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Transport and relaxation processes in superfluid ^{3}He close to the transition temperature*

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We derive the Boltzmann equation describing long-wavelength low-frequency transport and relaxation processes in a superfluid Fermi liquid close to the transition temperature T_c. In the superfluid, the quasiparticle number is not conserved and therefore one has to take into account the decay of one quasiparticle into three and the inverse process, as well as the two-quasiparticle scattering process. We calculate the collision term in the Boltzmann equation to first order in the superfluid gap Δ and show that it is related rather simply to the corresponding normal-state collision term. As applications of the Boltzmann equation, we solve it exactly to calculate the shear viscosity, and the intrinsic spin relaxation rate. The shear viscosity drops as $\Delta \propto (T_c-T)^{1/2}$ for temperatures just below T_c, and we determine the coefficient of Δ as a function of the normal-state collision probability. Leggett and Takagi's characteristic spin-relaxation time is shown to be equal to the relaxation time of a normal-state quasiparticle at the Fermi energy at T_c. The results provide one with a useful consistency check on experimental measurements which is independent of any assumption about the normal-state collision probability.

I. INTRODUCTION

Transport and relaxation processes in superfluid ^{3}He have been studied experimentally primarily through measurements of the viscosity[1] and the thermal conductivity[2] and through observations of NMR line broadening[3] and magnetic ringing experiments.[4] The present paper, which is the first of two longer papers dealing with transport and relaxation in superfluid ^{3}He, provides a unified theoretical treatment of these phenomena in the experimentally interesting temperature region close to T_c where the maximum gap in the excitation spectrum is small in comparison with the thermal energy k_BT. The second paper in this series deals with transport phenomena at temperatures at which k_BT is small in comparison with the gap. Brief accounts of some of the results have been reported previously.[5-7]

To describe kinetic phenomena in Fermi superfluids one generally has to work with coupled equations for the normal and anomalous parts of the quasiparticle distribution and the corresponding parts of the quasiparticle energy. However, for disturbances whose wave number is small compared with the inverse superfluid coherence length $\Delta/\hbar v_F$, where Δ is the superfluid gap and v_F is the Fermi velocity, and whose frequency is small compared to $\Delta/\hbar$, the response of the gap matrix (the anomalous part of the quasiparticle energy) may be assumed to be instantaneous and local in space. Under these conditions one does not have to consider the anomalous averages explicitly, provided one always works with quasiparticle states which diagonalize the instantaneous quasiparticle energy matrix. The evolution of the quasiparticle distribution may then be described by a Boltzmann equation of the standard form, as Betbeder-Matibet and Nozières have shown in detail.[8] The Boltzmann equation is expected to fail when the width of quasiparticle states $\sim\hbar/\tau$, where τ is a typical quasiparticle collision time, is comparable to or larger than Δ, and this has been confirmed by explicit calculations by Wölfle.[9] However, such effects will be important only at temperatures that differ from T_c by amounts less than the temperature resolution of most present experiments.

In calculating transport coefficients there are two problems to solve. The first is to derive the correct form of the Boltzmann equation. This problem has previously been discussed by Shumeiko[10] for the BCS state of an s-wave superfluid, by Seiden[11] for the Balian-Werthamer (BW) state of a p-wave superfluid at temperatures well below the transition temperature, and by Soda and Fujiki.[12] Recently, Geilikman and Chechetkin[13] and Shahzamanian[14] have also considered this problem. The streaming terms in the Boltzmann equation have the standard form, but the collision term is more complicated. In a normal Fermi liquid at low temperatures the only important collision process is the scattering of pairs of quasiparticles, but in a superfluid the quasiparticle number is not conserved, so one also has to take into account decay processes in which a single quasiparticle decays into three, and the inverse processes, in which three quasiparticles coalesce to form one. We shall evaluate the collision probabilities for these processes in terms of normal-state quasiparticle scattering amplitudes. Our calculations go beyond the earlier ones in that we calculate the collision probabilities in the superfluid state in terms of the normal state collision probability, without making any detailed assumptions about the form of the normal state collision probability. Also we fully take into account the superfluid coherence factors in the superfluid state.

The second problem in calculating transport coefficients is to solve the Boltzmann equation. For a normal Fermi liquid the quasiparticle Boltzmann equation has been solved exactly by Højgaard Jensen, Smith, and Wilkins,[15] and by Brooker and Sykes.[16] Close to T_c in the superfluid state the Boltzmann equation differs little from the normal-state equation, since the superfluid correlations affect only the small number of excitations with energies $\lesssim \Delta$, whereas typical quasiparticle energies are of order $k_B T_c$. We show that the Boltzmann equation may be solved exactly by treating the difference between the collision terms for the normal and superfluid states as a perturbation. Among the results we obtain are that the shear viscosity drops as $(T_c - T)^{1/2}$ for temperatures just below T_c, in agreement with approximate calculations,[10,12,14] and the coefficient of $(T_c - T)^{1/2}$ is expressed as a function of normal-state properties. The diffusive thermal conductivity has no square-root singularity, due to the fact that the low-energy quasiparticles affected by the superfluid correlations close to T_c carry a negligible heat current. The intrinsic spin-relaxation process is discussed, and the relaxation time introduced by Leggett and Takagi[17] is shown to be equal to the relaxation time of a normal-state quasiparticle at the Fermi surface at T_c. Without making any assumptions about the normal-state scattering amplitudes, we also find a relationship between a number of experimentally measurable quantities; this should provide a useful check on the consistency of the experimental measurements.

The paper is organized as follows. Section II contains a discussion of the collision integral. The shear viscosity is calculated in Sec. III, and spin relaxation is considered in Sec. IV. Section V contains a discussion of some of the results, and the consistency condition. A number of calculational details are discussed in a series of appendices. Estimates of parameters for liquid ^{3}He both close to T_c and well below T_c are given in the companion paper.[18]

II. COLLISION INTEGRAL

For the discussion of collision processes in the superfluid at temperatures just below the transition temperature it is most convenient to work in terms of quasiparticles which are related as closely as possible to the quasiparticles in the normal state. Accordingly we take the quasiparticle energy to be

$$E_{\vec{p}} = (\xi_{\vec{p}}^2 + \Delta_{\hat{p}}^2)^{1/2} \operatorname{sgn}\xi_{\vec{p}}, \tag{2.1}$$

where $\xi_{\vec{p}}$ is the normal-state quasiparticle energy measured with respect to the chemical potential, and $\Delta_{\hat{p}}$ is the magnitude of the gap in the direction $\hat{p}$ on the Fermi surface. In writing the energy in the form (2.1) we have implicitly limited ourselves to unitary states, since both the Anderson-Brinkman-Morel (ABM) state and the BW state, which seem to correspond to the A and B phases of liquid ^{3}He, respectively, are unitary states. We shall indicate how our results are modified for the nonunitary state describing the A_1 phase.

For frequencies ω much less than a typical gap frequency $\Delta/\hbar$, and for wave numbers much less than the inverse coherence length $\Delta/\hbar v_F$, the low-lying excited states of a superfluid may be characterized in terms of a quasiparticle distribution function n_p as discussed in Sec. I. The streaming terms in the Boltzmann equation have the standard form and need not be discussed at length, but the collision terms must be considered in some detail.

In a normal Fermi liquid the total quasiparticle number is conserved, and therefore the only allowed scattering processes are those in which the number of quasiparticles in the final state is the same as the number in the initial state. At temperatures well below the degeneracy temperature the density of excitations is low, and consequently the most important processes are those in which two quasiparticles scatter. In a superfluid the quasiparticle number is not conserved, and therefore processes other than scattering processes similar to those in a normal Fermi liquid can occur. Thus, for example, one quasiparticle can decay into three, or three can coalesce to produce one. On the grounds of phase space alone one would expect all processes having the same *total* number of quasiparticles in the initial and final states to be equally important in a superfluid, and that the most important processes would be those with a total of four quasiparticles in the initial and final states, as in the case of a normal Fermi liquid. Thus in a superfluid one generally has to take into account processes in which two quasiparticles scatter, the decay and coalescence processes mentioned above, and also processes in which four quasiparticles are created from the condensate or scattered into it. The collision term in the Boltzmann equation due to the two-particle scattering process has the standard form

$$\left(\frac{\partial n_1}{\partial t}\right)_{\rm coll} = -\sum_{2,3,4} W_s'(1,2;3,4)\,[n_1 n_2(1-n_3)(1-n_4) - (1-n_1)(1-n_2)n_3 n_4] \times \delta_{\vec p_1+\vec p_2,\,\vec p_3+\vec p_4}\,\delta(E_1+E_2-E_3-E_4)\,, \tag{2.2}$$

where n_i is the quasiparticle distribution function, and the indices $i=1-4$ denote both momentum ($\vec{p}$) and spin (σ) variables. In (2.2) the energies occurring in the delta function include Fermi-liquid effects, since the normal-state energies ξ_i in (2.1) contain them. $W_s'(1,2;3,4)$ is the transition probability for the transition in which quasiparticles in states 1 and 2 scatter into states 3 and 4.

In the applications studied here, we shall need to consider only small deviations from local equilibrium. It is therefore convenient to work in terms of a deviation function ψ_i defined in terms of the local equilibrium distribution $n_i^{\rm l.e.}(E_i)$, where the E_i includes Fermi-liquid effects, by the relation

$$n_i = n_i^{\rm l.e.}(E_i) + n_i^{\rm l.e.}(E_i)\,[1-n_i^{\rm l.e.}(E_i)]\psi_i\,. \tag{2.3}$$

Written in terms of ψ the linearized collision integral for two-quasiparticle scattering processes is

$$\left(\frac{\partial n_1}{\partial t}\right)_{\rm coll} = -\sum_{2,3,4} W_s'(1,2;3,4) n_1 n_2 \times (1-n_3)(1-n_4)\,\delta_{\vec p_1+\vec p_2,\,\vec p_3+\vec p_4} \times \delta(E_1+E_2-E_3-E_4)(\psi_1+\psi_2-\psi_3-\psi_4)\,. \tag{2.4}$$

Here the distribution functions and quasiparticle energies may be taken to be the values for *global* equilibrium, but to avoid complicating the notation we shall not indicate this explicitly.

The collision term for the process in which quasiparticle 1 decays into quasiparticles -2, 3, and 4 (and the inverse process) is

$$\left(\frac{\partial n_1}{\partial t}\right)_{\rm coll} = -\frac{1}{3}\sum_{2,3,4} W_s'(1;-2,3,4) \times [n_1(1-n_{-2})(1-n_3)(1-n_4) - (1-n_1)n_{-2}n_3 n_4] \times \delta_{\vec p_1+\vec p_2,\,\vec p_3+\vec p_4}\,\delta(E_1-E_{-2}-E_3-E_4)\,, \tag{2.5}$$

where $W_s'(1;-2,3,4)$ is the transition probability. Here by -2 we mean the state with momentum opposite that of 2, but the same spin. When linearized Eq. (2.5) becomes

$$\left(\frac{\partial n_1}{\partial t}\right)_{\rm coll} = -\frac{1}{3}\sum_{2,3,4} W_s'(1;-2,3,4) \times n_1(1-n_{-2})(1-n_3)(1-n_4) \times \delta_{\vec p_1+\vec p_2,\,\vec p_3+\vec p_4} \times \delta(E_1-E_{-2}-E_3-E_4) \times (\psi_1-\psi_{-2}-\psi_3-\psi_4)\,. \tag{2.6}$$

We have adopted the convention that the sum over states 2, 3, and 4 is to be made over all states 2 and all distinguishable states for the pair 3 and 4. That is the sum is half the unrestricted sum over all states for the quasiparticles 2, 3, and 4. The factor of $\frac{1}{3}$ in (2.5) and (2.6) results from the fact that for the decay process one must sum only over distinguishable states of the three final quasiparticles, which introduces the extra symmetry factor $\frac{1}{3}$.

For the process in which quasiparticles 1, 2, and -3 coalesce to give quasiparticle 4 the collision term is

$$\left(\frac{\partial n_1}{\partial t}\right)_{\rm coll} = -\sum_{2,3,4} W_s'(1,2,-3;4) \times [n_1 n_2 n_{-3}(1-n_4) - (1-n_1)(1-n_2)(1-n_{-3})n_4] \times \delta_{\vec p_1+\vec p_2,\,\vec p_3+\vec p_4}\,\delta(E_1+E_2+E_{-3}-E_4)\,, \tag{2.7}$$

where $W_s'(1,2,-3;4)$ is the transition probability for this process, which must be identical to $W_s'(4;1,2,-3)$. The linearized version of (2.7) is

$$\left(\frac{\partial n_1}{\partial t}\right)_{\rm coll} = -\sum_{2,3,4} W_s'(1,2,-3;4) \times n_1 n_2 n_{-3}(1-n_4)\,\delta_{\vec p_1+\vec p_2,\,\vec p_3+\vec p_4} \times \delta(E_1+E_2+E_{-3}-E_4) \times(\psi_1+\psi_2+\psi_{-3}-\psi_4)\,. \tag{2.8}$$

As we shall see in detail below when we consider the transition probabilities, to calculate the leading correction to the transport properties just below T_c only the scattering, coalescence and decay processes need be taken into account. Creation of four quasiparticles from the condensate and the inverse process do not contribute to the leading correction.

To simplify the total collision integral, obtained by adding (2.4), (2.6), and (2.8) we must consider the transition probabilities in some detail. The matrix elements of the residual interaction between quasiparticles in the superfluid differ from the corresponding quantities in the normal state for two reasons. First, the quasiparticles in the superfluid are linear superpositions of quasiparticles and quasiholes in the normal state, and therefore to obtain the matrix elements of the residual interaction between quasiparticles in the superfluid one must perform a Bogoliubov transformation on the matrix elements of the residual interaction between quasiparticles in the normal state. As we shall see, close to T_c the effect of the Bogoliubov transformation is to give changes in the collision integral of relative order Δ/k_BT_c, where Δ is a typical superfluid gap. A second effect is due to the modification of the effective interaction by the superfluid correlations, which is the type of strong-coupling effect considered by Anderson and Brinkman,[19] and Brinkman, Serene, and Anderson[20] in their calculations to account for the stability of the ABM state. Examination of these strong-coupling effects indicates that their leading contribution to transport and relaxation properties is of higher order than Δ/k_BT and therefore may be neglected. Thus the effective interaction between quasiparticles in the superfluid may be obtained from the interaction between normal-state quasiparticles just by performing a Bogoliubov transformation.

Let us now consider in more detail the calculation of the transition probabilities in the superfluid state. The Bogoliubov transformation between the normal quasiparticle creation and annihilation operators $a^\dagger_{\vec{p}\sigma}$ and $a_{\vec{p}\sigma}$ and the creation and annihilation operators $\alpha^\dagger_{\vec{p}\sigma}$ and $\alpha_{\vec{p}\sigma}$ in the superfluid may be written in the form

$$a_{\vec{p}\sigma}=u(\vec{p})_{\sigma\sigma'}\alpha_{\vec{p}\sigma'}-v(\vec{p})_{\sigma\sigma'}\alpha^\dagger_{-\vec{p}\sigma'},$$
$$a^\dagger_{-\vec{p}\sigma}=v^*(\vec{p})_{\sigma\sigma'}\alpha_{\vec{p}\sigma'}+u(\vec{p})_{\sigma\sigma'}\alpha^\dagger_{-\vec{p}\sigma'}, \tag{2.9}$$

where repeated spin indices are summed over. u is even under reversal of $\vec{p}$,

$$u(-\vec{p})_{\sigma\sigma'}=u(\vec{p})_{\sigma\sigma'}, \tag{2.10}$$

and v is even for singlet pairing and odd for triplet pairing;

$$v(-\vec{p})_{\sigma\sigma'}=v(\vec{p})_{\sigma\sigma'} \quad \text{(singlet pairing)},$$
$$v(-\vec{p})_{\sigma\sigma'}=-v(\vec{p})_{\sigma\sigma'} \quad \text{(triplet pairing)}. \tag{2.11}$$

As we mentioned earlier, we define the quasiparticle energy by Eq. (2.1). This is equivalent to the condition that v should be small for p differing from the Fermi momentum by amounts greater than $\Delta_{\hat{p}}/v_F$, where $\Delta_{\hat{p}}$ is the gap in the direction $\hat{p}$ on the Fermi surface.

It is convenient to choose as the normal-state quasiparticle states ones which have their spins parallel or antiparallel to some fixed direction in space (independent of $\hat{p}$). For all applications of interest to us it is then possible to work with superfluid quasiparticle states such that $u(\vec{p})_{\sigma\sigma'}$ tends to the unit matrix for $|p-p_F|\gg\Delta_{\hat{p}}/v_F$.

The interaction between quasiparticles in the superfluid is found by performing a Bogoliubov transformation on the normal-state interaction, and is

$$\begin{aligned}H&=\frac{1}{4}\sum_{1,2,3,4}\langle 3,4|T|1,2\rangle a^\dagger_4a^\dagger_3a_1a_2\\&=\frac{1}{4}\sum_{1,2,3,4}\langle 3,4|T|1,2\rangle\\&\qquad\times(u^*_4\alpha^\dagger_4-v^*_4\alpha_{-4})(u^*_3\alpha^\dagger_3-v^*_3\alpha_{-3})\\&\qquad\times(u_1\alpha_1-v_1\alpha^\dagger_{-1})(u_2\alpha_2-v_2\alpha^\dagger_{-2}).\end{aligned} \tag{2.12}$$

Note that the Hamiltonian contains terms which convert one quasiparticle into three or three into one, and others which create and destroy four quasiparticles, in addition to terms which scatter two quasiparticles.

Because we are interested only in changes in the collision integral of order Δ, we need to retain only terms involving no more than a single v factor. This follows from the fact that the v factors are appreciable only for quasiparticle energies within ~ $\Delta_{\hat{p}}$ of the Fermi energy, and therefore for small $\Delta_{\hat{p}}$ the terms containing larger numbers of v factors are relatively less important. Also in the expressions for the transition probabilities all interference terms can be dropped since they contain at least two v factors associated with different quasiparticle states, and are therefore negligible near T_c. The transition probabilities for the various processes of interest are thus

$$W'_s(1,2;3,4)=W_N(1,2;3,4)\times|u_1|^2|u_2|^2|u_3|^2|u_4|^2, \tag{2.13}$$

or, with spin indices written out explicitly,

$$\begin{aligned}&W'_s(\vec{p}_1\sigma_1,\vec{p}_2\sigma_2;\vec{p}_3\sigma_3,\vec{p}_4\sigma_4)\\&\quad=\sum_{\sigma'_1,\sigma'_2,\sigma'_3,\sigma'_4}W_N(\vec{p}_1\sigma'_1,\vec{p}_2\sigma'_2;\vec{p}_3\sigma'_3,\vec{p}_4\sigma'_4)\\&\qquad\times|u(\vec{p}_1)_{\sigma'_1\sigma_1}|^2|u(\vec{p}_2)_{\sigma'_2\sigma_2}|^2\\&\qquad\times|u(\vec{p}_3)_{\sigma'_3\sigma_3}|^2|u(\vec{p}_4)_{\sigma'_4\sigma_4}|^2\end{aligned} \tag{2.14}$$

for the two quasiparticle scattering process. For the decay and coalescence processes one finds

$$W_s'(1;-2,3,4)=W_N(1,2;3,4)|v_2|^2+W_N(1,-3;-2,4)|v_3|^2+W_N(1,-4;-2,3)|v_4|^2, \quad (2.15)$$

where

$$W_N(1,2;3,4)|v_2|^2=\sum_{\sigma_2'} W_N(\vec{p}_1\sigma_1,\vec{p}_2\sigma_2';\vec{p}_3\sigma_3,\vec{p}_4\sigma_4)|v(\vec{p}_2)_{\sigma_2'\sigma_2}|^2 \quad (2.16)$$

and

$$W_s'(1,2,-3;4)=W_s'(4;1,2,-3). \quad (2.17)$$

In the transition probabilities for the coalescence and decay processes the differences between the $|u|^2$ factors and the unit matrix have been neglected since these are small except for quasiparticle energies small compared with Δ. Because the probabilities already contain one $|v|^2$ factor the corrections from the $|u|^2$ factors do not contribute to the leading corrections to the collision rate close to T_c.

In the present paper we shall be interested almost exclusively in unitary states, and in situations where the driving term in the Boltzmann equation and the deviation function ψ have a definite symmetry with respect to reversal of the quasiparticle momentum, spin, and energy. If one denotes by ϵ_p, ϵ_σ, and ϵ_E the parities under these various interchanges one finds

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}}=-\sum_{2,3,4} W_N n_1 n_2(1-n_3)(1-n_4)\times\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(\xi_1+\xi_2-\xi_3-\xi_4)\times[\psi_1+f_1^{(\nu)}(f_2^{(\nu)}\psi_2-f_3^{(\nu)}\psi_3-f_4^{(\nu)}\psi_4)], \quad (2.18)$$

where ν stands for ϵ_σ and the product $\epsilon_p\epsilon_E$, since the form of the collision integral depends not on the symmetry of ψ under reversal of the momentum and energy separately, but only on the symmetry under reversal of both simultaneously. Here

$$f_i^{(\nu)}=f_{\vec{p}}^{(\nu)}=|u(\vec{p})_{\uparrow\uparrow}|^2+\epsilon_\sigma|u(\vec{p})_{\uparrow\downarrow}|^2-\epsilon_p\epsilon_E[|v(\vec{p})_{\uparrow\uparrow}|^2+\epsilon_\sigma|v(\vec{p})_{\uparrow\downarrow}|^2], \quad (2.19)$$

and

$$\psi_{\vec{p}\sigma}^{(\nu)}=\tfrac{1}{4}\{\psi_{\vec{p}\sigma}+\epsilon_\sigma\psi_{\vec{p}-\sigma}-\epsilon_p\epsilon_E[\psi_{-\vec{p}\sigma}(-E_{\vec{p}})+\epsilon_\sigma\psi_{-\vec{p}-\sigma}(-E_{\vec{p}})]\}$$

(we use the symbols $\uparrow$, $\downarrow$ synonymously with the values $+1$, -1 for the spin index σ). Details of the derivation of (2.18) are given in Appendix A. In the form (2.18) for the collision integral we have replaced the superfluid quasiparticle energies E_i by the corresponding normal-state energies ξ_i, since this does not affect contributions of order Δ. If $f^{(\nu)}$ is put equal to unity in (2.18) one recovers the normal state collision operator. In more general situations where ψ does not have a definite symmetry under reversal of $\sigma,\vec{p}$, and E the collision term is a sum over ν of terms of the type (2.18), with ψ there replaced by that part of ψ having the symmetry specified by the index ν.

III. VISCOSITY

In the hydrodynamic limit the distribution functions in the streaming terms in the Boltzmann equation may be replaced by local equilibrium distribution functions. To calculate the viscosity one needs to consider a local equilibrium distribution function corresponding to a spatially varying velocity $\vec{u}$,

$$n_1=(e^{(E_1-\vec{p}_1\cdot\vec{u})/k_BT}+1)^{-1}. \quad (3.1)$$

If $\vec{u}$ has only an x component, which varies in the y direction, the Boltzmann equation reduces to

$$-(p_1)_x\cdot(v_1)_y\frac{\partial n_1}{\partial E_1}\frac{\partial u_x}{\partial y}=\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} \quad (3.2)$$

in terms of the quasiparticle group velocity v_1.

It is obvious that the driving term (3.2) is unchanged when $\vec{p}$ is reversed, so the parity eigenvalues of (2.19) are

$$\epsilon_\sigma=\epsilon_p=\epsilon_E=1. \quad (3.3)$$

The renormalization factor f defined in (2.19) is therefore

$$f_{\vec{p}}=|u(\vec{p})_{\uparrow\uparrow}|^2+|u(\vec{p})_{\uparrow\downarrow}|^2-|v(\vec{p})_{\uparrow\uparrow}|^2-|v(\vec{p})_{\uparrow\downarrow}|^2. \quad (3.4)$$

This is just the quasiparticle group velocity divided by the Fermi velocity, since the velocity associated with the normal-state quasiparticle components of the superfluid quasiparticle is v_F, and the velocity associated with the quasihole components is $-v_F$. Thus

$$f_{\vec{p}}=v_{\vec{p}}/v_F=\xi_{\vec{p}}/E_{\vec{p}}\equiv V_{\vec{p}}. \quad (3.5)$$

Note that V is anisotropic in the ABM state, where $\Delta_{\hat{p}}=\Delta\sin\Theta$, with Θ being the angle between $\hat{p}$ and the orbital anisotropy axis $\hat{l}$,

$$V_{\text{ABM}}=|\xi|/(\xi^2+\Delta^2\sin^2\Theta)^{1/2} \quad (3.6)$$

according to (2.1). In the BW state V is isotropic and is given by

$$V_{\text{BW}}=|\xi|/(\xi^2+\Delta^2)^{1/2}. \quad (3.7)$$

E_i may be replaced by ξ_i in the Fermi function in the driving term in the Boltzmann equation, since this does not affect the leading corrections to the viscosity close to T_c. The Boltzmann equation for viscosity with f put equal to V in the collision term is therefore

$$-(p_1)_x(\hat{v}_1)_y V_1 v_F \frac{\partial n_1}{\partial \xi_1}\frac{\partial u_x}{\partial y} = -\sum_{2,3,4} W_N(1,2;3,4) n_1 n_2(1-n_3)(1-n_4)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \times \delta(\xi_1+\xi_2-\xi_3-\xi_4) \times[\psi_1+V_1(V_2\psi_2-V_3\psi_3-V_4\psi_4)]. \quad (3.8)$$

One would like to express ψ in terms of the corresponding quantity for the normal state. However, because of the V_1 factors occurring in both the driving term and the collision term, ψ differs appreciably from its normal-state value for quasiparticle energies $\lesssim\Delta$. It is more convenient to introduce the function $\chi_i=\psi_i/V_i$. This satisfies the equation

$$-(p_1)_x(\hat{v}_1)_y v_F \frac{\partial n_1}{\partial \xi_1}\frac{\partial u_x}{\partial y} = -\sum_{2,3,4} W_N(1,2;3,4) n_1 n_2(1-n_3)(1-n_4) \times\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(\xi_1+\xi_2-\xi_3-\xi_4) \times(\chi_1+V_2^2\chi_2-V_3^2\chi_3-V_4^2\chi_4). \quad (3.9)$$

Although V^2-1 has structure on an energy scale $\sim\Delta$, the right-hand side of (3.9) has none due to V^2 factors, since these all occur in summations. It is therefore clear that χ can have no structure on an energy scale Δ. The deviation functions in (3.9) may be written

$$(\chi_1+\chi_2-\chi_3-\chi_4)+[(V_2^2-1)\chi_2-(V_3^2-1)\chi_3-(V_4^2-1)\chi_4]. \quad (3.10)$$

The contribution to the right-hand side of (3.9) from the first of these terms is just the normal-state collision operator acting on χ, and is (apart from a Fermi function derivative) of order χ/τ, where τ is a typical quasiparticle relaxation time. Now V^2-1 is of order unity for $\xi\lesssim\Delta$, and is essentially zero otherwise, and therefore the contribution fo (3.9) from the second term in (3.10) is of order Δ/k_BT times that from the first term. Thus the contribution from the second term is small compared with that from the first for all values of ξ_1, and therefore the second term may be treated as a perturbation. Consequently, in the superfluid close to T_c, χ differs from the normal-state value only by amounts of order Δ/k_BT_c. Note that the same cannot be said of ψ.

Let us for simplicity first consider states with an isotropic gap. Then V^2 is isotropic, and χ has the same angular dependence as the driving term in the Boltzmann equation. It is therefore convenient to introduce a function Q, which depends only on ξ, defined by

$$\psi_1=(p_1)_x(\hat{v}_1)_y v_F V_1 \times\frac{\partial u_x}{\partial y}\tau_0 Q(\xi_1) 2\frac{\cosh(\xi_1/2k_BT)}{k_BT}, \quad (3.11)$$

where τ_0 is defined in Eq. (B5) in Appendix B. The Boltzmann equation (3.9) may be reduced to a one-dimensional integral equation in essentially the same way as for the normal state. One finds

$$\frac{1}{\cosh(\frac{1}{2}t)}=(\pi^2+t^2)Q(t) -\alpha_\eta\int_{-\infty}^{\infty}dt'\,F(t-t')V^2(t')Q(t'), \quad (3.12)$$

where

$$t=\xi/k_BT, \quad (3.13)$$

$$\alpha_\eta=2\left[\int\frac{d\Omega}{4\pi}\frac{W_N(\theta,\phi)}{\cos(\frac{1}{2}\theta)}\left(1-3\sin^4\frac{\theta}{2}\sin^2\phi\right)\right] \times\left(\int\frac{d\Omega}{4\pi}\frac{W_N(\theta,\phi)}{\cos(\frac{1}{2}\theta)}\right)^{-1}, \quad (3.14)$$

and $F(x)=x/2\sinh(\frac{1}{2}x)$. Note that α_η is $2\lambda_\eta$ in the notation of Ref. 16. In (3.14), $W_N(\theta,\phi)$ is the normal-state collision probability as a function of the usual angles θ and ϕ, as explained in Appendix B.

The momentum flux Π_{xy} is given by

$$\Pi_{xy}=\sum_1(p_1)_x(\hat{v}_1)_y V_1 v_F n_1(1-n_1)\psi_1=-\eta\frac{\partial u_x}{\partial y}, \quad (3.15)$$

and thus, from using Eq. (3.11), one can see that the viscosity η is proportional to

$$Y=\int_{-\infty}^{\infty}dt\frac{V^2(t)}{\cosh(\frac{1}{2}t)}Q(t). \quad (3.16)$$

As we shall be interested only in ratios of viscosities the proportionality factor is unimportant, since changes in it give changes in the viscosity proportional to T_c-T, while the changes in Y give terms proportional to $(T_c-T)^{1/2}$.

To solve (3.12) we observe first that V^2 is essentially equal to unity, except for quasiparticles with energies less than or on the order of the gap, that is for $t\lesssim\Delta/k_BT$. Thus for any function $G(t)$ having no structure on a scale Δ/k_BT we may write

$$\int_{-\infty}^{\infty}G(t)[1-V^2(t)]\,dt=G(0)\int_{-\infty}^{\infty}[1-V^2(t)]\,dt =\pi G(0)\frac{\Delta}{k_BT}, \quad (3.17)$$

where we have used Eq. (3.7) for V in evaluating the integral. Since the kernel in the integral equation has no structure on a scale $t' \sim \Delta/k_BT$ we may write the integral equation (3.12) in the form

$$X=(H_0+H_1)Q\,, \tag{3.18}$$

where $X=1/\cosh(\frac{1}{2}t)$, H_0Q is the right-hand side of (3.12) with $V=1$, and

$$H_1Q=\alpha_\eta\int_{-\infty}^{\infty}dt'\,F(t-t')[1-V^2(t')]Q(t')$$
$$=\pi\alpha_\eta\,\tilde{\Delta}\,F(t)Q(0)\,,$$

with $\tilde{\Delta}=\Delta/k_BT$. The dimensionless viscosity Y is then given by

$$Y=(VX,VQ)=(X,Q)-(X,(1-V^2)Q)$$
$$=(X,Q)-\pi\tilde{\Delta}X(0)Q(0)\,, \tag{3.19}$$

where (A,B) denotes the scalar product $\int_{-\infty}^{\infty}dtA(t)B(t)$. Writing $Q=Q_0+Q_1$, where Q_0 is the unperturbed solution and $Q_1\propto\tilde{\Delta}$ is the change due to the perturbation, we obtain, by equating to zero the terms independent of $\tilde{\Delta}$ and those linear in $\tilde{\Delta}$,

$$X=H_0Q_0 \quad\text{and}\quad 0=H_1Q_0+H_0Q_1\,. \tag{3.20}$$

The first of these is the normal-state Boltzmann equation, and from the second one finds $Q_1=-H_0^{-1}H_1Q_0$. Then to lowest order in $\tilde{\Delta}$,

$$Y=(X,Q_0)-\pi\tilde{\Delta}X(0)Q_0(0)-(X,H_0^{-1}H_1Q_0)\,, \tag{3.21}$$

the last term in (3.21) may be written

$$(X,H_0^{-1}H_1Q_0)=(Q_0,H_1Q_0)$$
$$=\pi\tilde{\Delta}Q_0(0)\alpha_\eta\int_{-\infty}^{\infty}dt\,F(t)Q_0(t)\,. \tag{3.22}$$

Thus

$$Y=(X,Q_0)-\pi\tilde{\Delta}Q_0(0)\left(X(0)+\alpha_\eta\int_{-\infty}^{\infty}dt\,F(t)Q_0(t)\right). \tag{3.23}$$

From the integral equation for Q_0, the term in large parentheses is just $\pi^2Q_0(0)$, and therefore

$$Y=(X,Q_0)-\pi^3\tilde{\Delta}Q_0^2(0)\,, \tag{3.24}$$

and the change in the viscosity is

$$\frac{\delta\eta}{\eta}=\frac{\delta Y}{Y}=-\pi^3\frac{\Delta}{k_BT_c}\frac{Q_N^2(0)}{Y_N}\,, \tag{3.25}$$

where

$$Y_N=\frac{1}{3}+\frac{4}{\pi^2}\alpha_\eta\sum_{n=1,3,\ldots}\frac{2n+1}{n^2(n+1)^2}\frac{1}{n(n+1)-\alpha_\eta} \tag{3.26}$$

is the exact (reduced) viscosity of the normal state[15] and $Q_N(0)[\equiv Q_0(0)]$ is the value of the normal-state solution $Q_N(t)$ at the Fermi energy ($t=0$). As we show in Appendix B, this is given by

$$Q_N(0)=\frac{1}{\pi^2}\left[1+\alpha_\eta\sum_{n=1,3,\ldots}\frac{2n+1}{n^2(n+1)^2}\frac{1}{n(n+1)-\alpha_\eta}\times\left(\frac{n!!}{(n-1)!!}\right)^2\right]. \tag{3.27}$$

The series converges very rapidly, so for practical purposes it is sufficient to include only the first few terms.

The relative change in the viscosity may therefore be written in the form

$$\delta\eta/\eta=-D(\alpha_\eta)\Delta/k_BT_c\,, \tag{3.28}$$

where $D=\pi^3Q_N^2(0)/Y_N$ is a function of the normal-state parameter α_η alone. It is plotted in Fig. 1 as a function of α_η. Note the divergence near $\alpha_\eta=2$, where D behaves asymptotically as $\frac{3}{8}\pi/(2-\alpha_\eta)$. This limit corresponds to a collision probability which is strongly peaked in the direction of small momentum transfers. Since calculated values of α_η for ^{3}He are typically around 1, it follows that $\delta\eta/\eta$ is quite different from the value obtained by neglecting the integral terms in (3.12) ($\alpha_\eta=0$). Note also that the viscosity always falls on entering the superfluid phase.

If one makes a relaxation-time approximation for the collision term in the Boltzmann equation and replaces it by $-\delta n_1/\tau$, where δn_1 is the deviation from local equilibrium and τ is an energy- and temperature-independent relaxation time, the viscosity is proportional to

$$\sum_{\vec{p}}V^2\left(-\frac{\partial n_{\vec{p}}}{\partial\xi_{\vec{p}}}\right)\propto 1-\frac{\pi}{4}\frac{\Delta}{k_BT_c}\,, \tag{3.29}$$

$\{Q\propto[\cosh(\frac{1}{2}t)]^{-1}$ is a solution of the Boltzmann equation in both the normal and superfluid phases$\}$ and therefore

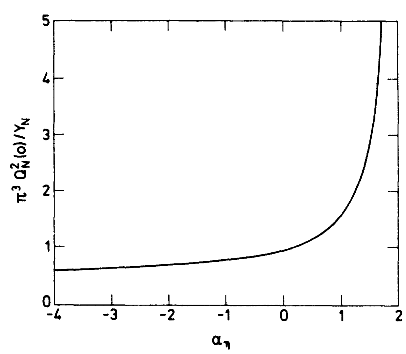

FIG. 1. Function $D(\alpha_\eta)=\pi^3Q_N^2(0)/Y_N$ of Eq. (3.28) as a function of α_η. Note the rapid variation with α_η in the region of main physical interest for ^{3}He, $1\lesssim\alpha_\eta\lesssim1.5$.

$$\delta\eta/\eta = -\tfrac{1}{4}\pi\Delta/k_B T_c . \quad (3.30)$$

The results above are adequate for dealing with the BW state, which has an isotropic gap. However for the ABM state the gap is anisotropic, and as a consequence the viscosity is a tensor. If one chooses as the z axis the direction along which the gap vanishes, the viscosity has two different components: η_{xy} for shearing in the plane perpendicular to the anisotropy axis, and $\eta_{xz}=\eta_{yz}$ for shearing in planes containing the anisotropy axis. When the gap is anisotropic χ does not have the same angular dependence as the driving term in the Boltzmann equation, but as we show in Appendix C one can still solve the Boltzmann equation by a perturbative approach similar to the one we described above for isotropic states. One finds

$$\frac{\delta\eta_{xy}}{\eta} = -D(\alpha_\eta)15\int\frac{d\Omega_{\hat{p}}}{4\pi}\hat{p}_x^2\hat{p}_y^2\left(\frac{\Delta_{\hat{p}}}{k_BT_c}\right), \quad (3.31)$$

and

$$\frac{\delta\eta_{xz}}{\eta}=\frac{\delta\eta_{yz}}{\eta} = -D(\alpha_\eta)15\int\frac{d\Omega_{\hat{p}}}{4\pi}\hat{p}_x^2\hat{p}_z^2\left(\frac{\Delta_{\hat{p}}}{k_BT_c}\right) \quad (3.32)$$

where the angular integrals are to be performed over all directions on the Fermi sphere. For the ABM state, where $\Delta\propto\sin\Theta$,

$$\frac{\delta\eta_{xy}}{\eta} = -D(\alpha_\eta)\frac{75\pi}{256}\frac{\Delta_{\max}}{k_BT_c} \quad (3.33)$$

and

$$\frac{\delta\eta_{xz}}{\eta}=\frac{\delta\eta_{yz}}{\eta}=\frac{4}{5}\frac{\delta\eta_{xy}}{\eta}, \quad (3.34)$$

where $\Delta_{\max}$ is the maximum value of the gap. The anisotropy of the change in the viscosity is therefore only 25%. We have made estimates of α_η for liquid ^{3}He at various pressures using the s- and p-wave approximation[21] for the collision probability. The expression for α_η is given in Appendix E, but the results of the calculations are to be found in the following paper.[18]

To calculate the viscosity in the presence of a magnetic field one must be able to make calculations for nonunitary states. In general this is complicated, but for the ABM state, with different superfluid gaps for the two spin populations, the calculations are relatively straightforward. The final result is that in the expressions for the viscosity the gap must be replaced by the average gap for the two spin populations. In a previous publication[6] we applied these results to analyze Otaniemi measurements[1] of the viscosity in the A and A_1 phases at the melting pressure.

Since $\delta\eta/\eta$ depends on pressure through the normal-state parameter α_η one may learn about the pressure dependence of the latter through measurements of $\delta\eta/\eta$ as a function of pressure. Such information cannot be obtained from normal-state measurements, because the normal-state viscous relaxation time τ_η is a function of α_η and $\tau(0)$, the quasiparticle relaxation time at the Fermi energy. Measurements of the pressure dependence of α_η may therefore lead to a better understanding of quasiparticle scattering amplitudes in liquid ^{3}He.

As shown in detail in Sec. IV measurements of spin relaxation close to T_c make it possible to obtain the value of the quasiparticle relaxation time $\tau(0)$. This provides one with additional motivation for measuring the pressure dependence of $\delta\eta/\eta$, since α_η and $\tau(0)$ together uniquely determine τ_η, which is known experimentally as a function of pressure in the normal state. The fact that our calculations give relationships between various experimentally determined quantities, independent of any assumptions about the scattering amplitude, is particularly important since our present understanding of the scattering amplitude is rather limited. In the companion paper[18] we give the results of using the s- and p-wave approximation[21] for calculating $\tau_\eta T^2$ as a function of pressure in the range 0–34 bar. It is evident from comparing these theoretical calculations with experiment that this approximation to the normal-state quasiparticle scattering amplitudes does not completely account for either the pressure dependence or the magnitude of $\tau_\eta T^2$.

IV. SPIN RELAXATION

In many NMR experiments the most important dissipation mechanism is the intrinsic process discussed by Leggett and Takagi,[17] in which quasiparticle spin is converted into the spin of Cooper pairs. The dipole forces change the spin of the Cooper pairs thereby producing a disequilibrium between the spin of Cooper pairs and the quasiparticle spin. The return towards equilibrium is brought about by collisions, which convert quasiparticle spin into the spin of Cooper pairs (or vice versa).

To see in more detail how the relaxation process comes about, let us consider the particular case of longitudinal resonance. The operator S_{op} for the z component of the total spin is

$$S_{op}=\sum_{\vec{p},\sigma}\frac{\hbar}{2}\sigma a^\dagger_{\vec{p}\sigma}a_{\vec{p}\sigma}, \quad (4.1)$$

where $a^\dagger_{\vec{p}\sigma}$ and $a_{\vec{p}\sigma}$ create and destroy normal-state quasiparticles. Written in terms of creation and destruction operators for quasiparticles in the superfluid, S_{op} will generally have terms like $\alpha^\dagger\alpha^\dagger$ and $\alpha\alpha$, as well as ones involving the quasiparticle number operator $\alpha^\dagger\alpha$. However if one works in terms of quasiparticle states which are eigenstates of the quasiparticle energy for the instantaneous value of the gap matrix (not the gap matrix in the unpolarized state), the $\alpha^\dagger\alpha^\dagger$ and $\alpha\alpha$ terms will not contribute to the expectation value of S_{op}. Consider now the spin of a state which differs little from the unpolarized equilibrium state, so that we need to consider only first-order changes. The spin may then be written as the sum of two parts. The first is proportional to the deviation of the quasiparticle distribution from its value in the unpolarized state, and is what is referred to as the quasiparticle spin. The second is due to deviations of the superfluid coherence factors from their values in the unpolarized state, and is referred to as the spin of Cooper pairs. The z component of the quasiparticle spin S_q may be written in the form

$$S_q = \frac{\hbar}{2}\sum_{\vec{p}\sigma\sigma'} (s_{\vec{p}})_{\sigma\sigma'}(\delta n_{\vec{p}})_{\sigma\sigma'}, \tag{4.2}$$

where $(s_{\vec{p}})_{\sigma\sigma'}$ is a 2×2 matrix in spin space and $(\delta n_{\vec{p}})_{\sigma\sigma'}$ is the 2×2 matrix quasiparticle distribution function. It is convenient to work in terms of quasiparticle states in which $(s_{\vec{p}})_{\sigma\sigma'}$ is diagonal. Using the fact that for a unitary state $(s_{\vec{p}})_{\uparrow\uparrow} = -(s_{\vec{p}})_{\downarrow\downarrow}$ one may therefore write for the z component of the spin

$$S_q = \frac{\hbar}{2}\sum_{\vec{p}\sigma} \sigma s_{\vec{p}}\delta n_{\vec{p}\sigma}, \tag{4.3}$$

where $\delta n_{\vec{p}\sigma}$ is the diagonal part of the matrix $(\delta n_{\vec{p}})_{\sigma\sigma'}$, and is the only part of interest in the case of longitudinal resonance. $s_{\vec{p}}$ is just $(s_{\vec{p}})_{\uparrow\uparrow} = -(s_{\vec{p}})_{\downarrow\downarrow}$. If one works with these eigenstates, calculations are particularly simple since one never has to deal with off-diagonal spin components of the quasiparticle distribution function. The quantity $s_{\vec{p}}$ is just the renormalization factor f of Eq. (2.19) for the spin operator, corresponding to $\epsilon_\sigma = -1$, $\epsilon_p = \epsilon_E = +1$, and gives the z component of spin carried by a quasiparticle in the superfluid compared with that carried by a particle. For the ABM state a quasiparticle in the superfluid has a probability $|u_{\vec{p}}|^2 = \frac{1}{2}(1+\xi_p/E_{\vec{p}})$ of being a normal-state quasiparticle of the same spin and a probability $|v_{\vec{p}}|^2 = \frac{1}{2}(1-\xi_p/E_{\vec{p}})$ of being a normal-state hole of the same spin. $s_{\vec{p}}$ is therefore $|u_{\vec{p}}|^2 - |v_{\vec{p}}|^2 = \xi_p/E_{\vec{p}}$. In the case of s-wave pairing $s_{\vec{p}}$ is just 1, since the normal-state hole component of the superfluid quasiparticle has the opposite spin, or, put in other words, the condensate carries no spin.

The collision integral may be obtained straightforwardly from the general results derived in Sec. II by putting $f_{\vec{p}}^{(\nu)} = s_{\vec{p}}$.

One important point to note is that the collision term will vanish when the distribution function is equal to the local equilibrium distribution function corresponding to the instantaneous value of the difference between the chemical potentials for up- and down-spin particles in the condensate. Any external magnetic field is irrelevant so far as the question of to what distribution the system is relaxing is concerned. The fact that the system is relaxing to a local equilibrium distribution function rather than a global one makes itself felt through the quasiparticle energies which occur in the energy conservation condition. Since the total spin (not just the quasiparticle spin) is conserved in collisions, if one neglects the very small effects due to the magnetic dipole-dipole interaction, the total energy associated with coupling of the system to the external field is conserved and therefore it must drop out of the conservation condition. Note however that if one wished to include explicitly the coupling energy to the external field one must remember that the spin of the Cooper pairs generally changes in a collision; the change in the coupling energy of the pairs to the external field must therefore also be included in the conservation condition. The local equilibrium distribution function to which $n_{\vec{p}\sigma}$ is relaxing may be written in the form

$$n_{\vec{p}\sigma}^{\text{l.e.}} = (e^{\beta E_{\vec{p}}(H)} + 1)^{-1}, \tag{4.4}$$

where $E_{\vec{p}\sigma}(H)$ is the quasiparticle energy evaluated when the difference between the up- and down-spin chemical potentials for particles in the condensate is finite, and equal to $2(\frac{1}{2}\hbar)H$. Note that H must not include the contribution from the external field. For small values of H we may write

$$n_{\vec{p}\sigma}^{\text{l.e.}} = n_{\vec{p}\sigma}^{0} - \frac{\hbar}{2}\sigma s_{\vec{p}}\frac{\partial n_{\vec{p}}}{\partial E_{\vec{p}}}H,$$

since (4.5)

$$\frac{\hbar}{2}\sigma s_{\vec{p}} = -\frac{\partial E_{\vec{p}\sigma}}{\partial H}.$$

This follows directly from the fact that the total energy, measured with respect to the chemical potential of the up and down spins is

$$E - \mu_\uparrow N_\uparrow - \mu_\downarrow N_\downarrow = E - \left(\frac{\mu_\uparrow + \mu_\downarrow}{2}\right)(N_\uparrow + N_\downarrow) - \left(\frac{\mu_\uparrow - \mu_\downarrow}{2}\right)(N_\uparrow - N_\downarrow), \tag{4.6}$$

where E is the total energy and $N_\uparrow$ and $N_\downarrow$ are the numbers of up-spin particles and down-spin particles, respectively.

We are interested only in spatially homogeneous situations, and therefore the terms in the Boltzmann equation containing spatial gradients may be dropped. Also we are considering the case of longitudinal resonance so the precession terms need not be included. The streaming terms in the Boltzmann equation thus reduce to just the time derivative, which we rewrite in terms of ψ:

$$\frac{\partial n_{\vec{p}\sigma}}{\partial t}=\frac{\partial}{\partial t}(n_{\vec{p}\sigma}-n^{\text{l.e.}}_{\vec{p}\sigma})+\frac{\partial}{\partial t}(n^{\text{l.e.}}_{\vec{p}\sigma}-n^{0}_{\vec{p}\sigma})$$

$$=\frac{\partial\psi_{\vec{p}\sigma}}{\partial t}n^0_{\vec{p}}(1-n^0_{\vec{p}})-\frac{\hbar}{2}\sigma s_{\vec{p}}\frac{\partial n_{\vec{p}}}{\partial E_{\vec{p}}}\frac{\partial H}{\partial t}. \quad (4.7)$$

The linearized Boltzmann equation is therefore

$$\frac{\partial\psi_1}{\partial t}n^0_1(1-n^0_1)-\frac{\hbar}{2}\sigma_1 s_1\frac{\partial n_1}{\partial E_1}\frac{\partial H}{\partial t}=-\sum_{2,3,4}W_N(1,2;3,4)n^0_1n^0_2(1-n^0_3)(1-n^0_4)$$
$$\times\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(\xi_1+\xi_2-\xi_3-\xi_4)[\psi_1+s_1(s_2\psi_2-s_3\psi_3-s_4\psi_4)]. \quad (4.8)$$

In the normal limit $s_i\to 1$, and therefore for $\psi_i\propto\sigma_i$ the collision term vanishes since quasiparticle spin is conserved in the normal state. This just reflects the fact that in the normal state the quasiparticle spin is the total spin.

Let us first consider the solution of Eq. (4.8) in the hydrodynamic limit in which the characteristic times for changes in H to occur are long compared with a characteristic relaxation time for the quasiparticle spin. (After having obtained the solution of the equation below we shall explore what this limit means physically.) In that case the $\partial\psi_1/\partial t$ term can be neglected, and the equation then has the same form as in the case of the viscosity. Also H on the left-hand side of (4.8) may be replaced by the local equilibrium value $\gamma^2 S_q/\chi_{q0}$, where χ_{q0} is the quasiparticle susceptibility, with Fermi liquid effects excluded. However for the spin relaxation problem the equation is singular in the normal limit, because of the conservation of quasiparticle spin in the normal state. To leading order in Δ/k_BT_c the solution $\psi_{\vec{p}\sigma}$ is given by

$$\psi_{\vec{p}\sigma}=-\sigma\frac{\hbar}{2}s_{\vec{p}}\tau(0)\frac{1}{\lambda}\frac{1}{k_BT}\gamma^2\frac{\partial S_q}{\partial t}\frac{1}{\chi_{q0}}, \quad (4.9)$$

where

$$\tau(0)=2\tau_0/\pi^2 \quad (4.10)$$

is the relaxation time of a normal-state quasiparticle at the Fermi energy at T_c. This is demonstrated in Appendix D. Note that this solution is equivalent to $\psi_i\propto\sigma_i$, the deviation function which in the normal state does not relax. The parameter λ is the susceptibility of the Cooper pairs relative to the total susceptibility, with Fermi-liquid effects neglected. To leading order in Δ it is

$$\lambda=\frac{1}{\nu(0)}\sum_{\vec{p}\sigma}(1-s^2_{\vec{p}})\left(-\frac{\partial n_{\vec{p}}}{\partial\xi_{\vec{p}}}\right), \quad (4.11)$$

where $\nu(0)$ is the density of quasiparticle states (of both spins) at the Fermi surface in the normal state. λ is of order Δ/k_BT_c, and therefore ψ diverges as the normal state is approached.

To derive an equation for the quasiparticle spin S_q, we calculate from ψ the deviation $n_{\vec{p}\sigma}-n^0_{\vec{p}\sigma}$ and insert it into Eq. (4.3), with the result

$$\sum_{\vec{p}\sigma}\sigma s_{\vec{p}}\frac{\hbar}{2}(n_{\vec{p}\sigma}-n^0_{\vec{p}\sigma})=S_q-\gamma^{-2}\chi_{q0}H$$
$$=-\frac{\tau(0)}{\lambda}\frac{\partial S_q}{\partial t} \quad (4.12)$$

or

$$\frac{\partial S_q}{\partial t}=-\frac{\lambda}{\tau(0)}(S_q-\gamma^{-2}\chi_{q0}H). \quad (4.13)$$

Equation (4.13) is of the same form as the equation one obtains by summing Combescot and Ebisawa's kinetic equation,[22] and shows that their characteristic relaxation time must be identified with $\tau(0)/\lambda$,[7] which diverges as $(T_c-T)^{-1/2}$ as T approaches T_c. If we now make use of the fact that the Cooper-pair magnetization responds essentially instantaneously, in a time $\sim\hbar/\Delta$, to changes in H, and introduce the Cooper-pair spin susceptibility function χ_{C0} we may write

$$S_p=S-S_q=\gamma^{-2}\chi_{C0}H \quad (4.14)$$

or

$$H=\gamma^2(S-S_q)/\chi_{C0}. \quad (4.15)$$

Substituting this result into (4.13) one finds, using $\chi_{C0}/\chi_{q0}=\lambda/(1-\lambda)$,

$$\frac{\partial S_q}{\partial t}=-\frac{1}{\tau(0)}(S_q-S_{q0}), \quad (4.16)$$

where $S_{q0}=(1-\lambda)S$ is the equilibrium value of S_q for the given total spin S. Finally if one writes an equation for $S_p=S-S_q$ one finds

$$\frac{\partial S_p}{\partial t}=\frac{\partial S}{\partial t}-\frac{\partial S_q}{\partial t} \quad (4.17)$$

$$=R_D-[1/\tau(0)](S_p-S_{p0}), \quad (4.18)$$

where $S_{p0}=\lambda S$ is the equilibrium value of S_p for given total spin S, and R_D is the dipolar torque.

So far we have considered only the hydrodynamic limit, and have neglected the $\partial\psi/\partial t$ term in (4.7). By using the solution (4.9) for ψ one can see that this term will no longer be negligible when the characteristic frequency ω of the variations of H is of the order of or greater than $\lambda/\tau(0)$. The characteristic frequencies are usually of the order of the longitudinal-resonance frequency, and therefore all experiments to date have been in the hydrodynamic regime. However, the Boltzmann equation (4.8) can be solved exactly outside the hydrodynamic limit. The equation is very similar to that for the frequency dependent conductivity.[23] In the extreme collisionless limit $[\omega \gg \lambda/\tau(0)]$ the solution is $\psi_i \propto \sigma_i$, and the characteristic spin relaxation time is again just $\tau(0)/\lambda$, as in the hydrodynamic limit. Generally the intermediate range of frequencies, when $\omega \sim \lambda/\tau(0)$ is difficult to treat, but for this problem one finds that the solution to the Boltzmann equation is again of the simple form $\psi_i \propto \sigma_i$. The reason for this is the existence of a nearly conserved quantity, here the quasiparticle spin. The calculation is analogous to that of the electrical resistivity of electrons in a metal when $\alpha_1(1,1)$ in the notation of Appendix B is close to 2 (electron momentum almost conserved),[23] or the attenuation of sound in a normal Fermi liquid when $\alpha_2(1,1)$ $(\equiv \alpha_\eta)$ is close to 2 (momentum flux almost conserved).[24] In summary then, the collision term in the Boltzmann equation may be replaced by $-\lambda(n_1 - n_1^{\text{l.e.}})/\tau(0)$ in the hydrodynamic, collisionless and intermediate regimes. As a consequence Eqs. (4.13) and (4.18) hold for all frequencies for which the Boltzmann equation is valid. Note that the justification for this is somewhat more complicated than stated in Ref. 7.

Expressions for $\tau(0)$ obtained using the s- and p-wave approximation are given in Appendix E, and estimates of $\tau(0)$ for liquid ^{3}He at various pressures are given in the companion paper.[18]

These estimates may be compared with the values of $\tau(0)$ extracted from the width of the longitudinal resonance[3] and magnetic ringing experiments.[4] Note that magnetic-relaxation measurements in the superfluid phases, like measurements of $\delta\eta/\eta$, enable one to obtain information about normal state properties which cannot be found from measurements in the normal state.

V. DISCUSSION

In the calculations described above we made no assumptions about the normal-state transition probability, and solved the Boltzmann equation exactly to lowest order in the gap. Thus measurements of transport and relaxation processes in superfluid ^{3}He close to T_c can now be used to provide precise information about quasiparticle interactions in normal ^{3}He, provided only that the magnitude of the superfluid gap is known. One particularly useful result is a consistency check relating the normal-state viscosity, the relative drop in the viscosity in the superfluid state close to T_c, and the intrinsic spin relaxation time at T_c. The relative drop of the viscosity in the superfluid may be used to determine α_η, the spin relaxation time is $\tau(0)$, and therefore the normal-state viscosity, which depends on the normal state collision probability only through α_η and $\tau(0)$, may be estimated in terms of directly measurable quantities, independent of any assumption about the angular dependence of the normal state collision probability. The form of this consistency relation becomes particularly simple if one limits oneself to the first two terms in the series (3.26) and (3.27), which is an excellent approximation for the values of α_η of interest. In terms of the normal-state viscous relaxation time τ_η, which is defined in the usual manner by $\eta = \frac{1}{5}nm^* v_F^2 \tau_\eta$, the relative change $\delta\eta/\eta$ in the viscosity on entering the B-phase may be expressed in terms of τ_η and $\tau(0)$ as

$$\frac{\delta\eta}{\eta} = -\frac{\pi}{4}\left(1 - \frac{\pi^2}{12} + \frac{\tau_\eta}{\tau(0)}\right)^2 \frac{\tau(0)}{\tau_\eta}\,\frac{\Delta}{k_B T} \tag{5.1}$$

for an arbitrary collision probability $W_N(\theta, \phi)$. Analogous relations hold in the A phase, with a slightly different numerical coefficient that depends on the particular orientation of the orbital vector $\hat{l}$ in a given experiment.

We have derived the collision integral close to T_c in a generally applicable form, which emphasized the role of the symmetry of the deviation function under reversal of spin, momentum and change in sign of energy. An important point is that the coalescence and decay processes play a significant role close to T_c.

The methods developed in this paper have also been applied to calculate the attenuation of low-frequency sound,[25] and orbital relaxation in ^{3}He-A.[26] In the latter case one finds that the characteristic quasiparticle relaxation time to be used is again $\tau(0)$, the intrinsic spin relaxation time. Thus measurement of orbital relaxation[27] close to T_c enables one to determine the value of $\tau(0)$, which must be consistent with that obtained from measurements of the normal state viscosity and the relative drop in the viscosity on entering the superfluid state, irrespective of any assumptions regarding the form of the quasiparticle scattering amplitude.

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APPENDIX A: DERIVATION OF THE COLLISION INTEGRAL

The contribution to the collision term from two quasiparticle scattering processes is given by Eqs. (2.4) and (2.13), and is

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_N(1,2;3,4)|u_1|^2|u_2|^2|u_3|^2|u_4|^2 n_1 n_2 (1-n_3)(1-n_4) \times \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(E_1+E_2-E_3-E_4)(\psi_1+\psi_2-\psi_3-\psi_4). \tag{A1}$$

The decay process gives a contribution which, from (2.6) and (2.15), is

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_N(1,2;3,4)|v_2|^2 n_1 (1-n_{-2})(1-n_3)(1-n_4) \times \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(E_1-E_{-2}-E_3-E_4)(\psi_1-\psi_{-2}-\psi_3-\psi_4), \tag{A2}$$

since the three terms in the expression for W_s' all give identical contributions to the sum. E_{-2} is identical with E_2, but we retain the minus sign to preserve the symmetry of the notation. Finally, for the coalescence process, one finds from (2.8) and (2.15)–(2.17),

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_N(1,2;3,4)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}[\,|v_1|^2 n_1(1-n_{-2})n_{-3}n_{-4}\delta(E_1-E_{-2}+E_{-3}+E_{-4})(\psi_1-\psi_{-2}+\psi_{-3}+\psi_{-4}) + |v_3|^2 n_1 n_2 n_{-3}(1-n_4)\delta(E_1+E_2+E_{-3}-E_4)(\psi_1+\psi_2+\psi_{-3}-\psi_4) + |v_4|^2 n_1 n_2 (1-n_3) n_{-4}\delta(E_1+E_2-E_3+E_{-4})(\psi_1+\psi_2-\psi_3+\psi_{-4})]. \tag{A3}$$

If we now make use of the particle-hole symmetry of degenerate Fermi systems, we may replace E_{-i} by $-E_i$ (and as a consequence n_{-i} by $1-n_i$) in (A2) and (A3).

The decay contribution then becomes

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_N(1,2;3,4)|v_2|^2 n_1 n_2 (1-n_3)(1-n_4)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \times \delta(E_1+E_2-E_3-E_4)[\psi_1-\psi_{-2}(-E_2)-\psi_3-\psi_4], \tag{A4}$$

and the coalescence term is, from Eqs. (2.7), (2.15), and (2.17),

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_N(1,2;3,4) n_1 n_2 (1-n_3)(1-n_4)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(E_1+E_2-E_3-E_4) \times \{|v_1|^2[\psi_1-\psi_{-2}(-E_2)+\psi_{-3}(-E_3)+\psi_{-4}(-E_4)] + |v_3|^2[\psi_1+\psi_2+\psi_{-3}(-E_3)-\psi_4] + |v_4|^2[\psi_1+\psi_2-\psi_3+\psi_{-4}(-E_4)]\}. \tag{A5}$$

Here by $\psi_{-i}(-E_i)$ we mean ψ evaluated for the quasiparticle state whose momentum is in the direction opposite that of $\vec{p}_i$, and whose energy is $-E_i$.

To illustrate how the various terms combine let us neglect the spin variable for the moment, and assume the coherence factors are just numbers satisfying the normalization condition $|u_i|^2+|v_i|^2=1$. Since near T_c we may neglect terms containing products of $|v_i|^2$ factors for different states, we may write $|u_1|^2|u_2|^2|u_3|^2|u_4|^2 \simeq 1-|v_1|^2-|v_2|^2-|v_3|^2-|v_4|^2$. The contribution proportional to ψ_1 in the sum of (A1), (A4), and (A5) involves no coherence factors, and is identical in form with the corresponding normal state collision term. The ψ_2 and $\psi_{-2}(-E_2)$ terms in the sum are

$$(1-|v_1|^2-|v_2|^2-|v_3|^2-|v_4|^2)\psi_2-|v_2|^2\psi_{-2}(-E_2)-|v_1|^2\psi_{-2}(-E_2)+|v_3|^2\psi_2+|v_4|^2\psi_2 = (1-|v_1|^2-|v_2|^2)\psi_2-(|v_1|^2+|v_2|^2)\psi_{-2}(-E_2). \tag{A6}$$

If $\psi_2=\psi_{-2}(-E_2)$, this reduces to $(|u_1|^2-|v_1|^2)(|u_2|^2-|v_2|^2)\psi_2$, and if $\psi_2=-\psi_{-2}(-E_2)$, to just ψ_2. This is in agreement with the expression (2.19) if the spin indices there are suppressed.

When spin sums are included the collision integral may be calculated in a way similar to the simple spinless case considered above. In deriving the final result (2.18) one has to make use of the invariance of the scattering probability under reversal of all the spins, and the fact that for unitary states

$$|u_{\sigma\sigma'}|^2=|u_{-\sigma-\sigma'}|^2 \quad\text{and}\quad |v_{\sigma\sigma'}|^2=|v_{-\sigma-\sigma'}|^2. \tag{A7}$$

APPENDIX B: NORMAL-STATE BOLTZMANN EQUATION

Here we discuss a number of important properties of the solution of the normal-state Boltzmann equation. Further details of some aspects of the calculations may be found in the original papers[15,16] and in a recent review.[28] Our aim here is to present the properties of the normal state equation in a unified notation which will enable us to apply it easily to a number of different physical problems. The normal-state collision operator is given by Eq. (2.18) with $f^{(\nu)}$ put equal to unity:

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\sum_{2,3,4} W_N(1,2;3,4) \times n_1 n_2 (1-n_3)(1-n_4)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \times \delta(\xi_1+\xi_2-\xi_3-\xi_4)(\psi_1+\psi_2-\psi_3-\psi_4). \tag{B1}$$

Let us now consider the result of acting with the collision operator on the components of ψ which have a definite symmetry under spin reversal and are either even or odd functions of the energy ξ; we specify these components by $\psi(\epsilon_\sigma, \epsilon_E, \hat{p}, t)$, where ϵ_σ and ϵ_E $(=\pm 1)$ give the symmetry under the two interchanges $\sigma \to -\sigma$ and $\xi \to -\xi$. Instead of the variable $\vec{p}$ we use $\hat{p}$ and $t = \xi_p / k_B T$. Next we expand ψ in terms of spherical harmonics

$$\psi(\epsilon_\sigma, \epsilon_E, \hat{p}, t) = \sum_{l,m} \psi_{lm}(\epsilon_\sigma, \epsilon_E, t) Y_{lm}(\hat{p}). \tag{B2}$$

The sums over $\vec{p}_2$, $\vec{p}_3$, and $\vec{p}_4$ in (B1) may be converted into integrals over the quasiparticle energies ξ_2, ξ_3, and ξ_4 and over angles. Since all quasiparticles have momenta essentially equal to the Fermi momentum, the transition probability is a function only of θ, the angle between $\vec{p}_1$ and $\vec{p}_2$, and ϕ, the angle between the plane containing $\vec{p}_1$ and $\vec{p}_2$, and the plane containing $\vec{p}_3$ and $\vec{p}_4$. There are two independent scattering probabilities, $W_{\uparrow\uparrow}(\theta,\phi)$ for scattering of quasiparticles with the same spin, and $W_{\uparrow\downarrow}(\theta,\phi)$ for scattering of quasiparticles with opposite spin. Performing the integrations one finds

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = -\frac{t_1^2+\pi^2}{2\tau_0} n_1(1-n_1)\psi_1 + \frac{1}{2\tau_0}\frac{1}{2\cosh(\frac{1}{2}t_1)} \times \int \frac{d\Omega_{\hat{p}'}}{4\pi}\alpha(\epsilon_\sigma, \epsilon_E, \hat{p}_1\cdot\hat{p}') \times \int_{-\infty}^{\infty} dt' F(t_1 - t') \frac{\psi(\epsilon_\sigma, \epsilon_E, \hat{p}', t')}{2\cosh(\frac{1}{2}t')} \tag{B3}$$

$$= -\frac{t_1^2+\pi^2}{2\tau_0} n_1(1-n_1)\psi_1 + \frac{1}{2\tau_0}\sum_{lm}\frac{\alpha_l(\epsilon_\sigma,\epsilon_E) Y_{lm}(\hat{p}_1)}{2\cosh(\frac{1}{2}t_1)} \times \int_{-\infty}^{\infty} dt' F(t_1-t')\frac{\psi_{lm}(\epsilon_\sigma,\epsilon_E,t')}{2\cosh(\frac{1}{2}t')}, \tag{B4}$$

where the characteristic relaxation time τ_0 is given by

$$\tau_0 = \frac{8\pi^4\hbar^6}{m^{*3}(k_B T)^2}\frac{1}{\langle W_N\rangle}, \tag{B5}$$

and $F(t) = t/2\sinh(\frac{1}{2}t)$. Also

$$W_N(\theta,\phi) = \tfrac{1}{2}[\tfrac{1}{2}W_{\uparrow\uparrow}(\theta,\phi) + W_{\uparrow\downarrow}(\theta,\phi)], \tag{B6}$$

$$\langle\cdots\rangle = \int \frac{d(\cos\theta)d\phi}{4\pi\cos(\frac{1}{2}\theta)}\cdots, \tag{B7}$$

and

$$\alpha(\epsilon_\sigma,\epsilon_E,\hat{p}_1\cdot\hat{p}') = \Big(\Big\langle\frac{W_{\uparrow\uparrow}(\theta,\phi)}{2}[-\epsilon_E\delta(\hat{p}'-\hat{p}_2)+\delta(\hat{p}'-\hat{p}_3)+\delta(\hat{p}'-\hat{p}_4)]\Big\rangle \tag{B8}$$

$$+\langle W_{\uparrow\downarrow}(\theta,\phi)[-\epsilon_E\epsilon_\sigma\delta(\hat{p}'-\hat{p}_2)+\delta(\hat{p}'-\hat{p}_3)+\epsilon_\sigma\delta(\hat{p}'-\hat{p}_4)]\rangle\Big)\Big/\langle W_N\rangle$$

$$= \sum_{lm}\alpha_l(\epsilon_\sigma,\epsilon_E) Y_{lm}(\hat{p}_1) Y^*_{lm}(\hat{p}'). \tag{B9}$$

Thus

$$\alpha_l(\epsilon_\sigma=+1,\epsilon_E) = 2\langle W_N(\theta,\phi)[-\epsilon_E P_l(\theta_{12}) + P_l(\theta_{13}) + P_l(\theta_{14})]\rangle/\langle W_N\rangle \tag{B10}$$

with θ_{ij} denoting the angle between $\vec{p}_i$ and $\vec{p}_j$, and

$$\alpha_l(\epsilon_\sigma=-1,\epsilon_E) = \Big\langle\frac{W_{\uparrow\uparrow}(\theta,\phi)}{2}[-\epsilon_E P_l(\theta_{12}) + P_l(\theta_{13}) + P_l(\theta_{14})] + W_{\uparrow\downarrow}(\theta,\phi)[\epsilon_E P_l(\theta_{12}) + P_l(\theta_{13}) - P_l(\theta_{14})]\Big\rangle\Big/\langle W_N\rangle. \tag{B11}$$

As a result of conservation of quasiparticle number, momentum, and spin in collisions between normal-state quasiparticles $\alpha_{l=0}(+1,+1)$, $\alpha_{l=1}(+1,+1)$, and $\alpha_{l=0}(-1,+1)$ are identically equal to 2. For $\epsilon_E=+1$ all other α's are less than or equal to 2 and for $\epsilon_E=-1$ the α's are less than or equal to 6, the equalities obtaining only for pathological forms of the scattering probability W. Note also that the quantity referred to in the body of the paper as α_η is just $\alpha_2(+1,+1)$. The expression (B3) is a particularly useful form for the collision term for the case of spin relaxation.

It is convenient to expand ψ_{lm} in terms of the functions $\phi_n(t)$ which satisfy the equation

$$\int_{-\infty}^{\infty} dt'\, F(t-t')\phi_n(t') = \frac{1}{\lambda_n}(t^2+\pi^2)\phi_n(t). \tag{B12}$$

This equation may be solved by Fourier transformation, and the eigenvalues are given by

$$\lambda_n = n(n+1), \quad n\ge 1, \tag{B13}$$

and the eigenfunctions are

$$\phi_n(t) = c_n \int_{-\infty}^{\infty} dq\, P_n^1(\tanh q) e^{iqt/\pi}, \tag{B14}$$

where c_n is a normalization constant, and P_n^1 is the associated Legendre polynomial. The ϕ_n may be taken to be real, and the c_n are chosen so that the orthonormality condition

$$\int_{-\infty}^{\infty} dt\,(t^2+\pi^2)\phi_n(t)\phi_{n'}(t) = \delta_{n,n'} \tag{B15}$$

is satisfied.

To calculate the viscosity in the superfluid state we need the solution of the normal-state Boltzmann equation at the Fermi energy. In the normal state the deviation function is given by (3.11) and (3.12) with V put equal to unity. The equation for Q in the normal state, which we denote by Q_N, is

$$\frac{1}{\cosh(\frac{1}{2}t)} = (\pi^2+t^2)Q_N(t) - \alpha_\eta \int_{-\infty}^{\infty} dt'\, F(t-t')Q_N(t'). \tag{B16}$$

The solution may be written

$$Q_N(t) = \frac{X}{\pi^2+t^2} + \sum_n a_n\phi_n(t), \tag{B17}$$

where $X = 1/\cosh(\frac{1}{2}t)$. Inserting this into (B16) and using the orthonormality condition (B15) one finds

$$a_n = [\alpha_\eta/(\lambda_n-\alpha_\eta)](X,\phi_n) \tag{B18}$$

[where the product notation is defined following Eq. (3.19)], whence

$$Q_N(t) = \frac{X}{\pi^2+t^2} + \alpha_\eta \sum_n \frac{1}{\lambda_n-\alpha_\eta}(X,\phi_n)\phi_n(t). \tag{B19}$$

Only odd n contribute to the sum since for even n ϕ_n is an odd function of t, and therefore (X,ϕ_n) vanishes.

The normal-state viscosity η_N may then be found from (3.15), (3.11), and (B19), with all V's put equal to one, and is

$$\eta_N = \tfrac{1}{5} n m^* v_F^2 \tau_0 (\tfrac{1}{2} Y_N), \tag{B20}$$

where

$$\begin{aligned} Y_N &= (X, Q_N) \\ &= \left(X, \frac{X}{\pi^2+t^2}\right) + \sum_{n=1,3,\ldots} \frac{\alpha_\eta}{n(n+1)-\alpha_\eta}\,|(X,\phi_n)|^2 \\ &= \frac{1}{3} + \frac{4\alpha_\eta}{\pi^2} \sum_{n=1,3,\ldots} \frac{2n+1}{n^2(n+1)^2}\,\frac{1}{n(n+1)-\alpha_\eta}, \end{aligned} \tag{B21}$$

since $(X,\phi_n) = 2c_n$, and $|c_n|^2 = (2n+1)/[n(n+1)\pi]^2$.

To evaluate $Q_N(0)$ one needs $\phi_n(0)$, which from (B14) may be shown to be

$$\phi_n(0) = \frac{c_n}{2}\left(\frac{n!!}{(n-1)!!}\right)^2. \tag{B22}$$

Thus

$$Q_N(0) = \frac{1}{\pi^2}\left[1 + \alpha_\eta \sum_{n=1,3,\ldots} \frac{2n+1}{n^2(n+1)^2}\left(\frac{n!!}{(n-1)!!}\right)^2 \times \frac{1}{n(n+1)-\alpha_\eta}\right]. \tag{B23}$$

The series (B23) converges rapidly, since for large n the terms in the sum behave asymptotically as n^{-4}, as compared with the terms in the series for Y_N (B21) which vary as n^{-5}.

APPENDIX C: VISCOSITY IN STATES WITH AN ANISOTROPIC GAP

In a superfluid with an anisotropic gap the deviation function does not have the same dependence on direction on the Fermi surface as the driving term in the Boltzmann equation. If one introduces a modified deviation function $Q(\hat{p},t)$, defined by

$$\psi_1 = \frac{v_F V_1}{k_B T}\frac{\partial u_x}{\partial y} p_F \tau_0 2\cosh(\tfrac{1}{2}t_1) Q(\hat{p}_1, t_1), \tag{C1}$$

one may show that this satisfies the equation

$$\frac{\hat{p}_i\hat{p}_j}{\cosh(\frac{1}{2}t)} = (\pi^2+t^2)Q_{ij}(\hat{p},t) - \int \frac{d\Omega_{\hat{p}'}}{4\pi}\alpha(1,1,\hat{p}\cdot\hat{p}') \times \int_{-\infty}^{\infty} dt'\, F(t-t')V^2(\hat{p}',t')Q_{ij}(\hat{p}',t'). \tag{C2}$$

Here the indices on Q indicate the symmetry of the driving term in the Boltzmann equation. Equation

(C2) is a straightforward generalization of Eq. (3.12) for the isotropic case, and may easily be derived from (3.9) and the properties of the normal-state collision operator discussed in Appendix B.

The viscosity is a tensor, and the momentum flux is given by

$$\Pi_{ij} = -\eta_{ij,kl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right). \quad \text{(C3)}$$

Here the x_i are spatial coordinates, and in this paper we restrict ourselves to the case where $i \neq j$, and $k \neq l$. Equation (C2) may be solved by a perturbative approach, just as in the isotropic case; the only difference is that in the anisotropic case the inner product must include an angular integral as well as an integral over t. The viscosities are given by

$$\frac{\eta_{ij,kl}}{\eta} = 1 - D(\alpha_\eta) 15 \int \frac{d\Omega_{\hat{p}}}{4\pi} \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l \frac{\Delta_{\hat{p}}}{k_B T_c}. \quad \text{(C4)}$$

The reason that no α's other than α_η [$=\alpha_2(1,1)$] enter is that the expression for the change in viscosity may be written in such a way that the normal-state collision operator acts only on the deviation function for the normal state, which has only $l=2$ terms.

APPENDIX D: SOLUTION OF THE BOLTZMANN EQUATION FOR SPIN RELAXATION

To reduce the Boltzmann equation (4.8) (with the $\partial\psi/\partial t$ term neglected) to a dimensionless form we introduce the deviation function Q which is related to ψ by

$$\psi_i = -\sigma_i \frac{\hbar}{2} s_i \tau_0 2\cosh\left(\frac{t_1}{2}\right) \frac{\partial H}{\partial t} \frac{Q_i}{k_B T}. \quad \text{(D1)}$$

The Boltzmann equation then becomes

$$\frac{\sigma_1}{\cosh(\frac{1}{2}t_1)} = (t_1^2 + \pi^2) Q_1 \sigma_1 - (Ks^2\sigma Q)_1 , \quad \text{(D2)}$$

where the operator K is defined by

$$(K\chi)_1 = \sum_{2,3,4} M(1,2;3,4) \times(-\chi_2 + \chi_3 + \chi_4) \Big/ \sum_{2,3,4} M(1,2;3,4) \quad \text{(D3)}$$

and

$$M(1,2;3,4) = W_N(1,2;3,4) n_1 n_2 (1-n_3)(1-n_4) \times \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \delta(\xi_1 + \xi_2 - \xi_3 - \xi_4). \quad \text{(D4)}$$

We now expand Q in terms of the eigenfunctions of the normal-state collision operator, defined by Eqs. (B12)–(B14):

$$Q(\hat{p}, t) = \sum_{nlm} a_{nlm} \phi_n(t) Y_{lm}(\hat{p}). \quad \text{(D5)}$$

Close to T_c the only term of importance in the sum is the one with $n=1$ and $l=m=0$. To show this one first writes

$$K\sigma s^2 Q = K\sigma Q - K\sigma(1-s^2)Q. \quad \text{(D6)}$$

The first of these may be evaluated by using the properties of the eigenfunctions ϕ_n discussed in Appendix B to show that

$$(K\sigma Q)_1 = \sigma_1(t_1^2 + \pi^2) \times \sum_{nlm} a_{nlm} \frac{\alpha_l(-1,1)}{n(n+1)} \phi_n(t_1) Y_{lm}(\hat{p}_1). \quad \text{(D7)}$$

The second term in (D6) may be written

$$K\sigma(1-s^2)Q_1 = \frac{t_1}{2\sinh(\frac{1}{2}t_1)} \times \sum_{lm} Y_{lm}(\hat{p}_1) \alpha_l(-1,+1) \times \int \frac{d\Omega_{\hat{p}}}{4\pi} Y^*_{lm}(\hat{p}) 4\lambda(\hat{p}) Q(\hat{p},0), \quad \text{(D8)}$$

where

$$\lambda(\hat{p}) = \frac{1}{4}\int_{-\infty}^{\infty} dt\,[1 - s^2(\hat{p},t)] = \int_{-\infty}^{\infty} d\xi_p \left(-\frac{\partial n_p}{\partial \xi_p}\right)(1 - s_p^2). \quad \text{(D9)}$$

This follows from the fact that $1 - s_p^2$ behaves essentially as a delta function at the Fermi surface, just as $1 - V^2$ did in the calculations of the viscosity [see Eq. (3.17)].

Multiplying Eq. (D2) by $\phi_n(t_1) Y^*_{lm}(\hat{p}_1)$ and integrating over t_1 and solid angles, one finds

$$\delta_{l,0} \int_{-\infty}^{\infty} \frac{dt}{\cosh(\frac{1}{2}t)} \phi_n(t) = \left(1 - \frac{\alpha_l(-1,1)}{n(n+1)}\right) a_{nlm} + 4\pi^2 \phi_n(0) \frac{\alpha_l(-1,1)}{n(n+1)} \times \int \frac{d\Omega_{\hat{p}}}{4\pi} Y^*_{lm}(\hat{p}) \lambda(\hat{p}) Q(\hat{p},0). \quad \text{(D10)}$$

The equation (D10) for $n=1$ and $l=m=0$ is

$$\int_{-\infty}^{\infty} \frac{dt}{\cosh(\frac{1}{2}t)} \phi_1(t) = 4\pi^2 \phi_1(0) \times \int \frac{d\Omega_{\hat{p}}}{4\pi} \lambda(\hat{p}) Q(\hat{p},0), \quad \text{(D11)}$$

since $\alpha_0(-1,+1) = 2$ as a consequence of quasiparticle spin conservation in the normal state. Since $\lambda(\hat{p})$ is of the order of magnitude of $\Delta/k_B T$, and the left-hand side of (D11) is of order unity, $Q(\hat{p},0)$ must be of order $k_B T/\Delta$, and therefore for all values of t, $Q(\hat{p},t)$ must be of this order, since $Q(\hat{p},t)$ has structure only on energy scales of order k_BT. From Eq. (D10) for components other than $n=1$, $l=m=0$ one sees immediately that a_{nlm} must be of order unity, since $\alpha_l(-1,+1)=n(n+1)$ only for $n=1$ and $l=m=0$. On the other hand Q must be of order k_BT/Δ, and therefore since all the eigenfunctions $\phi_n Y_{lm}$ are of the order of magnitude of unity, this implies that a_{100} must be of order k_BT/Δ, and consequently is the only term in the expansion that need be considered near T_c. We may therefore write $Q(\hat{p},t)=\gamma/2\cosh(\frac{1}{2}t)$ where the coefficient γ is independent of t. Substituting this into (D10) and using the fact that $\phi_1(t)\propto[\cosh(\frac{1}{2}t_1)]^{-1}$, one finds

$$\gamma = 2/\pi^2\lambda, \tag{D12}$$

where

$$\lambda = \int \frac{d\Omega_{\hat{p}}}{4\pi}\lambda(\hat{p}) \tag{D13}$$

is the ratio of the Cooper-pair susceptibility to the total susceptibility, both calculated neglecting Fermi-liquid effects, and near T_c it varies as $\Delta \sim (T_c-T)^{1/2}$ [cf. Eq. (4.10)]. From Eq. (D1) and this result for Q one immediately finds Eq. (4.9) for ψ.

APPENDIX E: s- AND p-WAVE APPROXIMATION (REF. 21)

If one neglects the small dipole-dipole interaction between nuclear spins and any other interactions that do not conserve the total spin, the quasiparticle scattering amplitude may be expressed in terms of the scattering amplitudes for pairs of quasiparticles in singlet and triplet states, T_s and T_t. The amplitudes are given by

$$\langle\uparrow\uparrow|T|\uparrow\uparrow\rangle = \langle\downarrow\downarrow|T|\downarrow\downarrow\rangle = T_t, \tag{E1}$$

$$\langle\uparrow\downarrow|T|\uparrow\downarrow\rangle = \langle\downarrow\uparrow|T|\downarrow\uparrow\rangle = \tfrac{1}{2}(T_s+T_t), \tag{E2}$$

and

$$\langle\uparrow\downarrow|T|\downarrow\uparrow\rangle = \langle\downarrow\uparrow|T|\uparrow\downarrow\rangle = \tfrac{1}{2}(-T_s+T_t). \tag{E3}$$

All other matrix elements vanish as a consequence of spin conservation.

The spin averaged collision probability is given by

$$W_N(\theta,\phi) = \frac{2\pi}{\hbar}\times\frac{1}{4}\times\frac{1}{2} \times \sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} |\langle\sigma_1\sigma_2|T|\sigma_3\sigma_4\rangle|^2 \tag{E4}$$

$$= \frac{2\pi}{\hbar}\frac{1}{8}(|T_s|^2+3|T_t|^2). \tag{E5}$$

In (E4) the factor of $\frac{1}{4}$ comes from averaging over initial spin states, and the factor of $\frac{1}{2}$ is to avoid double counting of final states.

For $\phi=0$ the scattering amplitude is given in terms of the Landau parameters F_l^s and F_l^a by

$$\nu(0)T_s(\theta,\phi=0) = \sum_l (A_l^s - 3A_l^a)P_l(\cos\theta) \tag{E6}$$

and

$$\nu(0)T_t(\theta,\phi=0) = \sum_l (A_l^s + A_l^a)P_l(\cos\theta), \tag{E7}$$

where $\nu(0)$ is the density of quasiparticle states for both spin directions at the Fermi surface and

$$A_l^{s,a} = \frac{F_l^{s,a}}{1+F_l^{s,a}/(2l+1)}. \tag{E8}$$

In the s- and p-wave approximation one assumes that the ϕ dependence is the simplest possible consistent with Fermi statistics, namely, that the singlet amplitude is only s wave, and hence independent of ϕ, and that the triplet amplitude is only p wave, and hence proportional to $\cos\phi$. If one further neglects Landau parameters with $l\geq 2$, about which there is rather little experimental information, one finds the following approximation for the scattering amplitude:

$$\nu(0)T_s = A_0^s - 3A_0^a + (A_1^s - 3A_1^a)\cos\theta \tag{E9}$$

and

$$\nu(0)T_t = [(A_0^s+A_0^a)+(A_1^s+A_1^a)\cos\theta]\cos\phi. \tag{E10}$$

If one inserts these expressions into the various averages required in calculating transport coefficients one finds

$$\nu(0)^2 W_N = (2\pi/\hbar)D/4, \tag{E11}$$

where

$$D = C_1^2 + \tfrac{7}{15}C_2^2 + \tfrac{3}{2}C_3^2 + \tfrac{7}{10}C_4^2 - \tfrac{2}{3}C_1C_2 - C_3C_4, \tag{E12}$$

with

$$C_1 = A_0^s - 3A_0^a,\quad C_2 = A_1^s - 3A_1^a,$$
$$C_3 = A_0^s + A_0^a,\quad C_4 = A_1^s + A_1^a. \tag{E13}$$

The characteristic relaxation time τ_0 [Eq. (B5)] [$=\frac{1}{2}\pi^2\tau(0)$, Eq. (4.10)] is

$$\tau_0^{-1} = \frac{\pi}{32}\frac{(k_BT)^2}{\hbar(p_F^2/2m^*)}D, \tag{E14}$$

and, from (B10), (E5), (E9), and (E10),

$$\alpha_\eta = 2\left(1 - \frac{\frac{4}{5}C_1^2 + \frac{52}{105}C_2^2 + \frac{3}{5}C_3^2 + \frac{13}{35}C_4^2 - \frac{8}{7}C_1C_2 - \frac{6}{7}C_3C_4}{D}\right). \tag{E15}$$

Finally, for completeness we give expressions for α_κ and α_D, the α parameters that enter the thermal conductivity and the spin diffusion coefficient:

$$\alpha_\kappa = \alpha_1(+1,-1) = (4/D)(\tfrac{1}{6}C_1^2 - \tfrac{1}{42}C_2^2 + \tfrac{1}{4}C_3^2 - \tfrac{1}{28}C_4^2 + \tfrac{3}{5}C_1C_2 + \tfrac{9}{10}C_3C_4), \tag{E16}$$

$$\alpha_D = \alpha_1(-1,+1) = 2[1 - (1/D)(\tfrac{4}{3}C_1^2 + \tfrac{76}{105}C_2^2 + \tfrac{2}{3}C_3^2 + \tfrac{38}{105}C_4^2 - \tfrac{8}{5}C_1C_2 - \tfrac{4}{3}C_1C_3 + \tfrac{4}{5}C_1C_4 + \tfrac{4}{5}C_2C_3 - \tfrac{76}{105}C_2C_4 - \tfrac{4}{5}C_3C_4)]. \tag{E17}$$

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