The scattering amplitude

TABLE II. Experimental and theoretical values of $\tau(0)\,T_c^2$.

Pressure (bar)	$\tau(0)\,T_c^2$ (μsec mK^2) Experiment	Theory (s- and p-wave approximation)
33.45	0.26 [a]	0.20
29.34	0.26 [a]	0.21
24.02	0.30 [a]	0.22
20.7	0.36 [b]	0.23

[a] Orbital relaxation (Ref. 8).
[b] Spin relaxation (Ref. 9).

for $\phi = 0$, and, because of the Fermi statistics, for $\phi = \pi$, may be expressed in terms of Landau parameters, but one has no direct way of investigating the scattering amplitude for $\phi \neq 0$. The interpolation used in the s- and p-wave approximation is the simplest consistent with the Fermi statistics, but for a system as dense as ^{3}He there is no reason to expect the contribution from higher partial waves to be negligible. One can understand why the discrepancy between theoretical and experimental values is greater for the viscous relaxation time than for the other relaxation times, because the viscous relaxation time depends most strongly on the scattering rate for $\phi \sim \frac{1}{2}\pi$, where our knowledge of the scattering amplitude is poorest.

One may ask whether it is possible to find a scattering amplitude which accounts for all the measured normal state properties. There are many ways in which one can modify the scattering amplitude when one goes beyond the simple s- and p-wave approximation. We have explored the consequences of treating A_1^a as an adjustable parameter in the s- and p-wave approximation rather than fixing it by imposing the forward-scattering sum rule. In Figs. 7 and 8 we show η at $T = 0$ and at $T = T_c$, and $\kappa_D T$ in the normal state, and in the superfluid close to $T = 0$ at the melting pressure as functions of A_1^a, the other Landau parameters being kept fixed at the values given by Wheatley.[3] For $A_1^a \simeq -1.8$ ($F_1^a = -1.1$) the normal-state properties agree with the experimental values. At other pressures it is possible to obtain agreement between theory and experiment by adjusting A_1^a. At zero pressure the value of A_1^a required is -0.9 ($F_1^a = -0.7$), which is close to the value needed to satisfy the forward-scattering sum rule if one neglects Landau parameters with $l > 1$.

The effect of making A_1^a more negative is to shift weight from $\theta \simeq \pi$ to $\theta \simeq 0$ in the singlet scattering amplitude, which is much more important than the triplet amplitude. Since the inverse viscosity and hence τ_η^{-1} to a good approximation is proportional to the angular average $\langle W_N(\theta,\phi)\sin^2\phi\sin^4(\frac{1}{2}\theta)\rangle$ of the normal-state collision probability W_N, a shift of weight from $\theta = \pi$

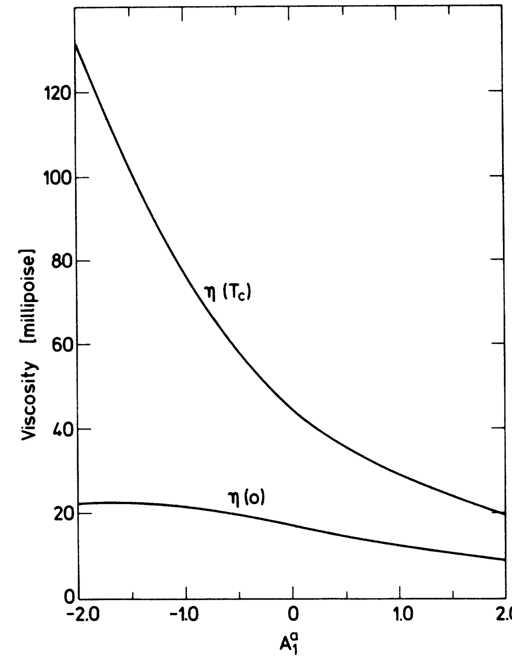

FIG. 7. Calculated normal-state viscosity $\eta(T_c)$ at the melting pressure shown as a function of the parameter $A_1^a = F_1^a/(1+\frac{1}{3}F_1^a)$. For comparison the corresponding dependence of the zero-temperature viscosity $\eta(0)$ is exhibited.

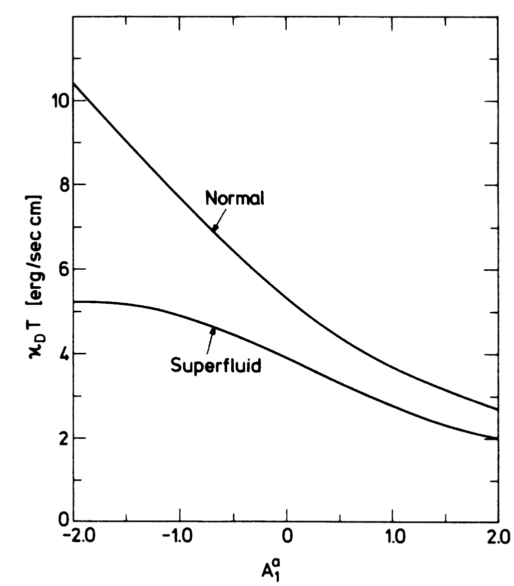

FIG. 8. Calculated A_1^a dependence of the quantity $\kappa_D T$ at the melting pressure in the normal state and in the superfluid at low temperatures (cf. Fig. 7).

to 0 will increase the viscosity by decreasing the magnitude of this average. For the thermal conductivity the difference in weighting of transitions with $\theta \simeq 0$ and $\theta \simeq \pi$ is less than for the viscosity, and therefore the relative change in the thermal conductivity with variation of A_1^a is not so great as the relative change in the viscosity. η and $\kappa_D T$ in the superfluid depend much less on A_1^a than do the normal-state properties. The essential reason is that the averages of the scattering amplitudes occurring in the relaxation time in the superfluid do not, in contrast to the viscous relaxation time in the normal state, weight strongly any particular angles for the normal-state scattering amplitude. The relaxation time of a normal-state quasiparticle at the Fermi energy $\tau(0)$ shows little A_1^a dependence for similar reasons.

The s- and p-wave scattering amplitudes with A_1^a adjustable are unphysical, in that they generally violate the forward-scattering sum rule and the values of F_1^a required are more negative than those obtained from spin-echo measurements.[10] However, the calculations suggest that it may be possible to find a scattering amplitude that can give a consistent account of normal-state transport measurements. Clearly when one goes beyond the simplest s- and p-wave approximation there are many possible forms one could use for the scattering amplitude: among the possibilities are to include higher Landau parameters, and higher partial waves.

All the transport quantities exhibited in Figs. 2–8 were calculated from Wheatley's tabulated Landau parameters.[3] Recently, Halperin *et al.*[11] made specific-heat measurements at the melting pressure in the normal state and obtained an effective mass ratio $m^*/m = 5.6$, which is 10% lower than the value given in Ref. 3, $m^*/m = 6.2$. To investigate the effect of this 10% difference we have calculated transport coefficients using a value of F_1^s obtained from the new value for m^*/m and F_0^s and F_0^a derived from the measured sound velocity and magnetic susceptibility using the new value of m^*/m. The results of the calculations are shown in Table III. The first column gives the results obtained using $m^*/m = 6.22$, and A_1^a is equal to the value -0.69 which satisfies the forward-scattering sum rule. The next two columns show results for $m^*/m = 5.6$. In both cases F_0^s, F_1^a, and F_0^a were altered to be consistent with the value of m^*/m, as explained above. For the second column we kept A_1^a at the old value -0.69, and the resulting scattering amplitude thus does not satisfy the forward scattering sum rule. For the third column A_1^a was set equal to -0.21 to satisfy the forward-scattering sum rule with A_0^s, A_1^s, and A_0^a having their altered values. The

TABLE III. Dependence of transport properties of ^{3}He at the melting pressure on m^*/m. The value $m^*/m = 6.22$ is taken from Ref. 3, and the value $m^*/m = 5.60$ from Ref. 11. The calculations were made using the s- and p-wave approximation. In the first two columns A_1^a was taken to be the value obtained from the forward-scattering sum rule, with $m^*/m = 6.22$. In the third column, A_1^a was obtained from the forward-scattering sum rule, with $m^*/m = 5.60$.

m^*/m	6.22	5.60	
ηT^2 (P mK^2)	0.44	0.38	0.31
κT (erg/sec cm)	6.9	6.3	5.4
DT^2 (cm^2 mK^2/sec)	0.111	0.099	0.084
α_η	1.05	0.978	0.802
α_κ	1.89	1.71	1.25
α_D	−0.350	−0.482	−0.770
$\tau(0)T^2$ (μsec mK^2)	0.20	0.17	0.16
$\eta(0)$ (mP)	21	20	18

changes in the normal-state transport coefficients can be as large as 30% for the viscosity, and 20% for the thermal conductivity, but unfortunately increase the discrepancy between theory and experiment. Note also the relatively large effect of allowing A_1^a to change to preserve the forward-scattering sum rule. These results indicate that the discrepancy between theory and experiment is not due to uncertainties in our knowledge of m^*/m.

Clearly the above discussion shows that more experimental and theoretical work is needed to arrive at a satisfactory expression for the scattering amplitude. Accurate measurements of the zero temperature viscosity and thermal conductivity in the superfluid state would be of value, since they constitute measurements of different angular averages of the collision probability than in the normal state. In this sense they complement the normal-state transport coefficients which have been so useful in giving insight into the nature of the quasiparticle interaction in mixtures of ^{3}He in ^{4}He as well as in pure ^{3}He.

The work described in this paper and the preceding one provides the framework for analyzing low-frequency long-wavelength transport and relaxation processes in superfluid Fermi systems. Since the transport equation has been solved exactly in a number of cases of experimental importance, it is possible to obtain quantitative information about microscopic properties of liquid

^{3}He. Also, the exact solutions serve as a useful starting point for variational calculations in the intermediate temperature range in which our results are not applicable.

ACKNOWLEDGMENTS

The authors wish to thank J. Serene for valuable discussions and correspondence that led to a considerable simplification of the calculation of the averaged collision probability. They are also grateful to Martti Salomaa for carrying out some numerical checks.

APPENDIX: TRANSITION PROBABILITY

The basic scattering amplitude may be written in the form of an interaction Hamiltonian

$$H = \frac{1}{4} \sum_{1,2,3,4} \langle 3,4| T |1,2\rangle a_4^\dagger a_3^\dagger a_1 a_2, \tag{A1}$$

where $a^\dagger$ and a are creation and annihilation operators for normal-state quasiparticles. Within the weak-coupling assumption we are making this Hamiltonian is the same in the superfluid and normal phases. According to standard time-dependent perturbation theory the rate at which transitions are being made from some state $|\Psi_i\rangle$ with a quasiparticle distribution n_i to other states is given by

$$\frac{2\pi}{\hbar} \langle \psi_i | H\delta(E_i - H_0) H | \psi_i\rangle , \tag{A2}$$

where H, given by (A1), is the interaction Hamiltonian, and H_0 is the unperturbed Hamiltonian. The simplest way to extract the transition probability for quasiparticles 1 and 2 to be scattered to states 3 and 4 is to pick out in (A2) the coefficient of $n_1 n_2(1-n_3)(1-n_4)$. Since the argument of the energy-conserving delta function may be easily read off, we may alternatively find the coefficient of $n_1 n_2(1-n_3)(1-n_4)$ in $\langle \psi_i | H^2 | \psi_i\rangle$, or, since we are interested in a thermal ensemble, in $\langle H^2\rangle_{\rm av}$ where $\langle \cdots \rangle_{\rm av}$ denotes a thermal average. The average of H^2 is

$$\langle H^2\rangle_{\rm av} = \frac{1}{16} \sum_{1\cdots 8} \langle 7,8| T| 5,6\rangle \times \langle a_8^\dagger a_7^\dagger a_5 a_6 a_4^\dagger a_3^\dagger a_1 a_2\rangle_{\rm av} \times \langle 3,4| T| 1,2\rangle. \tag{A3}$$

The thermal average in (A3) may easily be evaluated with the help of Wick's theorem, and the expressions for the elementary contractions:

$$\langle a_i^\dagger a_i\rangle_{\rm av} = |u_i|^2 n_i + |v_i|^2 (1-n_{-i}), \tag{A4}$$

$$\langle a_i a_i^\dagger\rangle_{\rm av} = |u_i|^2 (1-n_i) + |v_i|^2 n_{-i}, \tag{A5}$$

$$\langle a_i a_{-i}\rangle_{\rm av} = \frac{\Delta(\vec{p}_i)}{2E_i}[n_i - (1-n_{-i})], \tag{A6}$$

$$\langle a_{-i}^\dagger a_i^\dagger\rangle_{\rm av} = \frac{\Delta^\dagger(\vec{p}_i)}{2E_i}[n_i - (1-n_{-i})]. \tag{A7}$$

All these contractions depend on the spin variables, which we have not written out explicitly. $|u_i|^2$ and $|v_i|^2$ are diagonal in the spin index and have the values

$$|u_i|^2 = \tfrac{1}{2}(1+\xi_i/E_i) \tag{A8}$$

and

$$|v_i|^2 = \tfrac{1}{2}(1-\xi_i/E_i). \tag{A9}$$

Δ is the usual gap matrix. All contractions apart from the ones in (A4)–(A7) vanish.

The contributions to $\langle H^2\rangle_{\rm av}$ may be divided into three classes, according to whether they have 0, 2, or 4 anomalous contractions of the types (A6) and (A7). A typical term of the first class is

$$\langle 1,2| T| 3,4\rangle |u_1|^2 |u_2|^2 |u_3|^2 |u_4|^2 \langle 3,4| T| 1,2\rangle \times n_1 n_2 (1-n_3)(1-n_4). \tag{A10}$$

To complete the evaluation, one has only to calculate the spin trace implicit in (A10). This is precisely the same as in the normal state, and one finds a contribution

$$(|S_{\rm I}|^2 + 3|T_{\rm I}|^2) |u_1|^2 |u_2|^2 |u_3|^2 |u_4|^2 n_1 n_2 (1-n_3)(1-n_4). \tag{A11}$$

The notation for the singlet and triplet scattering amplitudes is explained in Sec. III of the text. There are five other terms related to (A11) which have some (or all) of the $|u_i|^2$ factors replaced by $|v_i|^2$; these are obtained from the $|v_i|^2$ terms in the contractions (A4) and (A5).

The next class of terms are those involving two anomalous contractions. With spin indices written explicitly a typical term is

$$\langle 1\alpha', -4\delta| T| 3\gamma, -2\beta'\rangle \times \frac{\Delta^\dagger(-\vec{p}_1)_{\alpha'\alpha}}{2E_1} \frac{\Delta(-\vec{p}_2)_{\beta'\beta}}{2E_2} |u_3|^2 |v_4|^2 \times \langle 3\gamma, -1\alpha| T| -4\delta, 2\beta\rangle n_1 n_2 (1-n_3)(1-n_4). \tag{A12}$$

The spin traces are most easily calculated if one uses the crossing relations to express the scattering amplitudes as follows:

$$\langle 1\alpha', -4\delta| T| 3\gamma, -2\beta'\rangle = (-\tfrac{1}{4}S_{\rm II} + \tfrac{3}{4}T_{\rm II})\delta_{\alpha'\beta'}\delta_{\delta\gamma} + (\tfrac{1}{4}S_{\rm II} + \tfrac{1}{4}T_{\rm II})\vec{\sigma}_{\alpha'\beta'}\cdot\vec{\sigma}_{\delta\gamma} \tag{A13}$$

and

$$\langle 3\gamma, -1\alpha|\, T\,| -4\delta, 2\beta\rangle = (\tfrac{1}{4}S_{III}+\tfrac{3}{4}T_{III})\delta_{\gamma\delta}\delta_{\alpha\beta} + (-\tfrac{1}{4}S_{III}+\tfrac{1}{4}T_{III})\vec{\sigma}_{\gamma\delta}\cdot\vec{\sigma}_{\alpha\beta}. \tag{A14}$$

The gap matrix for a p-wave superfluid may be expressed in terms of a vector $\vec{\Delta}(\vec{k})$ by means of the relation

$$\Delta(\vec{k})_{\alpha\beta} = i\vec{\Delta}(\vec{k})\cdot(\vec{\sigma}\sigma_y)_{\alpha\beta}, \tag{A15}$$

and thus the final task is to evaluate the spin traces. The required traces are

$$\delta_{\alpha'\beta'}\delta_{\delta\gamma}\Delta^*(-\vec{p}_1)_{\alpha'\alpha}\Delta(-\vec{p}_2)_{\beta'\beta}\delta_{\gamma\delta}\delta_{\alpha\beta} = \mathrm{Tr}(1)\,\mathrm{Tr}[\Delta^\dagger(-\vec{p}_1)\Delta(-\vec{p}_2)] = 4\vec{\Delta}^*(-1)\cdot\vec{\Delta}(-2) \tag{A16}$$

and

$$\vec{\sigma}_{\alpha'\beta'}\cdot\vec{\sigma}_{\delta\gamma}\Delta^*(-\vec{p}_1)_{\alpha'\alpha}\Delta(-\vec{p}_2)_{\beta'\beta}\vec{\sigma}_{\gamma\delta}\cdot\vec{\sigma}_{\alpha\beta} = \mathrm{Tr}(\sigma_i\sigma_j)\,\mathrm{Tr}[\tilde{\sigma}_j\Delta^\dagger(-\vec{p}_1)\sigma_i\Delta(-\vec{p}_2)] = 4\vec{\Delta}^*(-1)\cdot\vec{\Delta}(-2). \tag{A17}$$

The trace involving a single $\vec{\sigma}\cdot\vec{\sigma}$ operator vanishes. Collecting the results together, one finds the contribution (A12) is

$$\tfrac{1}{8}(S_{II}S_{III} - T_{II}S_{III} + S_{II}T_{III} - 5T_{II}T_{III}) \times \frac{\vec{\Delta}^*(1)\cdot\vec{\Delta}(2)}{E_1E_2}\,|u_3|^4|\,v_4|^2 n_1 n_2(1-n_3)(1-n_4).$$

The final class of terms are those involving four anomalous contractions. A typical contribution is

$$\langle -3\gamma', -4\delta'|\, T\,| -1\alpha', -2\beta'\rangle \frac{\Delta(\vec{p}_1)_{\alpha'\alpha}}{2E_1}\frac{\Delta(\vec{p}_2)_{\beta'\beta}}{2E_2} \times\frac{\Delta^*(\vec{p}_3)_{\gamma'\gamma}}{2E_3}\frac{\Delta^*(\vec{p}_4)_{\delta'\delta}}{2E_4} \times\langle 3\gamma, 4\delta|\, T\,|1\alpha, 2\beta\rangle n_1n_2(1-n_3)(1-n_4). \tag{A18}$$

If one writes

$$\langle 3\gamma, 4\delta|\, T\,|1\alpha, 2\beta\rangle = (\tfrac{1}{4}S_I+\tfrac{3}{4}T_I)\delta_{\alpha\gamma}\delta_{\beta\delta} + (-\tfrac{1}{4}S_I+\tfrac{1}{4}T_I)\vec{\sigma}_{\alpha\gamma}\cdot\vec{\sigma}_{\beta\delta}, \tag{A19}$$

the spin traces may be performed straightforwardly. One finds

$$\delta_{\alpha'\gamma'}\delta_{\beta'\delta'}\Delta(\vec{p}_1)_{\alpha'\alpha}\Delta(\vec{p}_2)_{\beta'\beta}\Delta^*(\vec{p}_3)_{\gamma'\gamma}\Delta^*(\vec{p}_4)_{\delta'\delta}\delta_{\alpha\gamma}\delta_{\beta\delta} = \mathrm{Tr}[\Delta(1)\Delta(2)]\,\mathrm{Tr}[\Delta^\dagger(3)\Delta^\dagger(4)] = 4[\vec{\Delta}(1)\cdot\vec{\Delta}(2)][\vec{\Delta}^*(3)\cdot\vec{\Delta}^*(4)]. \tag{A20}$$

The trace with one of the $\delta\delta$ terms in (A20) replaced by $\vec{\sigma}\cdot\vec{\sigma}$ is

$$\mathrm{Tr}[\Delta(1)\sigma_i\Delta^\dagger(3)]\,\mathrm{Tr}[\sigma_i\Delta^\dagger(4)\Delta(2)] = -4[\vec{\Delta}^*(3)\times\vec{\Delta}(1)]\cdot[\vec{\Delta}^*(4)\times\vec{\Delta}(2)] = -4\{[\vec{\Delta}^*(3)\cdot\vec{\Delta}^*(4)][\vec{\Delta}(1)\cdot\vec{\Delta}(2)] - [\vec{\Delta}^*(3)\cdot\vec{\Delta}(2)][\vec{\Delta}(1)\cdot\vec{\Delta}^*(4)]\}. \tag{A21}$$

The term with the $\vec{\sigma}\cdot\vec{\sigma}$ and $\delta\delta$ terms reversed is the same, and the term with two $\vec{\sigma}\cdot\vec{\sigma}$ operators is

$$\begin{aligned}\mathrm{Tr}[\tilde{\sigma}_i\Delta(1)\sigma_j\Delta^\dagger(3)]\,\mathrm{Tr}[\sigma_i\vec{\Delta}^\dagger(4)\tilde{\sigma}_j\vec{\Delta}(2)] &= 4\Delta_k(1)\Delta_l^*(3)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl})\Delta_m^*(4)\Delta_n(2)(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\delta_{mn}\delta_{ij})\\ &= 4\{2[\vec{\Delta}(1)\cdot\vec{\Delta}^*(4)][\vec{\Delta}(2)\cdot\vec{\Delta}^*(3)] + 2[\vec{\Delta}(1)\cdot\vec{\Delta}(2)][\vec{\Delta}^*(3)\cdot\vec{\Delta}^*(4)] - [\vec{\Delta}(1)\cdot\vec{\Delta}^*(3)][\vec{\Delta}(2)\cdot\vec{\Delta}^*(4)]\}.\end{aligned} \tag{A22}$$

In the BW state the vector $\vec{\Delta}(\vec{p}_i)$ has a constant magnitude Δ, and a direction obtained by performing a ($\vec{p}_i$ independent) rotation on $\vec{p}_i$. Thus $\vec{\Delta}(\vec{p}_i)\cdot\Delta(\vec{p}_j) = \Delta^2\hat{p}_i\cdot\hat{p}_j$. Substituting this expression into the various contributions, and adding them one obtains Eq. (3.1) for the transition probability.

Our result, Eq. (3.1), differs from that of Geilikman and Chechetkin[12] in a number of important ways, as can be seen by making the identifications appropriate for potential scattering described by the scattering amplitude $V_{\vec{q}}$,

$$S_I = V_{\vec{p}_3-\vec{p}_1} + V_{\vec{p}_4-\vec{p}_1}, \qquad T_I = V_{\vec{p}_3-\vec{p}_1} - V_{\vec{p}_4-\vec{p}_1}, \quad \text{etc.} \tag{A23}$$

For instance, when all four momenta are parallel the collision probability should vanish in the constant singlet approximation [cf. Eq. (3.2)]. This property is not shared by the collision probability exhibited below Eq. (16) of Ref. 12. Also the terms in Eq. (3.1) of the type $u_1^2u_2^2u_3^2u_4^2$, which are independent of angle but do depend on energy seem to have been replaced by a constant, equal to their value at $\xi_i = 0$.

A number of other results may easily be obtained using the above calculation. One simple case is the transition probability for the Anderson-Brinkman-Morel state, which is obtained by inserting the appropriate values of $\vec{\Delta}(\vec{p}_i)\cdot\vec{\Delta}(\vec{p}_j)$ in the expressions above. A second example is the transition probability for a process in which a quasiparticle in the state 1 decays into three quasiparticles in the states −2, 3, and 4. This is obtained by finding the coefficient of $n_1(1-n_{-2})(1-n_3)(1-n_4)$ in (A3). From the general form for the contractions (A4)–(A7), it is clear that the transition probability for this process is obtained from the one we have calculated by replacing $|u_2|^2$ by $|v_2|^2$, $|v_2|^2$ by $|u_2|^2$, and changing the sign of terms containing $\vec{\Delta}(\vec{p}_2)$, $\vec{\Delta}(-\vec{p}_2)$, or their Hermitian conjugates. Finally we remark that results may easily be obtained for the BCS state if one uses for the gap matrix $i\sigma_y\Delta$, where Δ is the magnitude of the gap.

As an alternative to the procedure described above we have also determined the collision probability (3.1) directly from the superfluid quasiparticle scattering amplitude, obtained by expressing $a^\dagger$ and a in (A1) in terms of the creation and annihilation operators for superfluid quasiparticles. The two methods give identical results for the collision probability.

---

*Research was supported in part by the NSF under Grant No. DMR-72-0326.

†Supported in part by the Alfred P. Sloan Foundation.

‡Present address: Dept. of Physics, Cornell University, Ithaca, N. Y. 14853.

[1]P. Bhattacharyya, C. J. Pethick, and H. Smith, preceding paper, Phys. Rev. B 15, 3367 (1977) (referred to as I).

[2]R. Balian and N. R. Werthamer, Phys. Rev. 131, 1553 (1963).

[3]For a general review of the experimental situation, see J. C. Wheatley, Rev. Mod. Phys. 47, 415 (1975).

[4]C. J. Pethick, H. Smith, and P. Bhattacharyya, Phys. Rev. Lett. 34, 643 (1975).

[5]K. S. Dy and C. J. Pethick, Phys. Rev. 185, 373 (1969).

[6]See, for example, the recent review by G. Baym and C. J. Pethick, in *The Physics of Liquid and Solid Helium*, edited by K. H. Bennemann and J. B. Ketterson (Wiley, New York, to be published), Vol. II.

[7]W. F. Brinkman, J. Serene, and P. W. Anderson, Phys. Rev. A 10, 2386 (1974).

[8]D. N. Paulson, M. Krusius, and J. C. Wheatley, Phys. Rev. Lett. 36, 1322 (1976).

[9]R. A. Webb, R. E. Sager, and J. C. Wheatley, Phys. Rev. Lett. 35, 1164 (1975).

[10]L. R. Corruccini, D. D. Osheroff, D. M. Lee, and R. C. Richardson, J. Low Temp. Phys. 8, 229 (1972).

[11]W. H. Halperin, C. N. Archie, F. B. Rasmussen, T. A. Alvesalo, and R. C. Richardson, Phys. Rev. B 13, 2124 (1976).

[12]B. T. Geilikman and V. R. Chechetkin, Zh. Eksp. Teor. Fiz. 69, 286 (1975) [Sov. Phys.-JETP 42, 148 (1976)].