

## Critical dynamics of dilute $^3\text{He}$ - $^4\text{He}$ mixtures\*

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(Received 1 November 1976)

The universal scaling functions that describe the crossover of the thermal conductivity and diffusion constant of  $^3\text{He}$  above  $T_\lambda$ , from their behavior in pure helium to that characteristic of  $^3\text{He}$ - $^4\text{He}$  mixtures, along the  $\lambda$  line, are calculated to first order in  $\epsilon = 4 - d$  along with the leading correction terms. The crossover exponent is determined to all orders in  $\epsilon$  which implies that the thermal conductivity at  $T_\lambda$  varies inversely with the concentration.

The critical dynamics of  $^3\text{He}$ - $^4\text{He}$  mixtures at the tricritical point and along the  $\lambda$  line for  $T > T_\lambda$  have recently been studied by renormalization-group methods.<sup>1</sup> This note demonstrates how that description of the  $\lambda$  line maps onto a similar model of pure helium and gives a number of dynamic crossover scaling functions. Note that, while in appropriate units, the static-order-parameter fluctuations remain unchanged as the concentration of  $^3\text{He}$ ,  $c_0$ , tends to zero, transport coefficients can change dramatically.<sup>2</sup> The thermal conductivity diverges strongly in pure helium, but is finite in the mixtures. The diffusion constant of the  $^3\text{He}$  impurities diverges as the  $\lambda$  line is approached at constant  $c_0$  or chemical potential  $\mu$ , but is finite as  $c_0 \rightarrow 0$  at fixed  $T > T_\lambda$ .

The critical dynamics of  $^3\text{He}$ - $^4\text{He}$  mixtures can be developed from the interactions among the "slow" modes, i.e., the order parameter  $\psi(x, t)$ , the entropy  $\sigma(x, t)$ , and the concentration  $c(x, t)$ .<sup>1</sup> It is convenient to work with combinations of the hydrodynamic modes whose fluctuations in thermal equilibrium are independent. For  $^3\text{He}$ - $^4\text{He}$ , these are an entropylike mode  $q$ , with susceptibility  $C_R = (\partial\sigma/\partial T)_{c,p}$  and the concentration with susceptibility  $\chi_R = (\partial c/\partial\mu)_{T,p}$ . The pure-helium fixed point is unstable under the addition of  $^3\text{He}$  impurities, which simply means that for any  $c_0 > 0$ , the ultimate critical behavior is that of the mixtures. If the only coupling between the concentration and order parameter were nondissipative, i.e.,  $g_{2,0}$  in Eqs. (1b) and (1c) below, one could calculate the crossover simply by following the growth in a single effective coupling constant from a small value  $\sim c_0$  to its fixed point value,  $\epsilon = 4 - d$ . While this is the approach we will follow, it requires some justification when the concentration susceptibility or specific heat are singular as they nearly are in three dimensions.<sup>2</sup>

In pure helium, it is well known that in the experimentally accessible range of temperatures there are important corrections to the ultimate

critical singularity, originating from the almost-singular entropy fluctuations.<sup>2,3</sup> The corrections may be calculated by including in the free energy a coupling  $\gamma_0$  between the entropy and order parameter which is relevant when  $\alpha > 0$ , but relaxes slowly to zero in three dimensions ( $\alpha > 0$ ).<sup>3</sup> When the fixed-point value of  $\gamma_0$  is nonzero, the corresponding susceptibility diverges.

In contrast to pure helium, along the  $\lambda$  line it is the concentration that is weakly coupled to the order parameter [ $\chi_R = (\partial c/\partial\mu)_{T,p} \sim t^{-\alpha}$  for  $\alpha > 0$ ], while the fluctuations in  $q$  are always finite, but it is  $q$  that becomes the entropy as  $c_0 \rightarrow 0$ . This seems to imply that to understand the leading correction terms as  $c_0 \rightarrow 0$  both densities should be coupled to  $\psi$ . Fortunately, this is not the case because the temperature scale on which  $q$  decouples from  $\psi$ , or  $\lambda_R$  begins to diverge, is very different from the temperature range of the dynamic crossover. For positive  $\alpha$ , we believe that the thermodynamic susceptibilities whose singular behavior near  $t = [T - T_\lambda(c_0)]/T_\lambda = 0$  is different for  $c_0 = 0$  and  $c_0 > 0$  can be represented in the form  $c_0^\alpha f(c_0^\alpha t^{-\alpha})$ , for small  $c_0$  and  $t$ , with  $x=1$  as a consequence of dilute-solution theory and an explicit calculation for  $(\partial c/\partial\mu)_{T,p}$ .<sup>4</sup> It is true, more generally above  $T_\lambda$  and in an isotropic system, that  $\alpha$  is the leading temperature exponent for any density and that  $c_0$  is the only small parameter. In the absence of corrections from  $\gamma_0$ , ( $\lambda_R$  and  $C_R$  constant), it is easy to show that the transport coefficients cross over when  $c_0 t^{-\epsilon\nu/2} \sim 1$ . Corrections to this simple result may therefore be calculated with the free energy appropriate to pure helium with the concentration mode statically decoupled from the order parameter. At a fixed concentration, the additional correction terms signaled by the growth of  $\lambda_R$  occur only at much smaller reduced temperatures and may be ignored in the dynamic-scaling functions which are calculated in the limit  $c_0 \rightarrow 0$ ,  $c_0 t^{-\epsilon\nu/2} \sim \text{const}$  (see Fig. 1).

The model we propose to investigate is defined

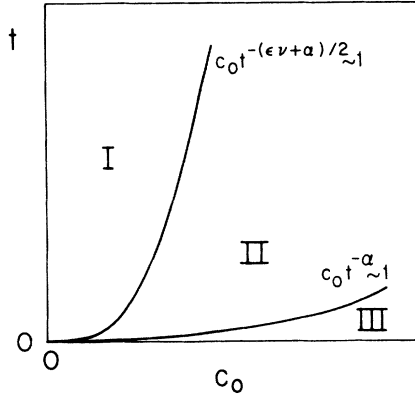


FIG. 1. Concentration-reduced temperature plane showing, schematically, the two distinct scales for dynamic and thermodynamic quantities for  $\alpha > 0$ . The  $\lambda$  line is given by  $t=0$ . Transport coefficients display pure-helium behavior in region I but only attain their  $\lambda$  line dependence in region III. In region II, the leading singularity is characteristic of the  $\lambda$  line but the correction terms are different. Only the crossover from I to II has been calculated. The distinction between regions II and III continues to exist in three dimensions ( $\alpha < 0$ ) with  $t^{-\alpha}$  replaced by  $(\partial\sigma/\partial T)_{\mu,p}$  because throughout the accessible temperature range, the specific heat is increasing and modifies the leading singularities. Thermodynamic quantities are characteristic of pure helium in region I and most of region II.

by a phenomenological thermodynamic potential  $W$ , and stochastic equations for  $\psi$ ,  $c$ , and  $q$ . It includes the leading corrections to dynamic scaling in region I and part of region II, but not in region III (Fig. 1):

$$W = \int d^d x \left( \frac{1}{2} r_0 |\psi|^2 + \frac{1}{2} |\bar{\nabla} \psi|^2 + \tilde{u}_0 |\psi|^4 + \frac{1}{2} C_0^{-1} q^2 + \gamma_0 q |\psi|^2 + \frac{1}{2} \chi_0^{-1} c^2 \right), \quad (1a)$$

$$\frac{\partial \psi}{\partial t} = -2\Gamma_0 \frac{\delta W}{\delta \psi^*} - i g_{1,0} \psi \frac{\delta W}{\delta q} - i g_{2,0} \psi \frac{\delta W}{\delta c} + \zeta, \quad (1b)$$

$$\frac{\partial c}{\partial t} = \lambda_0 \nabla^2 \frac{\delta W}{\delta c} + L_0 \nabla^2 \frac{\delta W}{\delta q} + 2g_{2,0} \text{Im} \left( \psi^* \frac{\delta W}{\delta \psi^*} \right) + \theta, \quad (1c)$$

$$\frac{\partial q}{\partial t} = K_0 \nabla^2 \frac{\delta W}{\delta q} + L_0 \nabla^2 \frac{\delta W}{\delta c} + 2g_{1,0} \text{Im} \left( \psi^* \frac{\delta W}{\delta \psi^*} \right) + \varphi. \quad (1d)$$

The noise sources  $\zeta(x, t)$ ,  $\theta(x, t)$ , and  $\varphi(x, t)$  satisfy

$$\begin{aligned} \langle \zeta(1) \zeta^*(1') \rangle &= 4 \text{Re} \Gamma_0 \delta(1-1'), \\ \langle \theta(1) \theta(1') \rangle &= -2\lambda_0 \nabla^2 \delta(1-1'), \\ \langle \varphi(1) \theta(1') \rangle &= -2L_0 \nabla^2 \delta(1-1'), \end{aligned}$$

$$\langle \varphi(1) \varphi(1') \rangle = -2K_0 \nabla^2 \delta(1-1').$$

It differs from the model in Ref. 1 only in that  $\gamma_0$  couples  $q$  and  $\psi$  rather than  $c$  and  $\psi$ . The renormalized susceptibilities can be identified with the physical thermodynamic derivatives<sup>5</sup>

$$C_R = \left. \frac{\partial \sigma}{\partial T} \right|_{c,p}, \quad (2)$$

$$\chi_R = \chi_0 = \left. \frac{\partial c}{\partial \mu} \right|_{T,p} \sim \frac{c_0}{k_B T \lambda}.$$

The weak temperature dependence of  $\chi_R$  in regions I and II is not contained in our model although  $C_R$  is of course given correctly. The renormalized transport coefficients may be combined to yield the experimentally accessible quantities

$$D = \lambda_R / \chi_R, \quad (3a)$$

$$k_T = \chi_R L_R / \lambda_R, \quad (3b)$$

$$\kappa_{tr} = K_R - L_R^2 / \lambda_R. \quad (3c)$$

Our notation agrees with that of Landau and Lifshitz<sup>6</sup> except that the thermal conductivity  $\kappa_{tr}$  carries a subscript to distinguish it from the inverse correlation length. The strengths of the mode couplings are not renormalized except for scale changes and are determined by hydrodynamics,<sup>1</sup>

$$g_{1,0} = k_B T \lambda \bar{\sigma}_0 / \hbar, \quad (4a)$$

$$g_{2,0} = k_B T \lambda c_0 / \rho_0 \hbar. \quad (4b)$$

It is only necessary to know that as  $c_0 \rightarrow 0$ ,  $g_{1,0}$  is constant and  $g_{2,0} \sim c_0$ .

The recursion relations generated by (1) are readily found by comparison with the results of Ref. 1 and Appendix C of Ref. 3.<sup>7</sup> They are most simply written in terms of the effective coupling constants  $v(l)$ ,  $f_1(l)$ , and  $f_2(l)$ , and the ratios  $w_1(l)$ ,  $w_2(l)$ , and  $w_3(l)$ , ( $K_4 = 1/8\pi^2$ ):

$$v(l) = K_4 C(l) \gamma^2(l), \quad (5a)$$

$$f_1(l) = \frac{K_4 g_1^2(l)}{K(l) \text{Re} \Gamma(l)}, \quad (5b)$$

$$f_2(l) = \frac{K_4 g_2^2(l)}{\lambda(l) \text{Re} \Gamma(l)}, \quad (5c)$$

$$w_1(l) = w_1'(l) + i w_1''(l) = \Gamma(l) C(l) / K(l), \quad (5d)$$

$$w_2(l) \equiv w_2'(l) + i w_2''(l) = \Gamma(l) \chi(l) / \lambda(l), \quad (5e)$$

$$w_3(l) = L(l) / \lambda^{1/2}(l) K^{1/2}(l). \quad (5f)$$

The recursion relations become

$$\frac{dv(l)}{dl} = \frac{\alpha}{\nu} v(l) - 4v^2(l), \quad (6a)$$

$$\frac{df_1(l)}{dl} = f_1(l) \left[ \epsilon - \frac{1}{2} f_1(l) - \text{Re} \left( \frac{w_1(l) A(l)}{B(l) w_1'(l)} \right) \right], \quad (6b)$$

$$\frac{df_2(l)}{dl} = f_2(l) \left[ \epsilon - \frac{1}{2}f_2(l) - \operatorname{Re} \left( \frac{w_1(l)A(l)}{B(l)w_1'(l)} \right) \right], \quad (6c)$$

$$\frac{dw_1(l)}{dl} = w_1(l) \left( \frac{A(l)}{B(l)} - \frac{1}{2}f_1(l) + 4v(l) \right), \quad (6d)$$

$$\frac{dw_2(l)}{dl} = w_2(l) \left( \frac{A(l)}{B(l)} - \frac{1}{2}f_2(l) \right), \quad (6e)$$

$$\frac{dw_3(l)}{dl} = \frac{1}{2} [f_1(l)f_2(l)]^{1/2} - \frac{1}{4}w_3(l) [f_1(l) + f_2(l)], \quad (6f)$$

where

$$\begin{aligned} A(l) = & \frac{w_1'(l)}{w_1(l)} \{ f_1(l)[1 + w_2(l)] + f_2(l)[1 + w_1(l)] \\ & - 2f_1^{1/2}(l)f_2^{1/2}(l)w_3(l) \\ & - 4v(l)w_1(l)[1 + w_2(l)] - 4i[f_1(l)v(l)w_1'(l)]^{1/2} \\ & \times [1 + w_2(l)] + 4i[f_2(l)v(l)w_1'(l)]^{1/2}w_3(l) \} \end{aligned}$$

and

$$B(l) = [1 + w_1(l)][1 + w_2(l)] - w_3^2(l).$$

In (5a),  $\alpha$  is taken from experiment to be slightly negative, and  $v(l)$  decays slowly to zero but generates the essential correction terms. To make contact with (3), some additional relations are needed<sup>4</sup>;

$$C_R \sim \exp \left( 4 \int_0^{l^*} v(l) dl \right), \quad (7a)$$

$$\lambda_R \sim \exp \left( \frac{1}{2} \int_0^{l^*} f_2(l) dl \right), \quad (7b)$$

$$K_R \sim \exp \left( \frac{1}{2} \int_0^{l^*} f_1(l) dl \right), \quad (7c)$$

where  $l^*$  is determined from  $l^\nu e^{l^*} \sim 1$ .

Equations (6b)–(6f) are to be solved at some small concentration  $c_0$ . Integrating on  $l$  we expect (6) will first relax to values appropriate to pure helium and then cross over to their  $\lambda$  line values. The integration requires the initial values of  $v$ ,  $f_i$ , and  $w_i$ ; in particular those that depend on  $c_0$ . They follow from (2) and (4), and the knowledge that at any fixed temperature the diffusion constant,  $\lambda_0/\chi_0$ , tends to a constant as  $c_0 \rightarrow 0$ . We thus have:

$$f_{2,0} \sim c_0, \quad \lambda_0 \sim c_0, \quad w_{3,0}^2 \sim c_0^x. \quad (8)$$

The last relation depends on the value of  $k_T$  in a dilute solution and while  $\kappa_{tr} > 0$  implies  $x \geq 0$ , the precise value of  $w_{3,0}$  will not enter the dynamic scaling functions, even if  $x = 0$ . The remaining initial values are independent of  $c_0$ .

Equations (6b)–(6f) are rather formidable, but fortunately considerable simplifications occur in the limit  $c_0 \rightarrow 0$ ,  $c_0 l^{\epsilon\nu/2} \sim \text{const}$ . If  $v(l)$  is neglect-

ed, the equations for  $w_1$  and  $f_1$  imply  $f_1(l) \rightarrow \epsilon$  and  $A/B \rightarrow \epsilon/2$  relatively rapidly;  $f_2(l) \sim f_{2,0} e^{\epsilon l/2}$  and  $w_2(l) = w_{2,0} e^{\epsilon l/2}$  provided  $f_2(l) \ll \epsilon$ . Thus by the time  $f_2(l) \sim \epsilon$ , its fixed-point value;  $w_2(l) \sim c_0^{-1}$ . Dynamic crossover functions, including the corrections from the slow transient  $v(l)$ , can be computed with  $w_2$  infinite from the beginning.

In this limit, the equations for  $w_1$  and  $f_1$  become identical to those of model *F* that were analyzed in Appendix C of Ref. 3. Their solution with correction terms is<sup>7</sup>

$$f_1(l) = \epsilon + 4v(l) + \frac{20}{9\epsilon} \frac{dv(l)}{dl} + \operatorname{Re} \left( \frac{w_1(l)A(l)}{B(l)w_1'(l)} \right) \quad (9a)$$

$$= \frac{\epsilon}{2} - 2v(l) + \frac{50}{9\epsilon} \frac{dv(l)}{dl}. \quad (9b)$$

Only the first terms in an expansion in  $v(l)$  and its derivatives have been given. Recall that  $v(l) \ll \epsilon$ ,  $v_\infty = 0$ ,  $|d \ln v(l)/dl| \leq |\alpha|/\nu \ll \epsilon$ , and  $d[\ln C(l)]/dl = 4v(l)$ .<sup>7</sup>

The equation for  $f_2$  can now be solved to the same relative order as (9a)<sup>8</sup>:

$$f_2(l) = \frac{\delta c_0 e^{\epsilon l/2} C^{1/2}(l) [1 - 50v(l)/9\epsilon]}{1 + \delta c_0 e^{\epsilon l/2} C^{1/2}(l) [1 - 86v(l)/9\epsilon]}, \quad (10)$$

where  $\delta$  is a nonuniversal constant of integration independent of  $c_0$ . It will be seen shortly that the argument of the dynamic crossover functions is just the combination of  $c_0$  and temperature ( $l = l^* \sim \nu \ln t$ ), that appears in the denominator of  $f_2$ . Although the functions are determined only to first order in  $\epsilon$ , the crossover exponent  $\varphi$  is known to all orders, as will be shown:

$$\varphi = \epsilon/2 + \bar{\alpha}/2\nu, \quad \bar{\alpha} = \max(\alpha, 0). \quad (11)$$

Formally,  $\varphi$  is the exponent for the growth of  $f_2(l)$  when (6c) is linearized about  $f_2(l) = 0$ . Equation (6c) is formed from the recursion relations of  $\lambda(l)$  and  $\Gamma(l)$  ( $g_2$  is simply rescaled). The diagrammatic contributions to  $\lambda(l)$  are always  $O(f_2)$  and do not contribute to  $\varphi$  while  $\partial \ln \Gamma(l)/dl$  is known to all orders in model *F* by scaling.<sup>3</sup>

Equations (2), (3a), (7b), (8), and (10) imply

$$\mathfrak{D} = \mathfrak{D}_0 \left( 1 + \delta c_0 e^{\epsilon l^*/2} C^{1/2}(l^*) \frac{1 - 86v(l^*)}{9\epsilon} \right), \quad (12)$$

where  $\mathfrak{D}_0$  is a constant. To make contact with experiments, we set  $\epsilon = 1$ ,  $e^{l^*} \sim t^{-\nu}$ , and

$$C(l^*) \sim C_R = \left. \frac{\partial \sigma}{\partial T} \right|_{c,p} = \left. \frac{\partial \sigma}{\partial T} \right|_{\mu,p} + O(c_0), \quad (13)$$

$$v(l^*) = \frac{-\partial(\ln C_R)}{4\nu \partial \ln t^*}.$$

Equation (6f) can also be solved

$$w_3^2(l) = \frac{\delta c_0 e^{\epsilon^{1/2}} C^{1/2}(l) [1 - 86\nu(l)/9\epsilon]}{1 + \delta c_0 e^{\epsilon^{1/2}} C^{1/2}(l) [1 - 86\nu(l)/9\epsilon]},$$

and  $w_3 \rightarrow 1$  as  $l \rightarrow \infty$ .<sup>1</sup> The thermal conductivity can be found from  $K_R$ , Eq. (7c), which is just the thermal conductivity in pure helium,<sup>7</sup> and  $w_3^2(l^*)$

$$\kappa_{\text{tr}}(c_0) = \kappa_{\text{tr}}(0) / \left[ 1 + \delta c_0 \kappa_{\text{tr}}(0) \left( 1 + \frac{8}{3\epsilon\nu} \frac{\partial \ln C_R}{\partial \ln t} \right) \right]. \quad (14)$$

Equations (12) and (14) are both  $O(\epsilon)$  results and apply in region I and into region II of Fig. 1. If correction terms are neglected, they are valid uniformly for small  $c_0$ . The coefficient of  $c_0$ ,  $\delta$ , is not universal, although the quantity  $\kappa_{\text{tr}}(c_0) \mathfrak{D}$  is independent of  $c_0$  to  $O(\epsilon)$ . In region II, (14) predicts that

$$\kappa_{\text{tr}} \sim c_0^{-1} \quad (15)$$

with some weak temperature dependence arising from the correction term  $\partial \ln C_R / \partial \ln t$ . Actually (15) holds to all orders in  $\epsilon$  because  $\varphi$ , Eq. (11), is identical to the exponent of  $\kappa_{\text{tr}}(0)$ , also known by scaling to all orders.<sup>3</sup> In region III it was found in Ref. 1, that  $\kappa_{\text{tr}}$  takes the form:

$$\kappa_{\text{tr}} = \delta_1 + \delta_2 t^{\epsilon\nu/2}.$$

The leading confluent singularity has been shown to emphasize that there are no correction terms induced by the most singular thermodynamic susceptibility which is  $\chi_R$  in region III. Again the temperature dependence of the less singular susceptibility,  $(\partial\sigma/\partial T)_{c,B}$ , has been ignored. By continuity  $\delta_1 \sim c_0^{-1}$  and  $\delta_2 \sim c_0^{-2}$ , for  $c_0 \ll 1$  and arbitrary  $\epsilon$ . The result,  $\kappa_{\text{tr}} \sim c_0^{-1}$ , along the  $\lambda$  line has previously been found by applying dilute solution theory below  $T_\lambda$  and neglecting critical effects.<sup>9</sup>

It would be rather difficult to test (14) and (15) for small  $c_0$  because  $\kappa_{\text{tr}}$  is rapidly varying and the small temperature gradient needed for its measurement leads to some rounding of the transition.<sup>10</sup> The extrapolation to  $t=0$  is easier, experimentally, from below  $T_\lambda$  where  $\kappa_{\text{tr}}$  is effectively constant. Its hydrodynamic expression has been given by Khalatnikov, and is rather more complicated than (13c). We have not done a renormalization-group calculation below  $T_\lambda$ , but nevertheless wish to point out that it is not necessary for  $\kappa_{\text{tr}}$  measured macroscopically above and below  $T_\lambda$  to agree at  $t=0$ .<sup>11</sup> Furthermore the breakdown of the hydrodynamic expression for  $\kappa_{\text{tr}}$  below  $T_\lambda$  need not occur at  $k\xi \sim 1$ , where  $k$  would typically be the inverse of the sample cell height  $h$ . A deviation from the criterion  $k\xi \sim 1$ , although difficult to attain in a macroscopic sample, must

occur for sufficiently small  $c_0$  since  $\kappa_{\text{tr}}$  does not exist in pure He II. The relation which replaces  $k\xi \sim 1$  is given below.

In a static measurement of  $\kappa_{\text{tr}}$ ,  $P$ , and  $\mu_4$  are constant and  $\nabla c = [\sigma_0/k_B T_\lambda + O(c_0)] \nabla T$ .<sup>9</sup> There is a normal velocity which opposes the concentration gradient  $v_n = (\mathfrak{D}\sigma_0/c_0 k_B T) \nabla T$ , and because the total density is constant  $j=0$  or  $v_s = -\rho_0 v_n / \rho_s$ . Hydrodynamics will cease to be valid when  $v_s$  equals the intrinsic critical velocity estimated by Reppy and Langer to be  $v_c \sim 500 \rho_s / \rho_0$  cm/sec.<sup>12</sup> If we use the approximations<sup>2</sup>  $\sigma_0/k_B \sim 1$  and  $\rho_s \sim 2\rho_0 t^{2/3}$ , the critical temperature gradient expressed in terms of the temperature difference  $\Delta T$  across the sample becomes

$$\Delta T/hT_\lambda \sim 10^3 c_0 t^{4/3} / \mathfrak{D}, \quad (16)$$

where all units are cgs. If the transition is to remain sharp,  $\Delta T/T_\lambda \ll t$  or

$$10^3 c_0 h t^{1/3} / \mathfrak{D} \ll 1. \quad (17)$$

The diffusion constant is only known above  $T_\lambda$  where Ahlers and Pobell found in concentrated solutions,  $\mathfrak{D} \sim 10^{-4} t^{-1/3}$  cm<sup>2</sup>/sec.<sup>13</sup> If a similar value applies below  $T_\lambda$ , perhaps not temperature dependent, Eq. (17) would be difficult to satisfy. If (17) alone is satisfied the superfluid transition would appear sharp, but the applied thermal gradient must be less than the critical value (16) if the conventional definition of a thermal conductivity is to apply.

Kawasaki<sup>14</sup> has recently proposed a scaling relation for  $\kappa_{\text{tr}}$  which predicts  $\kappa_{\text{tr}} \sim c_0^{-1/3}$  along the  $\lambda$  line in disagreement with (15). He appears to assume that  $\varphi=1$  because the  $\lambda$  line is linear in the  $c_0$ - $t$  plane [compare with Eq. (11)]. We believe this assumption to be unwarranted particularly because  $T_\lambda(c_0)$  is a static quantity while  $\varphi$  pertains only to dynamics.

Tanaka and Ikushima<sup>15</sup> have measured the thermal conductivity along the  $\lambda$  line and fit their data for  $c_0 \lesssim 10\%$  to

$$\kappa_{\text{tr}}^{-1} = \delta_1 c_0 + \delta_2 c_0^x.$$

For  $c_0 \lesssim 1\%$ , they extrapolated to  $T = T_\lambda$  from below  $T_\lambda$ . They found  $x \sim \frac{1}{3}$ , in accord with Kawasaki's scaling argument, but with  $\delta_1/\delta_2 \sim 10^2$ . For  $c_0=0$ , their data disagree with published results of Ahlers.<sup>16</sup>

#### ACKNOWLEDGMENT

The author has benefited from several conversations with G. Ahlers, B. I. Halperin, and P. C. Hohenberg.

\*Work begun while the author was a summer visitor at Bell Labs, Murray Hill, N.J. 07974.

<sup>1</sup>E. D. Siggia and D. R. Nelson, Phys. Rev. B **15**, 1427 (1977).

Our notation coincides with that of this reference.

<sup>2</sup>G. Ahlers, in *The Physics of Liquid and Solid Helium*, edited by J. B. Ketterson and K. H. Benneman (Wiley, New York, 1976), Vol. I, Chap. II.

<sup>3</sup>B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B **13**, 1299 (1976).

<sup>4</sup>Thermodynamic analysis implies, for any fixed temperature, that  $(\partial c/\partial \mu)_{T,p} \sim c_0$  as the concentration tends to zero. Therefore, the bare value of the effective-coupling constant  $v = \chi \gamma^2/8\pi^2$ , between  $c$  and  $\psi$  is  $\sim c_0$ . When Eq. (3.8) of Ref. 1 is integrated, one finds  $(\partial c/\partial \mu)_{T,p} \sim c_0[1 + \delta c_0(t^{-\alpha} - 1)/\alpha]$ , where  $\delta$  is a nonuniversal constant. In three dimensions, one could estimate the importance of the coupling to  $\psi$ , or the temperature range where  $(\partial c/\partial \mu)_{T,p}$  would be parallel to  $(\partial \sigma/\partial T)_{\mu,p}$  on a log plot, by replacing  $(t^{-\alpha} - 1)/\alpha$  by  $(\partial \sigma/\partial T)_{\mu,p}$ . The temperature and concentration dependence of thermodynamic derivatives can in general be extracted from  $(\partial \sigma/\partial T)_{\mu,p}$  and derivatives along the  $\lambda$  line by means of Pippard relations (Ref. 2).

<sup>5</sup>A subscript  $R$  denotes the quantity obtained by solving (1) to all orders. It was introduced in Ref. 1 to emphasize that they were functions of the coupling constants, whatever their values might be. This notation

is redundant when applied to quantities which are functions of temperature only and for this reason was not used in Ref. 3.

<sup>6</sup>L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959), Chap. VI.

<sup>7</sup>The quantities  $\lambda$  and  $\chi$  of Ref. 3 are denoted here by  $K$  and  $C$ .

<sup>8</sup>Although (10) does not include terms of order  $dv(l)/dl$ ,  $K_R$  and  $\lambda_R$ , Eq. (7), are of the same order when calculated from (9a) and (10), respectively.

<sup>9</sup>I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), Chap. 24.

<sup>10</sup>G. Ahlers, Phys. Rev. Lett. **24**, 1333 (1970).

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