Scaling theory of phase transitions in diluted systems near the percolation threshold*

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A heuristic picture, due to de Gennes and to Skal and Shklovskii, of a diluted lattice is used to introduce a one-dimensional path length l which diverges more rapidly than the percolation correlation length ξ_p at the percolation threshold. It is argued that thermodynamic functions should be scaling functions of $\xi_1(T)/l$, where $\xi_1(T)$ is the correlation length of a *one*-dimensional spin system. The implications of this scaling ansatz are discussed.

Consider a magnetic system (e.g., Rb_2MnF_4) randomly diluted with nonmagnetic impurities (e.g., Mg). Let p be the probability that a given site is occupied with a magnetic ion and $1-p$ be the probability that it is occupied with a nonmagnetic ion. In a quenched system, the transition temperature $T_c(p)$ decreases as p is decreased below 1 as shown in Fig. 1. Eventually, $T_c(p)$ decreases to zero at a critical value p_c of p. p_c is the percolation probability (or concentration). For $p<p_c$, there is no infinite cluster and no phase transition. For $p>p_c$, there is an infinite cluster and $T_c(p)$ rises from zero as depicted in Fig. 1. The magnetic phase transition for $T_c(p)>0$ has recently received considerable theoretical attention[1] and is understood at least qualitatively. Similarly the purely statistical percolation transition[2] can be described in the language of phase transitions[3,4] and studied by the usual techniques used to study phase transitions.[2,5] The magnetic phase transition in vicinity of $p=p_c$ and $T=0$ is, however, less-well understood and is the subject of this paper.

Stauffer has argued[6] that the point $p=p_c$, $T=0$ should be viewed as a type of multicritical point and that the scaling fields for the Ising model should be $\mu_2=p-p_c$ and $\mu_1=e^{-2J/T}$, where J is the magnetic exchange energy. He then proposes that the free energy should be a function of the scaled variable $(\tilde{\mu}_2/\tilde{\mu}_1{}^{1/\varphi})$, where $\tilde{\mu}_2$ and $\tilde{\mu}_1$ are linear combinations of μ_1 and μ_2 and where $\varphi \geq 1$. This scaling form is in agreement with exact calculations of the susceptibility on a random Cayley tree.[7] In this paper, we will present a heuristic picture of the phase transition near $p=p_c$, $T=0$ that will enable us to introduce a scaling function valid for n-component (not just Ising) spin systems.

We begin by considering a picture of the randomly diluted lattice proposed by de Gennes[8] and by Skal and Shklovskii[9] following earlier work on elasticity of gels.[10] This picture emphasizes the existence of two divergent correlation lengths. For p just above p_c, the lattice can be viewed as a collection of nodes (sites with three or more independent paths leading to infinity) connected by links that can be thought of as random zig-zag paths as shown in Fig. 2. (For details see Ref. 8.) The distance between nodes is the percolation correlation length $\xi_p \sim (p-p_c)^{-\nu_p}$. The zig-zag path length l between the nodes is proportional to the number of steps in the segment connecting the two nodes. For $p<p_c$, l is a measure of the number of steps in an average path traversing an average cluster. We introduce a new exponent ζ to describe the divergence of l near p_c:

$$l \sim |p-p_c|^{-\zeta} . \tag{1}$$

From Fig. 2, one can immediately deduce reasonable bounds for l. One would certainly expect l to be at least as large as ξ_p. In addition, a reasonable guess for l in general is that it be proportional to l_s, the number of steps in a self-avoiding walk between nodes. This, however, ignores the existence of paths between other nodes which have to be avoided. Thus, in general, l should be no larger than l_s. In other words,

$$\xi_p \leq l \leq \xi_p^{1/\nu_s} , \tag{2}$$

where ν_s is the correlation length exponent for the self-avoiding walk. This yields the following relation for ζ:

$$\nu_p \leq \zeta \leq \nu_p/\nu_s . \tag{3}$$

Above $d=d_c=6$, the critical dimensionality for the percolation problem, we expect $\zeta=2\nu_p=1$. Below six dimensions, the value of ζ is uncertain.

The crucial argument that produces scaling functions for the phase transition at $p=p_c$, $T=0$ is that on a length scale small compared to l, the lattice appears to be a collection of noninteracting contorted one-dimensional chains. On a scale large compare to l, however, the true d-dimensional nature of the lattice becomes apparent. It

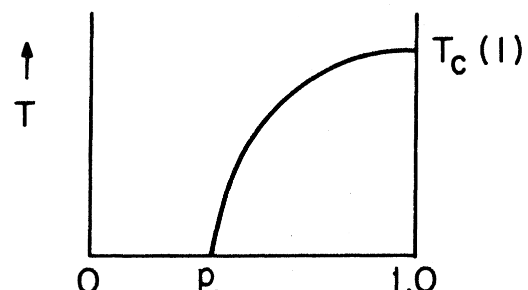

FIG. 1. Phase diagram for a quenched, diluted magnetic Ising lattice. p is the probability that a bond is present. T_c goes to zero at the percolation probability p_c.

is, therefore, relevant to compare the correlation length $\xi_1(T)$ of a one-dimensional system to l. $\xi_1(T)$ can of course be calculated exactly. For a classical n-component spin system, it is a function of Bessel functions of imaginary argument[11] which reduce at low temperature to

$$\begin{aligned} \xi_1 &= e^{2J/T} \quad (n=1), \\ \xi_1 &= \frac{2n}{n-1}\,\frac{J}{T} \quad (n>1) . \end{aligned} \tag{4}$$

If $\xi_1 \gg l$, the spins are essentially ordered and we expect the magnetic susceptibility χ and the magnetization m to be related to the mean-square cluster size $S(p)$ and the probability of being in the infinite cluster $P(p)$ via

$$\chi T \sim S(p) \sim |p-p_c|^{-\gamma_p} , \tag{5a}$$

$$m \sim P(p) \sim (p-p_c)^{\beta_p} . \tag{5b}$$

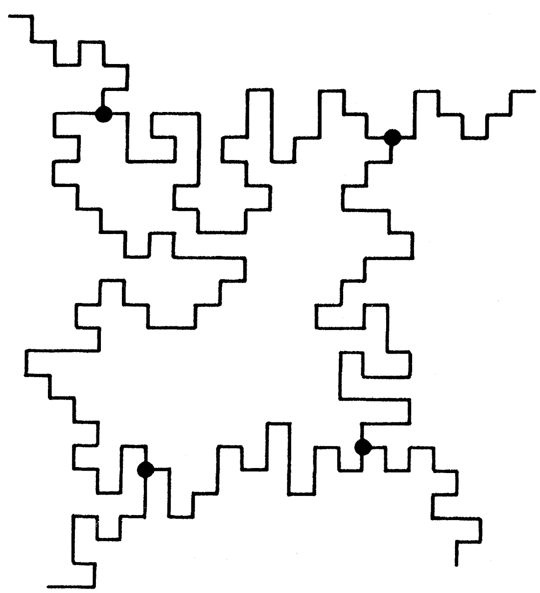

FIG. 2. Schematic drawing of a diluted lattice for $p=p_c^+$, showing nodes connected by zig-zag paths.

In addition, for $n \geq 2$, transverse susceptibility $\chi_\perp$ diverges for small wave number q as q^{-2} for $p>p_c$ and satisfies

$$\chi_\perp T = m^2/Aq^2 , \tag{6a}$$

$$A \sim (p-p_c)^{\mu} . \tag{6b}$$

This last equation is valid for all $n \geq 2$ (n is the number of spin components). To obtain behavior in other regimes, we assume that the Gibbs free energy G and spin correlation function Γ can be expressed in scaling forms

$$\frac{1}{T} G(p-p_c, T, H) = |p-p_c|^{2-\alpha_p} \times F^{\pm}\left(\frac{H}{T|p-p_c|^{\Delta_p}}, \frac{l}{\xi_1(T)}\right) , \tag{7}$$

$$\Gamma(k, (p-p_c), T) = k^{-2+\eta_p} \times f^{\pm}\left(\frac{k}{|p-p_c|^{\nu_p}}, \frac{l}{\xi_1(T)}\right) , \tag{8a}$$

$$\xi(p-p_c, T) = |p-p_c|^{-\nu_p} g^{\pm}(l/\xi_1(T)) , \tag{8b}$$

where H is the external magnetic field, the + (−) superscripts refer to $p>p_c$ ($p<p_c$), and ξ is the spin-correlation length [8(a) implies 8(b)]. α_p, η_p, and Δ_p are percolation exponents satisfying $\gamma_p = (2-\eta_p)\nu_p$, $\beta_p = 2-\alpha_p-\Delta_p$, and $\gamma_p = -2+\alpha_p - 2\Delta_p$. Note that the H/T rather than H is the appropriate scaling field to use near a $T=0$ transition. Equation (7) implies Eq. (5) when $\xi_1 \gg l$ and Eqs. (8) insure that the spin-correlation function is determined by the probability that two sites are in the same cluster. If $l \gg \xi_1$, Eqs. (7) and (8) imply

$$\chi T \sim (\xi_1)^{\gamma_p/\zeta} \equiv (\xi_1)^{\tilde{\gamma}} , \tag{9a}$$

$$\xi \sim (\xi_1)^{\nu_p/\zeta} \equiv (\xi_1)^{\tilde{\nu}} , \tag{9b}$$

and that the transition temperature satisfies

$$\xi_1(T_c(p)) = l \sim (p-p_c)^{-\zeta} \tag{10}$$

or

$$T_c(p) \sim \begin{cases} \dfrac{2J}{\zeta|\ln(p-p_c)|} , & n=1 \\[2ex] \dfrac{2n}{n-1} J (p-p_c)^{\zeta} , & n>1 . \end{cases} \tag{11}$$

Equation (11) for $n=1$ is in agreement with Stauffer.[6]

Equation (7) can also be used to obtain the scaling contribution to the specific heat $C = -T(\partial^2 G/\partial T^2)$. For $\xi_1 \gg l$, we obtain

$$C\sim\begin{cases}\left(\dfrac{2J}{T}\right)^2 e^{-2J/T}|p-p_c|^{d\nu_p-\zeta}, & n=1\\[2ex] T\left(\dfrac{n-1}{2nJ}\right)^2|p-p_c|^{d\nu_p-\zeta}, & n>1\end{cases}\tag{12}$$

and for $l\gg\xi_1$ and T small enough so that Eq. (4) is satisfied:

$$C\sim\begin{cases}\dfrac{4J^2}{T^2}\,\xi_1^{-x}, & n=1\\[2ex] \xi_1^{-x}, & n>1\end{cases}\tag{13}$$

where $x=d\nu_p/\xi$. The scaling relation $d\nu_p=2-\alpha_p$ was used in deriving Eqs. (12) and (13).

Recently, Birgeneau *et al.*[12] performed neutron scattering experiments from $Rb_2Mn_{0.5}Mg_{0.5}F_4$. This is an effective two-dimensional system which is Heisenberg-like down to very low temperature.[13] The percolation concentration[2] for two-dimensional site dilution is 59%. Birgeneau *et al.* find $\tilde{\nu}=0.75$ and $\tilde{\gamma}=1.25$. These numbers satisfy approximately the scaling relation implied by Eq. (9): $\tilde{\gamma}/\tilde{\nu}=1.67\sim\gamma_p/\nu_p=2.3/1.3=1.77$. ($\gamma_p$ and ν_p are the "best values" from Ref. 5.) Thus, experiment seems to document at least some of the scaling ideas presented here. Furthermore, $\tilde{\nu}=\nu_s=0.75$.[12,14] This would correspond to setting $\zeta=\nu_p/\nu_s$—which in turn corresponds to self-avoiding-walk zig-zag paths between nodes. This observation has been exploited theoretically with a slightly different approach than presented here by Stanley *et al.*[15]

The heuristic scaling picture presented here is far from complete and leads to some unanswered question. For example, why does $\tilde{\nu}=\nu_s$ experimentally? Is this a general, dimensionally independent result or is it only approximately satisfied in two dimensions? Is there a better way to define the length l? Computer location of nodes is very difficult, and a more local definition of l would be desirable. de Gennes argued that ζ should be related to the exponent μ describing the growth of the conductivity or spin-stiffness coefficient [Eq. (6)] above p_c via

$$\mu=\zeta+(d-\gamma)\nu_p\ .\tag{14}$$

In two dimensions, numerical calculations[16] yield $\mu=1.1$, which would give $\tilde{\nu}=1.3/1.1\sim1.2$ in clear contradiction with experiment. This raises the intriguing possibility that there are three (or more?) divergent lengths in percolation statistics. One length would be the correlation length ξ_p, another the geometrical length l introduced here and defined slightly differently in Ref. 15, and a third length would be a length l' defined as the ratio of the resistance between nodes to the fundamental bond resistance. Clearly parallel paths play a greater role in determining l' than in determining l. The existence of several divergent correlation lengths, of course, runs contrary to physical intuition developed in the study of thermodynamic phase transitions. This is particularly true since percolation statistics can be described by the thermodynamics of a one state Ashkin-Teller-Potts (ATP) model.[3,5] One would think, therefore, that any exponents appearing in percolation statistics should have their analog in the ATP model. This raises the question of whether crossover exponents in the ATP might correspond to exponents for l and l'. This question is currently being investigated. It is, of course, possible that some aspects of percolation statistics are not easily described by the ATP model. If so, it remains to understand why.

Note added in proof. Recently Stauffer[17] and Shender and Sklovskii[18] have studied the multicritical point discussed here using spin wave theory. They argue that the spin wave stiffness [defined as the coefficient of $(\nabla\theta)^2$ in the free-energy density where θ is the angular deviation of the magnetization from perfect order] goes to zero at p_c in the same way that the conductivity goes to zero [as $(d-2)\nu+\zeta$] and that this causes $T_c(p)$ to go to zero at p_c. If ζ is assumed to be determined by μ [Eq. (14) of this paper], the result they obtain for $T_c(p)$ is identical to that obtained here [Eq. (11)].

In a recent preprint, J. K. Bergstresser has derived rigorous upper and lower bounds on $T_c(p)$ in a two-dimensional diluted Ising model which becomes identical when $p\to p_c$. He predicts $T_c(p)=2J/\ln|p-p_c|$. Thus either $\zeta=1$ in two dimensions [cf. Eq. (11)] or the natural scaling axes are linear combinations of $|p-p_c|$ and $\xi_1(T)$.

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