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Exact solutions of the elliptic sine equation in two space dimensions with application to the Josephson effect*

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Exact time-independent solutions of the elliptic sine equation $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = \sin\psi$ are derived with the help of a new Bäcklund transformation and the associated Bianchi diagrams. A formula is developed which enables us to generate *without additional quadratures* an infinite number of real two-space dimensional solutions α. We apply these solutions to the propagation of magnetic flux through a large two-dimensional Josephson tunneling junction and discuss briefly the experimental implications. We call the α solutions *soliton-like* solutions, since they can be shown to carry through the Josephson junction an integral number n of positive flux quanta: $\Phi_n^{(+)} = n\Phi_0$, where Φ_0 denotes a single quantum of superconducting magnetic flux. The multiple soliton-like solutions possess infinite total energy and can be labeled by a topological quantum number. The analysis is strictly classical.

I. INTRODUCTION

Among the nonlinear dispersive wave equations that have gained prominence in recent years is the so-called sine-Gordon equation is one space and one time dimension. The properties of this famous equation—with applications in solid state physics, nonlinear optics, and differential geometry—and its intimate connection with the charge-zero sector of the massive Thirring model,[1] have by now been well documented.[2-4] The most important property of the sine-Gordon system, no doubt, is its complete separability in the Hamiltonian sense[5] with the existence of an infinite number of exact classical solutions.

The question logically arises whether the remarkable features of the sine-Gordon equation also survive in higher dimensions. To obtain at least a partial answer to this question, we examine here the "sine-Gordon" system in *two space* dimensions. We derive, in particular, an infinite number of *exact* time-independent classical solutions of the nonlinear dispersive equation

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = \sin\psi \ , \tag{1.1}$$

henceforth called the *elliptic sine equation*, in which ψ is a massless scalar field and where x, y are space variables.

The structure of our paper is as follows. In the first part of Sec. II we obtain, with the help of a new Bäcklund transformation and its associated Bianchi diagrams, *exact* solutions of Eq. (1.1). In the second half we develop, among other things, a formula which allows us to generate without additional quadratures an infinite number of real solutions. The latter are called *soliton-like* solutions, instead of soliton solutions, since they carry through a two-dimensional Josephson junction an integral number n of positive flux quanta: $\Phi_n^{(+)}=n\Phi_0$, where Φ_0 is a single quantum of superconducting magnetic flux (cf. Sec. IV).

A detailed analysis of the double soliton-like solution is then given in Sec. III, where we also construct, in analogy with the one-space dimensional sine-Gordon system, the important topological invariants.

In Sec. IV, we apply our exact solutions to superconductivity, specifically to the propagation of magnetic flux through a large two-dimensional Josephson tunneling junction. The Josephson effect has been under constant investigation for many years by both theorists[6] and experimentalists,[7] and it is therefore somewhat surprising to find that *exact* solutions of the appropriate "sine-Gordon" system are, to the best of our knowledge, still only available in *one* space dimension. We hope that the two-space dimensional solutions presented here will not only lead to a richer understanding of the behavior of the Josephson tunneling current and its many ramifications, but will also yield new qualitative characteristics of junction barriers. The Josephson effect is studied here strictly from the point of view of the macroscopic theory of superconductivity, rather than in the context of the Bardeen-Cooper-Schrieffer theory. We conclude the paper with a brief summary and discussion in Sec. V.

II. BÄCKLUND TRANSFORMATION AND GENERATING FORMULAS

A. Bäcklund transformation for the elliptic sine equation

The purpose of this section is to derive in two space dimensions exact multiple soliton-like solutions of the time-independent elliptic sine equation

$$\nabla^2\psi = \sin\psi, \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \ , \tag{2.1}$$

where x, y are space variables and ψ is a massless scalar field. To obtain these solutions we employ the powerful method of Bäcklund transformations which were studied in the 1880's in connection with pseudospherical surfaces, i.e., surfaces of constant negative curvature.[8] A Bäcklund transformation, which is more general than a contact transformation, may be regarded geometrically as a transformation of a surface S into a new surface S', where S represents a solution of a given partial differential equation and where the transformed surface S' may either be a solution of the original partial differential equation or of some other differential equation.

It has been known for some time[9] that certain classes of partial differential equations do not admit a Bäcklund transformation, while for other classes the associated Bäcklund transformation—though known to exist on theoretical grounds—has simply not been discovered so far. The advantages of possessing such a transformation are tremendous, especially in the case of higher nonlinear partial differential equations, since (i) a Bäcklund transformation reduces, under favorable circumstances, the task of solving a higher-order *partial* differential equation to the problem of solving instead a system of first-order *ordinary* differential equations; and (ii) given one solution of a partial differential equation one can readily obtain a second solution by integrating the system of first-order ordinary differential equations in (i). Other advantages and properties of Bäcklund transformations will become apparent during the course of the discussion.

The Bäcklund transformation associated with the elliptic sine Eq. (2.1) reads[10]

$$(\partial_x + i\partial_y)\left(\frac{\alpha - i\beta}{2}\right) = \sin\left(\frac{\alpha + i\beta}{2}\right) e^{i\phi}, \quad i = \sqrt{-1}, \tag{2.2}$$

where ϕ is a real Bäcklund transformation parameter, α and $i\beta$, with β real, are solutions of Eq. (2.1) and $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$. It is convenient to represent the transformation from α to $i\beta$ by the Bianchi diagram shown in Fig. 1 and to write accordingly

$$i\beta = B_\phi \alpha, \tag{2.3}$$

where $i\beta$ is called the Bäcklund *transform*, while B_ϕ is the Bäcklund transformation operator characterized by ϕ. Since (2.2) may also be written as

$$(\partial_x + i\partial_y)\left(\frac{i\beta - \alpha}{2}\right) = \sin\left(\frac{i\beta + \alpha}{2}\right) e^{i(\phi \pm N\pi)}, \quad N = 1, 3, 5, \ldots, \tag{2.4}$$

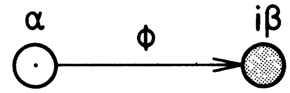

FIG. 1. The Bäcklund transformation from α to $i\beta$ is characterized by ϕ.

it follows from Fig. 2 and

$$\alpha = (B_\phi)^{-1} i\beta \tag{2.5}$$

that the *inverse* Bäcklund transformation operator $(B_\phi)^{-1}$ is given by

$$(B_\phi)^{-1} = B_{\phi \pm N\pi} = (-1)^N B_\phi, \quad N = 1, 3, 5, \ldots . \tag{2.6}$$

To derive nontrivial solutions for α and β we first replace Eq. (2.2) by the system

$$\tfrac{1}{2}(\partial_x \alpha + \partial_y \beta) = -\sin\phi \cos\frac{\alpha}{2} \sinh\frac{\beta}{2} + \cos\phi \sin\frac{\alpha}{2} \cosh\frac{\beta}{2}, \tag{2.7a}$$

$$\tfrac{1}{2}(\partial_y \alpha - \partial_x \beta) = \cos\phi \cos\frac{\alpha}{2} \sinh\frac{\beta}{2} + \sin\phi \sin\frac{\alpha}{2} \cosh\frac{\beta}{2}. \tag{2.7b}$$

Setting $i\beta = 0 \equiv i\beta_0$ in Eq. (2.7a), we obtain the α solution (for a *geometrical* representation, see Fig. 7)

$$\alpha_1(x, y, \phi) = 4\arctan[c \exp(x\cos\phi + y\sin\phi)], \quad c \text{ constant}, \tag{2.8}$$

which satisfies $\nabla^2\alpha_1 = \sin\alpha_1$. Similarly, setting $\alpha = 0 \equiv \alpha_0$ (vacuum solution) in Eq. (2.7a), we get the β solutions

$$\beta_1(x, y, \phi_1) = 4\,\text{arctanh}[\bar{c} \exp(x\cos\phi_1 + y\sin\phi_1)], \quad \bar{c} \text{ constant}, \quad \text{if } (x\cos\phi_1 + y\sin\phi_1) \le 0; \tag{2.9a}$$

or

$$\beta_1(x, y, \phi_1) = 4\,\text{arccoth}[\bar{c}' \exp(x\cos\phi_1 + y\sin\phi_1)], \quad \bar{c}' \text{ constant}, \quad \text{if } (x\cos\phi_1 + y\sin\phi_1) > 0, \tag{2.9b}$$

where $\nabla^2 i\beta_1 = \sin(i\beta_1)$. We observe that $\tanh(\beta_1/4)$ satisfies a *linear* equation,

$$\nabla^2 \tanh(\tfrac{1}{4}\beta_1) = \tanh(\tfrac{1}{4}\beta_1), \quad |\tanh(\tfrac{1}{4}\beta_1)| < 1 \tag{2.9c}$$

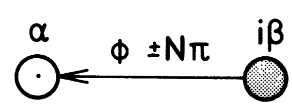

FIG. 2. The inverse Bäcklund transformation $\alpha = (B_\phi)^{-1} i\beta$.

whereas the general equation for either $u = \tanh(\frac{1}{4}\beta_{2n-1})$ or for $u=\coth(\frac{1}{4}\beta_{2n-1})$, $n=1,2,3,\ldots$ reads

$$\nabla^2 u = \frac{u(1+u^2) - 2u[(\partial_x u)^2 + (\partial_y u)^2]}{1-u^2}, \quad |u| \neq 1. \tag{2.9d}$$

Equations (2.9c) and (2.9d) are useful in discussing the analytic structure of the β solutions.

B. Generating formula

We next derive a generating formula for multiple soliton-like solutions of the elliptic sine equation (2.1).

Theorem: Suppose we are given three different solutions of Eq. (2.1), namely α_0, $i\beta_1^{(1)}$, and $i\beta_1^{(2)}$, which are related through $i\beta_1^{(1)} = B_{\phi_1}\alpha_0$ and $i\beta_1^{(2)} = B_{\phi_2}\alpha_0$. A new α solution, labeled α_2, is then given by

$$\tan\left(\frac{\alpha_2-\alpha_0}{4}\right) = \cot\left(\frac{\phi_1-\phi_2}{2}\right) \times \tanh\left(\frac{\beta_1^{(1)}-\beta_1^{(2)}}{4}\right), \quad \phi_2 \neq \phi_1 \pm j\pi, \quad j=0,1,2,\ldots$$

where $\nabla^2\alpha_2 = \sin\alpha_2$ and ϕ_1, ϕ_2 are Bäcklund transformation parameters. The superscript on β_1 indicates its dependence on ϕ_i, $i=1,2$.

Proof: The Bianchi diagram depicted in Fig. 3 represents the transformations

$$i\beta_1^{(1)} = B_{\phi_1}\alpha_0, \tag{2.10a}$$

$$i\beta_1^{(2)} = B_{\phi_2}\alpha_0, \tag{2.10b}$$

$$\alpha_2 = B_{\phi_2}(i\beta_1^{(1)}), \tag{2.10c}$$

$$\alpha_2 = B_{\phi_1}(i\beta_1^{(2)}), \tag{2.10d}$$

which correspond, in turn, to the following system of first-order differential equations:

$$(\partial_x + i\partial_y)\left(\frac{\alpha_0 - i\beta_1^{(1)}}{2}\right) = \sin\left(\frac{\alpha_0 + i\beta_1^{(1)}}{2}\right)\exp(i\phi_1), \tag{2.11a}$$

$$(\partial_x + i\partial_y)\left(\frac{\alpha_0 - i\beta_1^{(2)}}{2}\right) = \sin\left(\frac{\alpha_0 + i\beta_1^{(2)}}{2}\right)\exp(i\phi_2), \tag{2.11b}$$

$$(\partial_x + i\partial_y)\left(\frac{i\beta_1^{(1)} - \alpha_2}{2}\right) = \sin\left(\frac{\alpha_2 + i\beta_1^{(1)}}{2}\right)\exp(i\phi_2), \tag{2.11c}$$

$$(\partial_x + i\partial_y)\left(\frac{i\beta_1^{(2)} - \alpha_2}{2}\right) = \sin\left(\frac{\alpha_2 + i\beta_1^{(2)}}{2}\right)\exp(i\phi_1). \tag{2.11d}$$

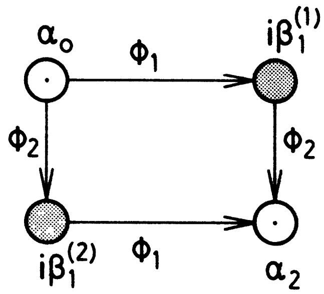

FIG. 3. Bianchi diagram used in the proof of the generating formula (2.12).

Adding Eqs. (2.11) according to the combination [(2.11a) − (2.11b) + (2.11c) − (2.11d)] and then performing some elementary manipulations,[11] we get

$$\exp(i\phi_1)\sin\left[\frac{(\alpha_2-\alpha_0) + (i\beta_1^{(2)} - i\beta_1^{(1)})}{4}\right] = \exp(i\phi_2)\sin\left[\frac{(\alpha_2-\alpha_0) - (i\beta_1^{(2)} - i\beta_1^{(1)})}{4}\right],$$

provided $\alpha_2 + \alpha_0 + i\beta_1^{(1)} + i\beta_1^{(2)} \neq 2\pi m$, $m = \pm1, \pm3, \pm5, \ldots$. The last equation reduces to

$$[\exp(i\phi_1) - \exp(i\phi_2)]\sin[\tfrac{1}{4}(\alpha_2-\alpha_0)]\cos[\tfrac{1}{4}(i\beta_1^{(1)} - i\beta_1^{(2)})] = [\exp(i\phi_1) + \exp(i\phi_2)] \times \sin[\tfrac{1}{4}(i\beta_1^{(1)} - i\beta_1^{(2)})]\cos[\tfrac{1}{4}(\alpha_2-\alpha_0)],$$

so that

$$\tan\left(\frac{\alpha_2-\alpha_0}{4}\right) = \cot\left(\frac{\phi_1-\phi_2}{2}\right)\tanh\left(\frac{\beta_1^{(1)}-\beta_1^{(2)}}{4}\right), \quad \phi_2 \neq \phi_1 \pm j\pi, \quad j=0,1,2,\ldots \tag{2.12}$$

which is the desired result.

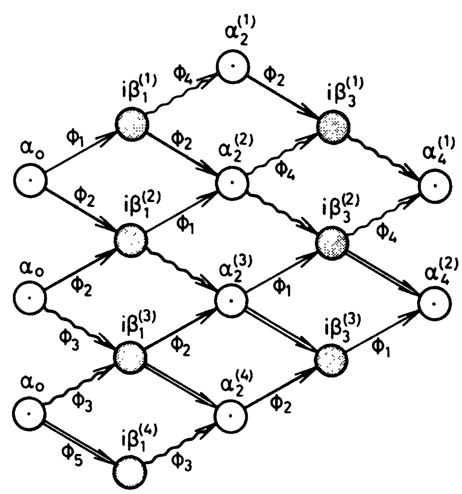

FIG. 4. Extended Bianchi diagram showing the generation of real α solutions [cf. Eqs. (2.12) and (2.14)] and purely imaginary β solutions [cf. Eq. (2.13)].

The analogous expression for a new β solution is

$$\tanh\left(\frac{\beta_3-\beta_1}{4}\right)=\cot\left(\frac{\phi_1-\phi_3}{2}\right)\tan\left(\frac{\alpha_2(\phi_3)-\alpha_2(\phi_1)}{4}\right)$$

$$\phi_3\neq\phi_1\pm j\pi,\quad j=0,1,2,\ldots\,. \qquad (2.13)$$

C. Multiple soliton-like solutions

Repeated application either of formula (2.12) or of (2.13) enables us to generate *without additional quadratures* other solutions of Eq. (2.1), as symbolized by the extended Bianchi diagram Fig. 4. The generalization of (2.12) to α_{2n}, for example, yields

$$\tan\left(\frac{\alpha_{2n}-\alpha_{2n-2}}{4}\right)=\cot\left(\frac{\phi_{2n-1}-\phi_{2n}}{2}\right)\times\tanh\left(\frac{\beta_{2n-1}^{(1)}-\beta_{2n-1}^{(2)}}{4}\right),$$

$$n=1,2,3,\ldots,\quad \phi_{2n}\neq\phi_{2n-1}\pm j\pi,\quad j=0,1,2,\ldots, \qquad (2.14)$$

where the two superscripts on β_{2n-1} refer to two *different* ϕ's (see Fig. 4 again). There exists a similar formula for β_{2n-1}, in terms of β_{2n-3} and α_{2n-2}, which will not be stated explicitly.

The technique of employing a Bäcklund transformation to generate other solutions is, of course, familiar from the treatment of the one-space dimensional sine-Gordon equation. The fundamental difference between the Bäcklund system for that equation and the Bäcklund transformation for the elliptic sine equation is that Eq. (2.2) connects two *different* sets of solutions: one set, α, being real, the other one, β, purely imaginary.

D. Commutativity

It is known[8] that successive Bäcklund transformations associated with the sine-Gordon equation in one space and one time dimension, commute. This commutative property is also shared by the Bäcklund transformations of the elliptic sine Eq. (2.1). [According to Eqs. (2.10c) and (2.10a), $\alpha_2=B_{\phi_2}B_{\phi_1}\alpha_0$. Similarly, it follows from Eqs. (2.10d) and (2.10b), as well as from $\alpha_0\to i\beta_1^{(2)}\to\alpha_2$, that $\alpha_2=B_{\phi_1}B_{\phi_2}\alpha_0$.]

III. DOUBLE SOLITON-LIKE SOLUTION

A. The solution α_2

We first show that α_2, as given in (2.12), is indeed a solution of $\nabla^2\alpha_2=\sin\alpha_2$. Let α_0 be the "vacuum" solution $\alpha_0\equiv 0$ for which the Bäcklund-generated solutions $\beta_1^{(1)}$ and $\beta_1^{(2)}$ are known to read [we use (2.9a)]

$$\beta_1^{(1)}(x,y,\phi_1)=4\,\mathrm{arctanh}[\bar{c}_1\exp(x\cos\phi_1+y\sin\phi_1)] \qquad (3.1a)$$

$$\beta_1^{(2)}(x,y,\phi_2)=4\,\mathrm{arctanh}[\bar{c}_2\exp(x\cos\phi_2+y\sin\phi_2)]\,. \qquad (3.1b)$$

Substituting the right-hand side of Eqs. (3.1) into

$$\alpha_2(x,y,\phi_1,\phi_2)=4\arctan\left[\cot\left(\frac{\phi_1-\phi_2}{2}\right)\tanh\left(\frac{\beta_1^{(1)}-\beta_1^{(2)}}{4}\right)\right], \qquad (3.2)$$

and calculating $\partial^2\alpha_2/\partial x^2$ and $\partial^2\alpha_2/\partial y^2$, we find, upon rewriting $\sin\alpha_2$ as

$$\sin\alpha_2=\frac{4\tan(\frac{1}{4}\alpha_2)[1-\tan^2(\frac{1}{4}\alpha_2)]}{[1+\tan^2(\frac{1}{4}\alpha_2)]^2}, \qquad (3.3)$$

that α_2 does indeed satisfy Eq. (2.1).

B. Asymptotic behavior

We begin with α_1 in Eq. (2.8) which, in polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$, reads

$$\alpha_1(r,\theta,\phi)=4\arctan\{c\exp[r\cos(\theta-\phi)]\},\quad c>0, \qquad (3.4a)$$

so that

$$\lim_{r\to\infty}\alpha_1(r,\theta)=2\pi,\quad \text{if } -\tfrac{1}{2}\pi<\theta-\phi<\tfrac{1}{2}\pi, \qquad (3.4b)$$

$$\lim_{r\to\infty}\alpha_1(r,\theta)=0,\quad \text{if } \tfrac{1}{2}\pi<\theta-\phi<\tfrac{3}{2}\pi. \qquad (3.4c)$$

In Fig. 7 we have attempted to draw a two-dimensional picture of α_1. A similar, though somewhat lengthier, analysis of the double soliton-like solution

$$\alpha_2(r,\theta,\phi_1,\phi_2)=4\arctan\left[\cot\left(\frac{\phi_1-\phi_2}{2}\right)\times\left(\frac{\bar{c}_1e^{r\cos(\theta-\phi_1)}-\bar{c}_2e^{r\cos(\theta-\phi_2)}}{1-\bar{c}_1\bar{c}_2e^{r\cos(\theta-\phi_1)+r\cos(\theta-\phi_2)}}\right)\right]$$

$$\phi_2\neq\phi_1\pm j\pi,\ j=0,1,2,\ldots, \qquad (3.5)$$

leads for $\phi_1=\pi/3$ and $\phi_2=0$, say, to the following asymptotic behavior ($\bar{c}_1=\bar{c}_2=1$):

(i) for $\pi+\phi_1<\theta<\frac{3}{2}\pi+\phi_1$ and $\phi_1<\theta<\frac{1}{2}\pi+\phi_1$,

$$\lim_{r\to\infty}\alpha_2(r,\theta)=0;\quad \theta\neq\tfrac{2}{3}\pi+n\pi,\ n=0,1,\ldots \qquad (3.6)$$

(ii) for $\frac{3}{2}\pi+\phi_1<\theta<2\pi+\phi_1$ and $\frac{1}{2}\pi+\phi_1<\theta<\pi+\phi_1$,

$$\lim_{r\to\infty}\alpha_2(r,\theta)=-4\pi;\quad \theta\neq\tfrac{2}{3}\pi+n\pi,\ n=0,1,\ldots\,. \qquad (3.7)$$

From (3.6) and (3.7) it is possible now to construct

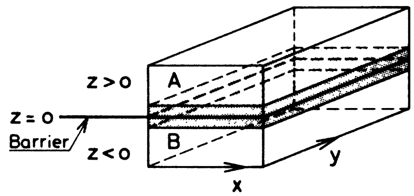

FIG. 5. Two-dimensional Josephson junction consisting of metals A, B, and a barrier of thickness b.

topological invariants, such as

$$\lim_{r\to\infty}[\alpha_2(r,\theta_1)-\alpha_2(r,\theta_0)]=+4\pi\,, \tag{3.8}$$

with

$$\theta \neq \tfrac{2}{3}\pi + n\pi,\quad n=0,1,2,\ldots$$

$$\pi+\phi_1<\theta_1<\tfrac{3}{2}\pi+\phi_1,\ \text{and}\ \tfrac{3}{2}\pi+\phi_1<\theta_0<2\pi+\phi_1\,,$$

which are reminiscent of the 4π pulse in nonlinear optics[12] and of the two-soliton solution of the sine-Gordon equation in one space dimension.[4] There is of course an essential difference between solutions in one and two space dimensions. Whereas the N-soliton solutions, $N=1,2,3,\ldots$, in one space dimension have a total energy which is *finite*, the total energy of our static solutions, such as (2.14), is *infinite*. For α_{2n} the total energy is given by

$$E(\alpha_{2n})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\,dy\{1-\cos(\alpha_{2n}) + \tfrac{1}{2}[(\partial_x\alpha_{2n})^2+(\partial_y\alpha_{2n})^2]\}\,. \tag{3.9}$$

So far we have verified $E(\alpha_{2n})\to\infty$ only for $n=1$ and 2, but it appears certain that $E(\alpha_{2n})\to\infty$ also for $n>2$. There exists a remote possibility that one of the higher α solutions, say α_6, possesses a total energy which, for a judicious choice of the parameters $\phi_1,\ldots,\phi_6$, is finite. But we have not been able to find such a magic combination of ϕ's. We are therefore inclined to believe that the α solutions generated from (2.14) possess infinite total energy, in agreement with Derrick's theorem.[13] A similar statement holds for the β solutions.

IV. PROPAGATION OF MAGNETIC FLUX THROUGH A TWO-DIMENSIONAL JOSEPHSON JUNCTION

A. Phenomenological description of junction barriers

In Sec. II we obtained exact time-independent solutions of the elliptic sine equation in two space dimensions. The purpose of this section is to apply these static solutions, such as (2.8) and (3.5), to the propagation of magnetic flux through a large two-dimensional Josephson tunneling junction. For the convenience of those readers who are not too familiar with Josephson's theory, we have taken the liberty of summarizing briefly those features of the Josephson effect which are particularly pertinent to the present investigation.[14]

A Josephson tunneling junction consists of two superconducting metals A and B which are separated by a very thin nonsuperconducting barrier of thickness b (Fig. 5). The barrier lies strictly in the xy plane, metal A goes from $z=\frac{1}{2}b$ to $z=+\infty$ and metal B from $z=-\frac{1}{2}b$ to $z=-\infty$.[15] The superconductors A, B may be described by wave functions (order parameters) of the form[3]

$$\psi_A(x,y,t)=(\rho_A)^{1/2}\exp[i\varphi_A(x,y,t)]\,,$$

$$\psi_B(x,y,t)=(\rho_B)^{1/2}\exp[i\varphi_B(x,y,t)]\,,$$

where φ_A, φ_B are the phases of metals A, B, while ρ_A, ρ_B are proportional to their electronic charge densities; ρ_A, ρ_B and φ_A, φ_B are both purely real and ρ_A, ρ_B are in addition positive definite.

The most important quantity in the macroscopic description of the Josephson effect is the phase *difference* $\varphi=\varphi_A-\varphi_B$ between metals A, B which causes a supercurrent, the Josephson tunneling current, to flow across the barrier. For a satisfactory description of the Josephson effect one requires, in addition to φ, the magnetic induction $\vec{B}$ or rather, the corresponding magnetic field strength $\vec{H}$, and the potential difference $V(x,y,t)$ across the barrier. We assume that $\vec{H}$ is constant in time and lies entirely in the xy plane:

$$\vec{H}(x,y)=\vec{i}\,H_x(x,y)+\vec{j}\,H_y(x,y)\,.$$

The three basic equations of the Josephson effect then read[16]

$$\partial\varphi/\partial t=2eV/\hbar\,, \tag{4.1}$$

which describes the variation of the relative phase φ with time t;

$$\frac{\partial\varphi}{\partial x}=\frac{2ed}{\hbar c}H_y\,,\quad \frac{\partial\varphi}{\partial y}=-\frac{2ed}{\hbar c}H_x \tag{4.2a}$$

or, equivalently,

$$\vec{\nabla}\varphi=\frac{2ed}{\hbar c}\vec{H}\times\vec{n}\,,\quad \vec{n}=(0,0,n_z)\,, \tag{4.2b}$$

which describes the space variation of φ; and finally,

$$j_z(x,y,t)=j_0(x,y)\sin\varphi(x,y,t)\,, \tag{4.3}$$

which gives the Josephson current per unit area across the barrier.[3] Here e is the electron charge, c the speed of light *in vacuo*, and $d=\lambda_A+\lambda_B+b$, where λ_A, λ_B denote, respectively, the London penetration depths for metals A, B; j_0 depends only on position and on the properties of the barrier. Equations (4.1), (4.2), and (4.3) give a satisfactory phenomenological description of junction barriers.

Substitution of Eqs. (4.1)–(4.3) into Maxwell's equation

$$\frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x - \frac{4\pi C}{c}\frac{\partial V}{\partial t} = \frac{4\pi}{c} j_z \tag{4.4}$$

yields the barrier equation

$$\nabla^2\varphi - \frac{\partial^2\varphi}{c_0{}^2\partial t^2} = \lambda_J^{-2}\sin\varphi , \tag{4.5}$$

where c_0 and λ_J, the two fundamental parameters of the theory, are given by

$$c_0 = c(4\pi C d)^{-1/2} , \tag{4.6}$$

$$\lambda_J = (\hbar/2edj_0)^{1/2} ; \tag{4.7}$$

c_0 is the speed of electromagnetic waves along the surface of the barrier if the Josephson current is absent, while λ_J is the Josephson penetration depth which, in general, is *not* a constant. C is the capacitance of the barrier per unit area.[16]

Equation (4.5) may be simplified by measuring *distance* in units of λ_J,

$$\overline{x} \equiv x/\lambda_J , \quad \overline{y} \equiv y/\lambda_J , \tag{4.8}$$

and *time* in units of

$$\tau_J = (\hbar C/2ej_0)^{1/2} , \overline{t} \equiv t/\lambda_J , \tag{4.9}$$

so that Eq. (4.5) reduces to

$$\left(\frac{\partial^2}{\partial \overline{x}^2} + \frac{\partial^2}{\partial \overline{y}^2} - \frac{\partial^2}{\partial \overline{t}^2}\right)\varphi = \sin\varphi , \tag{4.10}$$

or, in the *static* case, to

$$\left(\frac{\partial^2}{\partial \overline{x}^2} + \frac{\partial^2}{\partial \overline{y}^2}\right)\varphi = \sin\varphi . \tag{4.11}$$

Since Eq. (4.11) is just the elliptic Eq. (2.1), we know that it admits exact solutions of the form (2.8), (2.9), (2.14), etc. Before discussing some of these solutions in greater detail, we address ourselves to the question of magnetic flux.

B. Magnetic flux through a Josephson junction

The total magnetic flux through an area S is defined by

$$\Phi = \int\!\!\int_S \vec{B}\cdot d\vec{S} , \tag{4.12}$$

where S corresponds, in our case, to the area encircled by the curve $\gamma = \gamma_+ + \gamma_-$ as shown in Fig. 6.[16] Employing Stokes' theorem and utiliz-

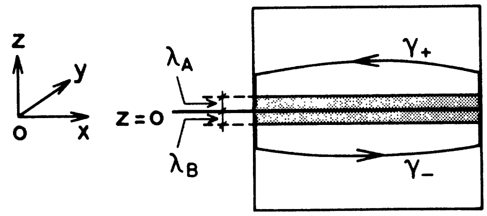

FIG. 6. Diagram shows area S encircled by curve $\gamma = \gamma_+ + \gamma_-$ through which flux Φ passes.

ing Eq. (4.2b), we can write the flux through γ as

$$\Phi_\gamma = (2\pi)^{-1}\Phi_0 \oint_\gamma d\vec{l}\cdot\vec{\nabla}\varphi , \quad d\vec{l} = \vec{i}\,dx + \vec{j}\,dy , \tag{4.13}$$

where

$$\Phi_0 = hc/2e = 2\times 10^{-7}\,\mathrm{G\,cm}^2 \tag{4.14}$$

is a single quantum of superconducting magnetic flux. The line integral in Eq. (4.13) is readily evaluated, since the value of the integral depends merely on its endpoints (we drop the subscript on Φ):

$$\Phi = (2\pi)^{-1}\Phi_0 \int_{P_0}^{P_1} d\varphi = (2\pi)^{-1}\Phi_0[\varphi(P_1) - \varphi(P_0)] . \tag{4.15}$$

If the φ's in Eq. (4.15) are static solutions, as they are in our case, the corresponding flux Φ should, strictly speaking, be referred to as *static* magnetic flux.

C. Flux quantization and topological quantum numbers

The condition of flux quantization in superconductivity reads

$$\Phi = N\Phi_0 , \tag{4.16}$$

where N, an integer, is sometimes called the winding number and where Φ is the total flux through the junction. Whether or not condition (4.16) can also be satisfied in "other situations" depends decisively on the existence of a *topological quantum number*[17] Q defined, in one space dimension, as

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx \frac{\partial\varphi}{\partial x} . \tag{4.17}$$

If, for example, $\varphi(x,t)$ is a finite-energy solution of the (1+1)-dimensional sine-Gordon equation, then

$$\varphi(\infty, t) - \varphi(-\infty, t) = 2\pi m , \quad m \text{ integer}, \tag{4.18}$$

and condition (4.16) is satisfied.

We show in the remainder of this section that condition (4.16) also holds for the static two-space dimensional solutions of Secs. II and III. To derive this result it is essential to verify—for the expression $[\varphi(P_1)-\varphi(P_0)]$ in Eq. (4.15)—the asymptotic relation

$$\lim_{r\to\infty}[\varphi(r,\theta_1)-\varphi(r,\theta_0)]=2\pi N,\quad N \text{ integer}, \tag{4.19}$$

where $P_0\equiv(r,\theta_0)$ and $P_1\equiv(r,\theta_1)$.

D. Total positive flux of φ solutions

The symbol φ shall henceforth denote an α solution as given, for example, in Eqs. (2.8), (2.12), or (2.14). Thus the single soliton-like solution of Eq. (4.11), which is depicted in Fig. 7 and which reads

$$\varphi_1(r,\theta,\phi)=4\arctan\{\exp[(r/\lambda_J)\cos(\theta-\phi)]\}, \tag{4.20}$$

satisfies the asymptotic conditions [cf. (3.4)]

$$\lim_{r\to\infty}\varphi_1(r,\theta)=2\pi,\quad \text{if } -\tfrac{1}{2}\pi<\theta-\phi<\tfrac{1}{2}\pi, \tag{4.21}$$

$$\lim_{r\to\infty}\varphi_1(r,\theta)=0,\quad \text{if } \tfrac{1}{2}\pi<\theta-\phi<\tfrac{3}{2}\pi, \tag{4.22}$$

where ϕ still denotes the Bäcklund parameter. It is convenient[10] to consider only the amount of *positive* flux that is being carried by $\varphi_1(r,\theta)$ through the Josephson junction. Accordingly we choose—for the single soliton-like solution—the angles of $P_1(r,\theta_1)$ and $P_0(r,\theta_0)$ in such a way that $\theta_0=\theta_1-\pi$ and $-\frac{1}{2}\pi<\theta_1-\phi<\frac{1}{2}\pi$, so that formula (4.15) becomes

$$\Phi=(2\pi)^{-1}\Phi_0[\varphi(r,\theta_1)-\varphi(r,\theta_1-\pi)]\equiv\Phi_1,$$

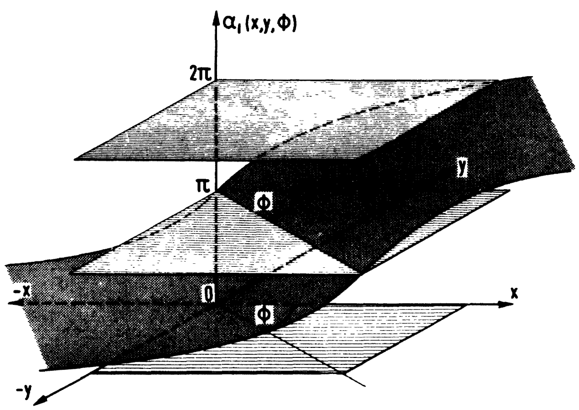

FIG. 7. The single solition-like solution $\alpha_1(x,y,\phi)$ in two space dimensions. The continuous Bäcklund transformation parameter ϕ gives the orientation in space. Notice the asymptotic behavior of α_1 as described by Eqs. (3.4).

or

$$\Phi_1=\frac{2}{\pi}\Phi_0\left\{\arctan\left[\exp\left(\frac{r}{\lambda_J}\cos(\theta_1-\phi)\right)\right]-\arctan\left[\exp\left(-\frac{r}{\lambda_J}\cos(\theta_1-\phi)\right)\right]\right\}. \tag{4.23}$$

As $r\to\infty$, the total amount of *positive* static flux through the junction is seen to be exactly one flux quantum:

$$\Phi_1^{(+)}=+\Phi_0,\quad \cos(\theta_1-\phi)>0. \tag{4.24}$$

To complete the picture we should point out that the total *negative* flux is

$$\Phi_1^{(-)}=-\Phi_0,\quad \cos(\theta_1-\phi)<0. \tag{4.25}$$

We have also examined, with the help of Eqs. (3.5)–(3.8), the flux carried by the double soliton-like solution of (4.11),

$$\varphi_2(r,\theta)=4\arctan\left[\cot\left(\frac{\phi_1-\phi_2}{2}\right)\times\left(\frac{e^{(r/\lambda_J)\cos(\theta-\phi_1)}-e^{(r/\lambda_J)\cos(\theta-\phi_2)}}{1-e^{(r/\lambda_J)[\cos(\theta-\phi_1)+\cos(\theta-\phi_2)]}}\right)\right]. \tag{4.26}$$

The total positive flux for the regions specified by $r\to\infty$, $\frac{1}{2}\pi+\phi_1<\theta<\pi+\phi_1$ and $3\pi/2+\phi_1<\theta<2\pi+\phi_1$, is again of the form (4.16), namely[18]

$$\Phi_2^{(+)}=2\Phi_0,\ \tan\theta\neq-[(1+\cos\phi_1)/\sin\phi_1]. \tag{4.27}$$

The other solutions of (4.11) such as φ_4, φ_6, etc., may be analyzed in much the same way.

E. Experimental considerations

Although the solutions $\varphi_{2n}(x,y)$ of Eq. (4.11) possess infinite total energy, they may provide additional information about the qualitative features of two-dimensional junction barriers. The reason is that several quantities in Josephson's tunneling theory depend through φ_{2n} in a nontrivial way on both x and y. Among these quantities are:

(i) the Josephson current [cf. Eq. (4.3)]

$$j_z(x,y)=j_0(x,y)\sin\varphi_{2n}(x,y),$$

j_0 depending only on the properties of the barrier;

(ii) the total current I through the junction,

$$I=\int d\varphi_{2n}j_0(x,y)\sin\varphi_{2n}(x,y),$$

where the integration is taken over the area of the junction[15];

(iii) the component H_x of the magnetic field strength which, according to Eq. (4.2a), is given by

$$H_x(x, y) = (-\hbar c/2ed)\partial\varphi_{2n}/\partial y\ ,$$

with a similar expression for H_y; and

(iv) the Josephson penetration depth

$$\lambda_J(x, y) = [\hbar/2edj_0(x, y)]^{1/2}$$

which, as we know, is not a constant in general. There are no doubt other features of junction barriers, such as the current-voltage characteristics of the Josephson current j_z which could also be studied with the φ_{2n}'s.

V. CONCLUDING REMARKS

We have studied the elliptic sine equation in two-space dimensions and derived for it an infinite number of exact time-independent classical solutions. The principal tools in the derivation were (i) a new Bäcklund transformation, (ii) its symbolic representation in terms of Bianchi diagrams, and (iii) two formulas, one generating an infinite number of real solutions α [Eq. (2.14)], the other and infinite number of purely imaginary solutions β [cf. Eq. (2.13)].

It was found, in analogy with the one-space dimensional sine-Gordon equation, that both types of solutions α, β can be labeled by topological quantum numbers. This was demonstrated in the text for the single and double soliton-like solutions α_1 and α_2, respectively. Although both α and β are mandatory for a self-consistent treatment of Eq. (2.1), only the *real* α solutions, labeled φ_{2n} in Sec. IV, are likely to be of any physical significance.

The two-dimensional static solutions derived here possess infinite total energy,[19] a somewhat discouraging result, but one which is nevertheless in agreement with Derrick's theorem.[13] A possible way out of this "energy dilemma" would be to solve, instead of (2.1), the three-dimensional equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{c^2\partial t^2} = \sin\psi\ . \qquad (5.1)$$

Since Derrick's theorem says nothing about time-*dependent* solutions,[20] it is at least within the realm of possibility that the total energy of the time-dependent solutions is *finite*. This problem is currently under investigation. In Sec. IV we applied the real φ_{2n} solutions to superconductivity and showed that the total positive static magnetic flux through a Josephson barrier is precisely

$\Phi_1^{(+)} = +\Phi_0$ for the single soliton-like solution,

and

$\Phi_2^{(+)} = +2\Phi_0$ for the double soliton-like solution.

Although the φ_{2n}'s were seen to possess infinite total energy, they can be expected to provide further information about certain quantities which are essential for a phenomenological description of junction barriers, such as the total current I through the junction, the Josephson penetration depth $\lambda_J(x, y)$, or the current-voltage characteristics of the Josephson current j_z.

Finally, it is reassuring to know that the elliptic sine equation (2.1) possesses not only an infinite number of conserved *local* currents,[10] as does the sine-Gordon equation in one space dimension, but in addition an infinite number of *nonlocal* conserved currents which will be reported elsewhere.

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[18]Notice that this flux is calculated relative to a *half* plane. In the other half plane the flux is of course *negative*.

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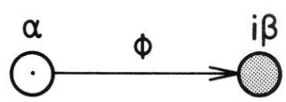

FIG. 1. The Bäcklund transformation from α to $i\beta$ is characterized by ϕ.

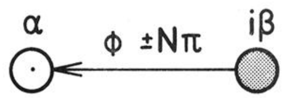

FIG. 2. The inverse Bäcklund transformation $\alpha = (B_\phi)^{-1} i\beta$.

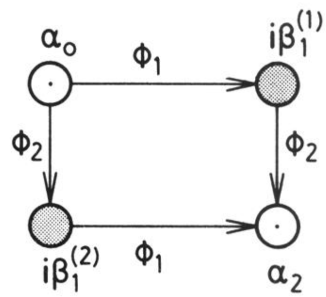

FIG. 3. Bianchi diagram used in the proof of the generating formula (2.12).

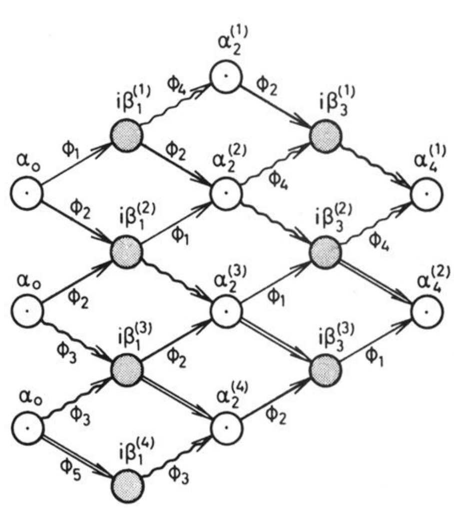

FIG. 4. Extended Bianchi diagram showing the generation of real α solutions [cf. Eqs. (2.12) and (2.14)] and purely imaginary β solutions [cf. Eq. (2.13)].

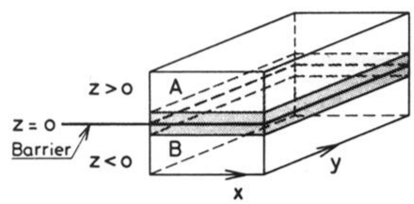

FIG. 5. Two-dimensional Josephson junction consisting of metals A, B, and a barrier of thickness b.

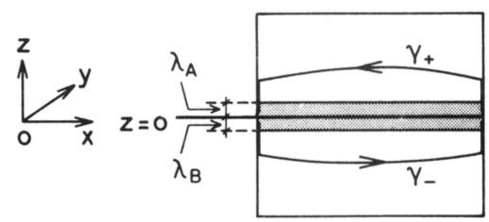

FIG. 6. Diagram shows area S encircled by curve $\gamma = \gamma_+ + \gamma_-$ through which flux Φ passes.

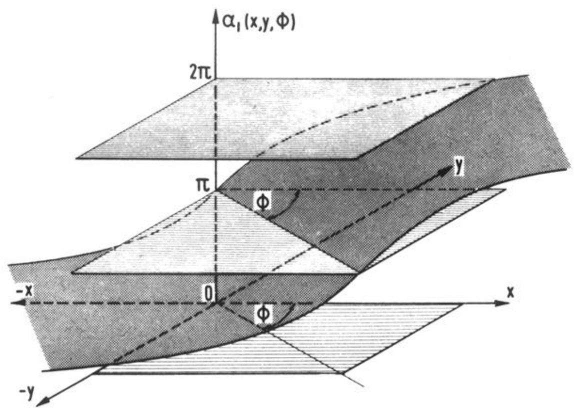

FIG. 7. The single solition-like solution $\alpha_1(x, y, \phi)$ in two space dimensions. The continuous Bäcklund transformation parameter ϕ gives the orientation in space. Notice the asymptotic behavior of α_1 as described by Eqs. (3.4).