Kinetic theory of transport coefficients near T_c in a dynamic spherical model*

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(Received 12 October 1976)

Kinetic transport coefficients are studied in the vicinity of the critical point for the dynamic spherical model. The solution of the kinetic equation leads to divergent transport coefficients at T_c for the thermal conductivity and sound attenuation in contrast to the earlier results of Ma and Senbetu. Their method is generalized to obtain the particle diffusion constant and viscosity, which are nondivergent at T_c. A comparison to mode-mode coupling theories is also given.

I. INTRODUCTION

Recently, some progress in deriving dynamic critical properties from first principles has been made.[1,2] The most notable has been for a system of m-coupled complex Bose fields in the limit $m \to \infty$. Ma and Senbetu[1] (MS) were able to derive a kinetic equation, using a microscopic Hamiltonian as the only input. Using this equation, they calculated the heat conductivity and sound attenuation above and below the critical temperature. Some of their results were unexpected. For instance, the heat conductivity and sound attenuation were not divergent at the critical point. This result was not in agreement with dynamic scaling predictions.[3,4] Sak[2] found that their approximation procedure neglected the conservation laws and this led to the unexpected results for the transport coefficients. A corrected procedure for $T > T_c$ was recently presented by Sak.[2]

Most of our present knowledge about transport coefficients comes from phenomenological theories. These are all based on mode-mode coupling theories[5,6] or on Langevin-type stochastic equations.[7,8] In general, they describe the existing experimental data[9] for critical exponents, but are not always good for the critical amplitudes and in particular do not do very well for the amplitude of the second-sound damping.[6] For this reason, it is interesting to investigate transport phenomena from a microscopic viewpoint, which is the purpose of the present work.

The spherical model is familiar to those who have studied critical phenomena, and the statics have been worked out in great detail.[10] The thermodynamics are known since they are exactly given by Hartree theory and are summarized in Table I of MS. Here, we investigate the dynamics of the model in the $m \to \infty$ limit. The procedure of MS is clarified and corrected results for the singular part of the transport coefficients are obtained. The analysis differs from that developed by Sak[2] since the kinetic equation used by MS differs from the one Sak used. It turns out that the two kinetic equations describe different physical quantities and this feature is discussed. The kinetic equation is also generalized to include particle diffusion.[11,12] This is of interest since the number of components in the system is greater than one and particle diffusion modes become important. In fact, from Halperin's mode-mode analysis,[6] it is believed that the $m = 1$ system (e.g., helium) is quite different from the $m \geq 2$ system, because of these extra modes. Thus it is of interest to derive these extra modes from a microscopic theory and that is done here. To date, the spherical model is the only model for which the kinetic equation has been derived and studied. A similar calculation in the ϵ expansion has not been performed.

The outline of the paper is as follows. In Sec. II, the kinetic equation for T greater than the critical temperature is presented and solved to obtain the singular parts of the thermal conductivity and sound attenuation. The viscosity and particle diffusion are constant at T_c and the volume viscosity and thermal diffusion vanish. Comparison to mode-mode theory results is also given. In Sec. III, a similar analysis for $T < T_c$ is presented. An unexpected infinity for the thermal conductivity and sound attenuation is found for $d \leq 3$ below T_c. The possible cause of this result is discussed. In Sec. IV, a short discussion of important results is given.

II. KINETIC COEFFICIENTS ABOVE T_c

The spherical model is completely described in MS, where the reader can find a review of the statics and thermodynamics. Here, we are interested in the dynamics. To obtain results from first principles, we use finite-temperature perturbation theory to derive a kinetic equation. The large-m limit is useful, as only a certain class of diagrams need be considered. The equation is derived in Appendix A of MS and the reader is referred to that discussion for details, which will not be repeated here.

We must distinguish two types of kinetic equa-

tions, that used by Sak and the one used here and derived by MS. In the large-m limit, one can directly write the kinetic equation for the one-particle distribution function. The linearized version of which was used by Sak[2] in his analysis of the transport coefficients for $T > T_c$. The one developed by MS does not correspond to a distribution function, but is related to the current-current correlation function as shown below. Here we will use the equations proposed by MS and comment briefly on its relation to Sak's.

We begin by defining the density-response function defined by

$$\mathfrak{F}(k,\omega) = -i \int dt\, d^d x\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \langle [\rho(x,t), \rho(0)] \rangle \Theta(t) \tag{2.1}$$

with $\mathrm{Im}\omega > 0$. The speed and damping of sound waves and heat conductivity are determined by the poles of $\mathfrak{F}(k,\omega)$ in the complex ω plane, or equivalently the zeros of the dielectric function $\epsilon(k,\omega)$,

$$\epsilon(k,\omega) = 1 + mu\Pi(k,\omega), \tag{2.2}$$

where Π is related to $\mathfrak{F}$ via $\mathfrak{F} = m\Pi/\epsilon$. (In the present paper m is the number of complex Bose fields and equals $\frac{1}{2}n$. We also use units in which the mass is unity, unlike MS who use $2m_p = 1$.) The task is to find the zeros of the dielectric function in the ω plane. The idea is to write a kinetic equation whose solution will be related to Π. The equation sums an infinite set of graphs which gives $\epsilon(k,\omega)$ to $O(1)$. Since most singular behavior often comes from the large boson populations with small momenta, it is convenient to use the current response function instead of the density-response function. This is because the current is proportional to the momentum and the large boson population for small momenta are less singular. Therefore, following MS, we express $\epsilon(k,\omega)$ in a more convenient form,

$$\epsilon(k,\omega) = 1 - (uk^2/\omega^2)[N - m\Pi^{jj}(k,\omega)], \tag{2.3}$$

where $\Pi^{jj}(k,\omega)$ is the irreducible part of the current response function. The procedure for calculating the sound velocity and damping takes two steps. First, we solve the kinetic equation and solve for $\Pi^{jj}(k,\omega)$ to leading order in $1/m$. Then, we calculate the zeros of the dielectric function. The sound velocity which results from the solution of the kinetic equation does not agree with the thermodynamic result and is not the real sound velocity. The procedure that Sak uses does not contain this second step because his kinetic equation gives the correct sound velocity directly. This is expected since his kinetic equation is for a particle-distribution function.

A. Kinetic equation

The derivation of the kinetic equation is given in MS. The result is the Boltzmann equation in the random-phase approximation which will be briefly described. Define

$$\langle \varphi, \chi \rangle = \frac{1}{m} \sum_{j=1}^{m} \int \frac{d^d p}{(2\pi)^d} s_p^{-2} \varphi_j(p) \chi_j(p), \tag{2.4}$$

$$s_p = 2\sinh(\tfrac{1}{2}\beta\epsilon_p), \quad \epsilon_p = \tfrac{1}{2}p^2 + r, \tag{2.5}$$

where φ and χ are any m-component functions of p and φ_j, χ_j are the jth components of φ, χ. r is the renormalized chemical potential. Equation (2.4) defines a scalar product in the vector space of functions of p. The result of summing an infinite set of leading-order graphs gives[1]

$$m\Pi^{jj}(k,\omega) = m\beta(\langle p_x, p_x \rangle - \omega \langle p_x, \varphi \rangle)$$
$$= N - m\beta\omega \langle p_x, \varphi \rangle, \tag{2.6}$$

where φ is the solution to the integral equation,

$$(\omega - \vec{k}\cdot\vec{p})\varphi_i(p) = p_x - iK\varphi_i(p), \tag{2.7}$$

and we have taken k along x direction. This form is more general than that given by MS. Their equation is in terms of φ and not φ_i. The form (2.7) is necessary to obtain the diffusion modes. The collision operator K is defined by

$$K\varphi_i = \sum_{j=1}^{m} (2\pi)^{-3d} \int d^d p' d^d p'' d^d p'''\, s(s's''s''')^{-1} R(pp'p''p''')[\varphi_i(p) + \varphi_j(p') - \varphi_i(p'') - \varphi_j(p''')], \tag{2.8}$$

and R is proportional to the rate of boson-boson scattering,

$$R(pp'p''p''') = (2\pi)^{d+1}\delta(\epsilon + \epsilon' - \epsilon'' - \epsilon''')\delta(p + p' - p'' - p''')|\tilde{u}(p - p'', \epsilon - \epsilon'')|^2, \tag{2.9}$$

$$\tilde{u}(p,\epsilon) = u/[1 + mu\Pi_0(p,\epsilon)]. \tag{2.10}$$

In the above formulas, $\epsilon \equiv \epsilon_p$, $\epsilon' \equiv \epsilon_{p'}$, etc., $s \equiv s_p$, $s' \equiv s_{p'}$, etc.; and $\varphi(p)$ is also a function of k, ω. The equation is derived under the explicit assumption $k \ll p$ and $\omega \ll \epsilon$ and $\omega, k \sim \mathcal{O}(1/m)$. It can be shown that for $r = 0$ and p, ϵ small,

$$\Pi_0(p,\epsilon) \sim p^{d-4} f(\epsilon/p^2),$$
$$\tilde{u}(p,\epsilon) \sim (1/m) p^{4-d} f^{-1}(\epsilon/p^2),$$

where $f(y)$ is a nonzero function except for $y \to \infty$. Thus, $\tilde{u}$ is just the random-phase approximation for the interaction, which is effectively short ranged and causes no divergences.

This equation differs from that of Sak, which is given by

$$(\omega - \vec{k}\cdot\vec{p})\chi_i(p) - \frac{mu}{T}\vec{k}\cdot\vec{p}\int \frac{d^d p'}{(2\pi)^d} s'^{-2}\chi_i(p') = -iK\chi_i(p). \tag{2.11}$$

The function $\chi_i(p)$ is related to the distribution function for the ith particle $f_i(p)$,

$$f_i(p) = f_0(p) + f_0(p)[1 + f_0(p)]\chi_i(p), \tag{2.12}$$

where $f_0(p) = 1/(e^{\beta\epsilon} - 1)$ and $\chi_i(p)$ is also a function of k, ω. The extra term on the right-hand side of (2.11) is the Vlasov or streaming term. It is essential to give the correct sound velocity, and eliminates the need to find the zeros of the dielectric function. Both methods are equivalent, but the procedure of MS will be followed here.

B. Speed of sound

To compute $\langle p_x, \varphi\rangle$, we must solve (2.8). The solution is obtained by inverting the operator $(\omega - \vec{k}\cdot\vec{p} + iK)$ such that

$$\langle p_x, \varphi\rangle = \langle p_x, (\omega - kp_x + iK)^{-1} p_x\rangle. \tag{2.13}$$

We will assume there is a complete set of orthonormal eigenvectors ϕ_α such that

$$(kp_x - iK)\phi_{\alpha j} = \omega_\alpha \phi_{\alpha j}, \tag{2.14}$$

where α labels the eigenvalue and the eigenvector, while j takes on values $1, 2, \ldots, m$ and labels the elements in the column matrix ϕ_α. If such a solution exists, we can write

$$\langle p_x, \varphi\rangle = \sum_\alpha \frac{\langle p_x, \phi_\alpha\rangle^2}{\omega - \omega_\alpha}. \tag{2.15}$$

The idea is to determine $\langle p_x, \varphi\rangle$ and ω_α as a power series in k, following the standard steps in kinetic theory. This method is usually referred to as the eigenfunction method.[13] The solutions of $\omega_\alpha(k)$ we are interested in decay exponentially in time and vanish as $k \to 0$.

First let $k = 0$ in (2.14). The resulting eigenvalue problem for ϕ_α has a $(d+m+1)$-fold zero eigenvalue corresponding to collisional invariants which form the null subspace and an infinite number of relaxational modes with nonzero decay frequencies. As usual, $d+2$ modes correspond to the conservation laws for particle number, momentum, and energy, while the additional $m-1$ modes are related to particle diffusion. These extra modes were not considered by either MS or Sak. In the spherical model, above T_c, there turns out to be only $m-1$ such modes, unlike the $m(m-1)$ predicted by Halperin.[6] This is because the model Hamiltonian contains only density-density interactions and does not allow for particles to change their component label. The extra modes found by Halperin probably exist, but only for a more general interaction.

The null space is spanned by the orthonormal eigenvectors,

$$\chi_0 = \frac{1}{\sqrt{A_0}}\begin{bmatrix}1\\1\\\cdot\\\cdot\\1\end{bmatrix}, \quad \chi_\epsilon = \frac{1}{\sqrt{A_\epsilon}}\begin{bmatrix}1\\1\\\cdot\\\cdot\\1\end{bmatrix} p^2 - \frac{dA_1\chi_0}{A_\epsilon^{1/2}A_0},$$

$$\chi_a = \frac{1}{\sqrt{A_1}}\begin{bmatrix}1\\1\\\cdot\\\cdot\\1\end{bmatrix} p_a, \quad a = x, y, z, \ldots,$$

$$\chi_1 = \left(\frac{m}{2A_0}\right)^{1/2}\begin{bmatrix}1\\-1\\0\\\cdot\\\cdot\\0\end{bmatrix}, \quad \chi_2 = \left(\frac{m}{6A_0}\right)^{1/2}\begin{bmatrix}1\\1\\-2\\0\\\cdot\\0\end{bmatrix}, \quad \ldots,$$

$$\chi_{m-1} = \frac{1}{[(m-1)A_0]^{1/2}}\begin{bmatrix}1\\1\\\cdot\\\cdot\\1\\-(m-1)\end{bmatrix},$$

where the functions $\chi_1, \ldots, \chi_{m-1}$ are the $m-1$ dif-

fusion modes and the A_i's are normalization constants,

$$A_0 = \langle 1, 1\rangle = \int \frac{d^d p}{(2\pi)^d} s_p^{-2}, \tag{2.16}$$

$$A_1 = NT/m, \tag{2.17}$$

$$A_\epsilon = NT(d+2)\langle p^2\rangle_{av}/m - d^2 A_1^2/A_0, \tag{2.18}$$

where $\langle p^2\rangle_{av}$ is the average of p^2 with the equilibrium function $f_0(p)$.

The next step is to let k be different from zero in (2.14) and treat terms linear in k. Because of the cylindrical symmetry around k, there are only three nonzero matrix elements $\langle p_x, \phi_\alpha\rangle$. These three eigenvectors χ_0, χ_x, χ_ϵ combine to give the sound wave velocity and thermal conductivity, while χ_y, χ_z, ... correspond to the $d-1$ transverse viscous modes and $\chi_1, \chi_2, \cdots \chi_{m-1}$ are the diffusion modes. To find ω_α to first order in k, use first-order degenerate perturbation theory to diagonalize kp_x in the three-dimensional subspace spanned by χ_0, χ_x and χ_ϵ, as the other modes decouple and have zero eigenvalues to this order. The eigenvalues ω_α for this subspace are

$$\omega_0 = 0 + O(k^2), \tag{2.19}$$

$$\omega_\pm = \pm c_0 k + O(k^2), \tag{2.20}$$

where the velocity c_0 is given by

$$c_0^2 = \frac{1}{d^2} A_\epsilon/A_1 + A_1/A_0 = \frac{d+2}{d^2}\langle p^2\rangle_{av}. \tag{2.21}$$

The eigenvectors are

$$\phi_0 = \frac{1}{c_0}\left[\frac{1}{d}\left(\frac{A_\epsilon}{A_1}\right)^{1/2}\chi_0 - \left(\frac{A_1}{A_0}\right)^{1/2}\chi_\epsilon\right], \tag{2.22}$$

$$\phi_\pm = \frac{1}{\sqrt{2}}\left[\pm\frac{1}{c_0}\left(\frac{A_1}{A_0}\right)^{1/2}\chi_0 + \chi_1 \pm \frac{1}{c_0 d}\left(\frac{A_\epsilon}{A_1}\right)^{1/2}\chi_\epsilon\right], \tag{2.23}$$

and corresponding scalar products,

$$\langle p_x, \phi_0\rangle = O(k), \tag{2.24}$$

$$\langle p_x, \phi_\pm\rangle = (\tfrac{1}{2}A_1)^{1/2} + O(k). \tag{2.25}$$

This result determines $\langle p_x, \varphi\rangle$ from (2.15),

$$\langle p_x, \varphi\rangle = A_1\omega/(\omega^2 - c_0^2 k^2). \tag{2.26}$$

Substituting this into (2.6) and then (2.3), we obtain

$$\epsilon(k, \omega) = 1 - uNk^2/(\omega^2 - c_0^2 k^2), \tag{2.27}$$

whose zeros are given by

$$\omega^2 = c^2 k^2, \quad c^2 = uN + c_0^2. \tag{2.28}$$

This corresponds exactly to the thermodynamic result $c^2 = (\partial P/\partial N)_s$ as expected.

C. Kinetic coefficients

We must now examine the $O(k)$ terms in $\langle p_x, \phi_\alpha\rangle$ and $O(k^2)$ terms in ω_α. Following MS, let Q be a projection operator excluding the null space of K and $P = 1 - Q$. To $O(k^2)$, MS find the following equation for $P\phi$:

$$[P\vec{k}\cdot\vec{p} - iP\vec{k}\cdot\vec{p}Q(1/K)Q\vec{k}\cdot\vec{p}]P\phi_{\alpha j} = \omega_\alpha P\phi_{\alpha j} \tag{2.29}$$

The eigenvalue solutions of (2.29) to $O(k^2)$ are easily calculated in the new basis $\phi_0, \phi_\pm$,

$$\omega_0 = -iD_0 k^2, \tag{2.30}$$

$$\omega_\pm = \pm c_0 k - i\tau_+ k^2, \tag{2.31}$$

$$\omega_\eta = -i\eta k^2/\rho, \tag{2.32}$$

$$\omega_D = -iDk^2, \tag{2.33}$$

where

$$D_0 = \sum_\beta{}' \langle \phi_0, p_x\chi_\beta\rangle^2 \frac{1}{\omega_\beta^{(0)}}, \tag{2.34}$$

$$\tau_+ = \sum_\beta{}' \langle \phi_\pm, p_x\chi_\beta\rangle^2 \frac{1}{\omega_\beta^{(0)}}, \tag{2.35}$$

$$\eta/\rho = \sum_\beta{}' \langle \chi_y, p_x\chi_\beta\rangle^2 \frac{1}{\omega_\beta^{(0)}}, \tag{2.36}$$

$$D = \sum_\beta{}' \langle \chi_1, p_x\chi_\beta\rangle^2 \frac{1}{\omega_\beta^{(0)}}, \tag{2.37}$$

ρ is the density, and the sum excludes χ_β from the null space of K. The scalar product needed to evaluate $\epsilon(k, \omega)$ is given by

$$\langle p_x, \phi_0\rangle = \frac{1}{dc_0}\left(\frac{A_\epsilon A_0}{A_1}\right)^{1/2} iD_0 k + O(k^2), \tag{2.38}$$

$$\langle p_x, \phi_\pm\rangle = \pm(\tfrac{1}{2}A_1)^{1/2}(1 \mp ik\tau_-/2c_0) + O(k^2), \tag{2.39}$$

where

$$\tau_- = \sum_\beta{}' \langle \phi_+, p_x\chi_\beta\rangle\langle \chi_\beta p_x, \phi_-\rangle \frac{1}{\omega_\beta^{(0)}}. \tag{2.40}$$

Again, because of the cylindrical symmetry, only these three matrix elements are nonzero. The real thermal conductivity and sound attenuation are not given by (2.34) and (2.35), but by the zeros of the dielectric function. The true viscosity and diffusion are given by (2.36) and (2.37), since they are completely decoupled from the calculation of the dielectric function. Substituting the results of (2.38), (2.39), and (2.15) into (2.6) and then (2.3) gives $\epsilon(k, \omega)$,

$$\epsilon(k, \omega) = 1 - \frac{uNk^2[1 + ik^2(\tau_+ - \tau_-)/\omega]}{\omega^2 - c_0^2 k^2 + 2i\tau_+ k^2\omega} + \frac{uNk^4}{\omega}\frac{A_\epsilon A_0}{d^2 c_0^2 A_1^2} D_0^2 \frac{1}{\omega + iD_0 k^2}. \tag{2.41}$$

The zeros are

$$\omega_0 = -iD_T k^2, \tag{2.42}$$

$$\omega_\pm = \pm ck - \tfrac{1}{2}i\Gamma k^2, \tag{2.43}$$

with

$$D_T = D_0 \frac{c_0^2 A_0}{A_1 c^2}(uN + A_1/A_0) = \frac{\kappa}{\rho C_p}, \tag{2.44}$$

$$\Gamma = 2\tau_+ + D_0 - D_T, \tag{2.45}$$

and where we have used

$$\tau_+ - \tau_- = D_0 A_\epsilon A_0 / d^2 c_0^2 A_1^2 .$$

These expressions are exact in the large-m limit, and we need now only evaluate the matrix elements or at least their singular temperature dependence as a function of $t = (T - T_c)/T_c$. This is the difficult part of the calculation.

Before doing this analysis, it is of interest to rewrite the transport coefficient in a more instructive form. The restriction on the sum over β, can be removed by noting that

$$\langle \chi_\alpha, k p_x \chi_{\alpha'} \rangle = \omega_\alpha^{(1)} \delta_{\alpha\alpha'},$$

and that the spectral resolution of $(K - \epsilon)^{-1}$ is

$$\frac{1}{K-\epsilon} = \sum_\beta \chi_\beta \frac{1}{\omega_\beta^{(0)} - \epsilon} \chi_\beta .$$

We then get

$$D_T = \frac{\kappa}{\rho C_p} = \lim_{\epsilon \to 0^+} \frac{C_v}{A_\epsilon C_p} \left\langle p_x \left(p^2 - \frac{d+2}{d}\langle p^2 \rangle_{av} \right), \right.$$
$$\left. \times \frac{1}{K-\epsilon} \left(p^2 - \frac{d+2}{d}\langle p^2 \rangle_{av} \right) p_x \right\rangle, \tag{2.46}$$

$$\Gamma = \lim_{\epsilon \to 0^+} \frac{1}{A_1} \left\langle (p_x^2 - p^2/d), \frac{1}{K-\epsilon}(p_x^2 - p^2/d) \right\rangle$$
$$+ \left(\frac{1}{C_v} - \frac{1}{C_p} \right) \frac{\kappa}{\rho}, \tag{2.47}$$

$$\eta = \lim_{\epsilon \to 0^+} \frac{\rho}{A_1} \left\langle p_x p_y, \frac{1}{K-\epsilon} p_x p_y \right\rangle, \tag{2.48}$$

$$D = \frac{\lambda_D}{\rho \chi} = \lim_{\epsilon \to 0^+} \frac{1}{A_0} \left\langle p_x, \frac{1}{K-\epsilon} p_x \right\rangle. \tag{2.49}$$

Note that the heat flux $p_x\{p^2 - [(d+2)/d]\langle p^2 \rangle_{av}\}$ coincides with the conjugate flux of Mori.[14] Thus the kinetic theory produces Kubo-type formulas with conjugate fluxes.

λ_D is the transport coefficient theory for diffusion and χ is a susceptibility.[6] From static scaling theory,[6]

$$\chi \sim \xi^x \quad \text{for } T, T_c , \tag{2.50}$$
$$x = 2\varphi/\nu - d ,$$

where ξ is the correlation length ($\tau \sim \xi^{-2+\eta} \sim \xi^{-2}$ in spherical model) and φ is the crossover exponent for a perturbation proportional to $[\rho_\alpha - (1/m)\rho]$,[15] $\varphi/\nu = 2 + O(1/m)$ in spherical model.

Finally, let us consider the expression for sound attenuation. Sak has shown that the first term in (2.47) just equals $[2(d-1)/d]\eta/\rho$. By comparing the hydrodynamic result,[11,16]

$$\Gamma = \frac{1}{\rho}\left(\left(\frac{1}{C_v} - \frac{1}{C_p} \right)\kappa + \frac{2(d-1)}{d}\eta + \zeta \right)$$

with (2.47), we see that the volume viscosity ζ vanishes as in the Boltzmann gas. Since the masses of all the components are equal, we find that the thermal diffusion[11] k_T also vanishes for this model.

D. Spectrum of K

To complete the analysis, we must find the eigenvectors and their corresponding eigenvalues which are not in the null space. This is impossible to do exactly and we must resort to approximations. This is where MS made a crucial approximation for K which violated the conservation laws. This violation led directly to their result of nondivergent transport coefficients for κ and Γ. Here we follow the analysis of Sak,[2] which while not exact, satisfies the conservation laws. Write the collision operator K as a sum of two terms

$$K\varphi_i = \omega(p)\varphi_i + H\varphi_i , \tag{2.51}$$

where

$$\omega(p) = \tfrac{1}{4} m \sinh(\tfrac{1}{2}\beta\epsilon_p)$$
$$\times \int\int\int \frac{d^d p' d^d p'' d^d p'''}{(2\pi)^{2d}} |\tilde{u}(p - p'', \epsilon - \epsilon'')|^2$$
$$\times \frac{\delta(p + p' - p'' - p''')\delta(\epsilon + \epsilon' - \epsilon'' - \epsilon''')}{\sinh(\frac{1}{2}\beta\epsilon')\sinh(\frac{1}{2}\beta\epsilon'')\sinh(\frac{1}{2}\beta\epsilon''')} \tag{2.52}$$

and H is the Hilbert operator consisting of the three remaining terms in K. Note that $\omega(p)$ is not an integral operator, just a multiplicative factor. The integral in K is well defined and vanishes exponentially at large moments. Counting powers shows that H is also well defined at small momenta. We will assume, that the relaxation spectrum consist of a continuous spectrum and a discrete spectrum. In some cases, when the Hilbert operator is completely continuous,[17] it can be shown that H can add only discrete values to the spectrum. While this is probably not true here, we will assume that the continuous spectrum is well approximated by the explicit p dependence in front of integral in (2.52). While this may not be exact, it should be sufficient for determining the singular part of the transport coefficient in the limit $\tau \to 0$, but could possibly be off by factors of logarithms. We proceed under the assumption that

the relaxation spectrum has the form

$$\omega(p) = Cu^2 \sinh(\tfrac{1}{2}\beta\epsilon_p) , \tag{2.53}$$

where C is some unknown constant.

The problem is now to determine $\chi_\beta(p)$. If we made a relaxation-time approximation as in MS, we would replace the collision operator by the multiplicative operator $K\,\chi_q(p) = \omega(p)\chi_q(p)$. However, this procedure would violate the conservation laws. In particular, the solutions could be taken in the form $\chi_q(p) \propto \delta(\vec{p} - \vec{q})$, which is not orthogonal to the null space of K. Sak used a generalized of the relaxation-time approximation by constructing a model collision operator K_m which has the spectrum $\omega(p)$, but preserves the conservation laws.[18] Unfortunately K_m is not equal to its transpose as K is and one should really go back to the initial stages of the calculations and include this fact. Instead, we shall add to the δ function, additional terms so that it will be orthogonal to the null space. This takes into account the conservation laws. Therefore, we solve the eigenvalue equation

$$K\chi_q(p) = \omega_q \chi_q(p) ,$$

which has the eigenvalue $\omega_q = \omega(q)$ given by (2.53). There are several corresponding eigenfunctions, which depend on the symmetry of the solution desired. The first solution, contains no concentration gradients and corresponds to uniform flow,

$$\chi_q(p) = B_q \delta(\vec{p} - \vec{q}) + P\frac{\omega(p)}{\omega(p) - \omega(q)} \times (C_0\chi_0 + C_x\chi_x + \cdots + C_\epsilon\chi_\epsilon) , \tag{2.54}$$

where P means principal value and the C_i's are constants determined from the orthogonality conditions $\langle \chi_\alpha, \chi_q \rangle = 0$. $\chi_q(p)$ does not contain any diffusion terms since the δ function is already orthogonal to χ_i, $i = 1, \ldots, m$. This eigenfunction gives nonzero contributions to D_T, Γ, and η, but because of its uniformity does not contribute to D. Following the analysis of Sak,[2] we find that the relevant constants for (q^2, r) small are

$$B_q = 2\sinh(\tfrac{1}{2}\beta\epsilon_q) A(r/q^2) , \tag{2.55}$$

$$C_x \approx -B_q \langle \chi_x , \delta(\vec{p} - \vec{q}) \rangle , \tag{2.56}$$

where $A(r/q^2)$ is a dimensionless function.

To determine the diffusion, we also need the eigenfunction which corresponds to nonuniform flow or a concentration gradient. The simplest of which has components 1, 2 moving in opposite directions. That particular eigenfunction is

$$\tilde{\chi}_{q1} = \tilde{B}_q \delta(\vec{p} - \vec{q}) + P\frac{\omega(p)}{\omega(p) - \omega(q)} (\tilde{C}_0\chi_0 + \tilde{C}_\epsilon\chi_\epsilon) ,$$
$$\tilde{\chi}_{q2} = \chi_{-q1}, \quad \chi_{q3} = \cdots = \chi_{qm} = 0 , \tag{2.57}$$

where $\tilde{B}_q = \frac{1}{2}mB_q$. $\chi_x, \chi_y, \ldots$ and $\chi_1, \ldots, \chi_{m-1}$ are orthogonal to $\tilde{\chi}_q$ and therefore do not contribute to (2.57). There are $m-2$ other eigenfunctions similar to (2.57) which describe concentration gradients for the remaining components, but the two given here are sufficient to determine the transport coefficients.

E. Singular part of the transport coefficients

The singular parts of the matrix elements in (2.46)–(2.49) are now easily determined. For the heat conductivity we need the matrix element {the enthalpy terms $[(d+2)/d]\langle p^2 \rangle_{av}$ do not contribute by orthogonality}

$$\langle p_x p^2, \chi_q \rangle = B_q \left(\langle p_x p^2, \delta(\vec{p} - \vec{q}) \rangle - \left\langle p_x p^2, P\frac{\omega(p)}{\omega(p) - \omega(q)} \chi_x \right\rangle \langle \chi_x, \delta(\vec{p} - \vec{q}) \rangle \right) . \tag{2.58}$$

Here only the δ function and C_x term contribute. For small q, the second term dominates and gives

$$\langle p_x p^2, \chi_q \rangle = \text{const}\,\frac{q_x}{\sinh(\frac{1}{2}\beta\epsilon_q)} . \tag{2.59}$$

This second term was not included in MS and since it dominates for small q, a different result is obtained. Since the specific heat is constant in this model, for κ we have

$$\kappa \sim \text{const}\,\frac{1}{u^2} \int \frac{d^d q}{(2\pi)^d} \frac{q_x^2}{\omega(q)\sinh^2(\frac{1}{2}\beta\epsilon_q)} \approx (\text{const}/u^2)\xi^{4-d} . \tag{2.60}$$

Comparing this result with the form expected from scaling,[19]

$$\kappa \sim \text{const}\,\xi^{2+z-d} \tag{2.61}$$

we conclude $z = 2$.

Similar calculations for viscosity and second attenuation show that[2]

$$\eta \sim (1/u^2)\,\text{const} + \text{const}'(1/u^2)\xi^{2-d} , \tag{2.62}$$

$$\Gamma \sim \text{const}(1/u^2)\xi^{4-d} . \tag{2.63}$$

Both results are also consistent with scaling.

The singular part for the diffusion is also easily calculated. The uniform eigensolution χ_q gives no contribution to D, but the solution $\tilde{\chi}_q$ do contribute. The matrix element we need is

$$\langle p_x, \tilde{\chi}_{q1}\rangle = \tilde{B}_q \langle p_x, \delta(\vec{p}-\vec{q})\rangle . \tag{2.64}$$

For small q,

$$\langle p_x, \tilde{\chi}_{q1}\rangle \approx \text{const}\, \frac{q_x}{\sinh(\frac{1}{2}\beta\epsilon_q)} . \tag{2.65}$$

Thus we get finally for D,

$$D = \frac{\text{const}}{A_0 u^2}\int \frac{d^d q}{(2\pi)^d}\, \frac{q_x^2}{\omega(q)\sinh(\frac{1}{2}\beta\epsilon_q)} \approx \frac{\text{const}}{u^2} , \tag{2.66}$$

where we have used $A_0 \sim r^{-(d-4)/2} \sim \xi^{4-d}$. This result is consistent with Halperin's since

$$D = \frac{\lambda_D}{\chi} \sim \xi^{2+z-d}/\xi^{2\varphi/\nu - d} \sim \xi^0 \tag{2.67}$$

for the spherical model. Halperin's claim that $z = \varphi/\nu$ for $m \geq 2$ and $D \sim \xi^{2-\varphi/\nu}$ cannot be checked by the present calculation since we do not know z beyond the lowest order.

The singular parts of the transport coefficients are the main result of this section. The results for κ, Γ, and η agree with those of Sak and that for D is new. The analysis also shows how important the conservation laws are in obtaining singular parts for κ and Γ.

III. KINETIC COEFFICIENTS BELOW T_c

A. Kinetic equation

For $T < T_c$, the Bose condensation gives rise to several important features, one of which is second sound. The overdamped thermal conductivity mode splits into two propagating modes known as second sound below T_c. The condensate introduces new collision processes and as a result, new terms in the collision operator. Also for $m > 1$, the correlations below T_c do not decay exponentially, as they do above T_c, but rather according to a power law.

We describe the condensate by a nonzero average of $\langle a_{10}\rangle = (N_0)^{1/2}$. In spin-vector language, we have a nonzero average of the spin in Reψ_1 direction in spin space.[20] This breaks the rotational symmetry in spin space and the number of bosons with component 1 are no longer conserved. The Green's function G_1 is different from $G_2 = G_3 = \cdots = G_m = G_\perp$ and we define

$$\lim_{p\to 0} G_1^{-1}(p,0) = r_1 , \qquad \lim_{p\to 0} G_\perp^{-1}(p,0) = r_\perp . \tag{3.1}$$

These are the inverse of the magnetic susceptibilities parallel and perpendicular to the direction of an "external field" h. In the limit $h \to 0$,[20] r_1 and $r_\perp$ vanish, and can be set equal to zero when there is no ambiguity. For spin systems, the 1 direction can unambiguously be chosen by applying a small field h and taking the limit $h \to 0$. For nonmagnetic systems, the component labels $1, \ldots, m$ can no longer be thought of a species of different particles as was done above T_c. The number 1-component is now the nonconserved combination of the m-original wave functions, while the other $m-1$ components are conserved combinations.

As in Sec. II, we determine Π^{jj} by solving the kinetic equation. To simplify the notation and analysis, we will solve only for the first- and second-sound attenuations. Since there is one less conserved component, there will be only $m-2$ diffusion modes. The diffusion and viscous modes are similar to the results above T_c. The interesting new feature below T_c is the second sound and we will restrict our attention to it. Thus we can write φ as a pair

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_\perp \end{pmatrix}, \tag{3.2}$$

where we have used $\perp$ for the equivalent components $2, 3, \ldots, m$. The scalar product is now

$$\langle \varphi, \chi\rangle = \frac{1}{m}\int \frac{d^d p}{(2\pi)^d}\, s^{-2}[\varphi_1\chi_1 + (m-1)\varphi_\perp\chi_\perp], \tag{3.3}$$

where

$$s = 2\sinh(\tfrac{1}{2}\beta\epsilon), \quad \epsilon = \tfrac{1}{2}p^2. \tag{3.4}$$

We write the new kinetic equation as

$$(\omega - \vec{k}\cdot\vec{p})\varphi = \hat{k}\cdot\vec{p} - iK\varphi , \tag{3.5}$$

where we can express K as 2×2 matrix[1]:

$$K\varphi_i = s\sum_{j=1}^{m}(2\pi)^{-3d}\int\int\int d^d p'\, d^d p''\, d^d p'''\,(s's''s''')^{-1} R(pp'p''p''')[\varphi_i(p) + \varphi_j(p') - \varphi_i(p'') - \varphi_j(p''')] \times\{1 + N_0(2\pi)^d[\delta(p')\delta_{j,1}s' + \delta(p'')\delta_{i,1}s'' + \delta(p''')\delta_{j,1}s''']\}. \tag{3.6}$$

In (3.6), $R(pp'p''p''')$ is still given by (2.9), we have set $[\varphi_1(p)]_{p=0} = 0$ and expanded in powers of N_0/N, since $N_0/N \sim (T_c - T)/T_c$.

The current response function is now given by

$$m\Pi^{jj}(k,\omega) = N' - m\beta\omega\langle p_x, \varphi\rangle . \tag{3.7}$$

Again, we solve for $\langle p_x, \varphi\rangle$ in powers of k, calculate the dielectric function $\epsilon(k,\omega)$ and look for

its zeros to obtain the sound velocity and attenuation. A generalization of Sak's kinetic equation for $T < T_c$ turns out to be more complex, as the Vlasov term includes contributions from fluctuations of the condensate. As a result, two additional equations, one for the superfluid velocity and one for the fluctuations in N_0 are necessary. These three coupled equations are not as straightforward to solve as the MS equation. This is because Eq. (3.5) describes only the above condensate particles and the dielectric function automatically takes into account the fluctuations in N_0, at least to lowest order in $1/m$. This makes the calculation very similar to that for $T > T_c$ and we follow the MS procedure here.

B. Speed of sound

To solve for φ, we follow the procedures for $T > T_c$. First set $k = 0$ and find the null-space eigenvector of K. Since we are only interested in uniform eigenvectors, there is a $(d+2)$-fold degenerate zero eigenvalue. The three-dimensional subspace in the null space with cylindrical symmetry around the x direction is

$$\chi_0 = A_0^{-1/2}\begin{pmatrix}0\\1\end{pmatrix},$$
$$\chi_\epsilon = A_\epsilon^{-1/2}\begin{pmatrix}1\\1\end{pmatrix}p^2, \tag{3.8}$$
$$\chi_x = A_1^{-1/2}\begin{pmatrix}1\\1\end{pmatrix}p_x .$$

The normalization factors are given by

$$A_\epsilon = T(d+2)\langle p^2\rangle_{\rm av} N'/m , \tag{3.9}$$
$$A_1 = N'T/m , \tag{3.10}$$
$$A_0 \to \infty . \tag{3.11}$$

Since $A_0 \to \infty$, the matrix elements of interest involving χ_0 drop out and we can safely ignore χ_0.

Now let k be nonzero and solve for ω_α to $O(k)$ and ϕ_α to $O(1)$. Using degenerate perturbation theory, we diagonalize $\vec{k}\cdot\vec{p}$ in the two-dimensional space spanned by χ_ϵ and χ_x. The eigenvalues ω_α are $\pm c_0 k$ and eigenvectors are

$$\phi_\pm = (1/\sqrt{2})(\chi_\epsilon \pm \chi_x), \tag{3.12}$$

where c_0 is given by (2.21). Putting this result into (2.15) to obtain $\langle p_x, \varphi\rangle$ and in turn into (3.7), we have

$$m\Pi^{ij} = N' - [mA_1\beta\omega^2/(\omega^2 - c_0^2k^2)]. \tag{3.13}$$

The dielectric function is then

$$\epsilon(k,\omega) = 1 - u\left(\frac{N_0k^2}{\omega^2} + \frac{N'k^2}{\omega^2 - c_0^2k^2}\right). \tag{3.14}$$

From $\epsilon(k,\omega) = 0$ we find

$$\omega^4 - \omega^2k^2(uN + c_0^2) + uN_0c_0^2k^4 = 0 , \tag{3.15}$$

which exactly agrees with Khalatinkov's[21] result after inserting the known thermodynamic functions.[1] To first order in N_0/N, we find

$$\omega^2/k^2 = c_1^2 = c_0^2 + uN - \frac{uN_0c_0^2}{uN + c_0^2} + O\left(\frac{N_0^2}{N^2}\right), \tag{3.16}$$

$$\omega^2/k^2 = c_2^2 = \frac{uN_0c_0^2}{uN + c_0^2} + O\left(\frac{N_0^2}{N^2}\right). \tag{3.17}$$

c_1 and c_2 are interpreted as the speed of first and second sounds, respectively.

C. Damping of sound waves

The next terms in ω_α and ϕ_α are obtained as in Sec. II. To $O(k^2)$, the eigensolutions of (2.29) in the new basis $\phi_\pm$ are

$$\omega_\pm = \pm c_0k - \tau_+k^2 , \tag{3.18}$$
$$\langle\phi_\pm, p_x\rangle^2 = \tfrac{1}{2}A_1(1 \mp ik\tau_-/c_0), \tag{3.19}$$

where τ_+ and τ_- are given by (2.35) and (2.40) with $A_0 \to \infty$. The dielectric function can be calculated from (2.15) and is given by

$$\epsilon(k,\omega) = 1 - uN_0k^2/\omega^2 - \frac{uN'k^2[1 + (ik^2/\omega)(\tau_+ - \tau_-)]}{\omega^2 - c_0^2k^2 + 2i\tau_+k^2\omega}, \tag{3.20}$$

the zeros of ϵ are

$$\omega = \pm c_1k - i\tfrac{1}{2}\Gamma_1k^2 , \qquad \omega = \pm c_2k - i\tfrac{1}{2}\Gamma_2k^2 , \tag{3.21}$$

with $\Gamma_{1,2}$ defined by

$$\Gamma_{1,2} = (\tau_+ + \tau_-)\frac{c_{1,2}^2 - uN_0}{c_{1,2}^2 - c_{2,1}^2} + (\tau_+ - \tau_-)\frac{c_{1,2}^2 - uN}{c_{1,2}^2 - c_{2,1}^2}. \tag{3.22}$$

These matrix elements are more illustrative in the form

$$\tau_\pm = \lim_{\epsilon\to 0^+}\frac{1}{2A_1}\left\langle\left(\frac{p_x^2 - p^2}{d}\right), \frac{1}{K-\epsilon}\left(\frac{p_x^2 - p^2}{d}\right)\right\rangle \pm \lim_{\epsilon\to 0^+}\frac{1}{2A_\epsilon}\left\langle p_x\left(p^2 - \frac{d+2}{d}\langle p^2\rangle_{\rm av}\right), \frac{1}{K-\epsilon}\left(p^2 - \frac{d+2}{d}\langle p^2\rangle_{\rm av}\right)p_x\right\rangle . \tag{3.23}$$

Within the proportionality factors, the first term is the viscosity and the second is the thermal conductivity. Expanding (3.22a) to first order in N_0/N and comparing it with the hydrodynamic result,[21,22] we find ζ_2, the second-volume viscosity, vanishes. This is to be expected since $\zeta = 0$ for $T > T_c$.

One advantage of the MS procedure, is that it directly calculates correlation functions and $\epsilon(k, \omega)$. The density-response function $\mathfrak{F}(k, \omega)$ and the structure factor $S(k, \omega)$ are then easily accessible. We find

$$\mathfrak{F}(k,\omega)=\frac{Nk^2\{-\omega^2+(N_0/N)c_0^2k^2+i\omega k^2[(N_0/N)(\tau_+ +\tau_-)+\tau_+ -\tau_-]\}}{(\omega^2-c_1^2k^2+i\Gamma_1k^2\omega)(\omega^2-c_2^2k^2+i\Gamma_2k^2\omega)}. \tag{3.24}$$

By comparing this with the usual hydrodynamic result,[21] in particular the numerator, we find that $\zeta_1 = \zeta_3 = \zeta_4 = 0$ for this model. The structure factor $S(k, \omega)$ is proportional to $\mathrm{Im}\mathfrak{F}(k, \omega)$ and is given by the usual hydrodynamic form.

D. Temperature dependence of Γ_1 and Γ_2

The singular parts of the matrix elements $\tau_\pm$ and hence $\Gamma_{1,2}$ can be determined as was done in Sec. II E above. For $T < T_c$, the continuous spectrum of K starts from zero and does not contain a gap. There is also several new features due to terms in K which appear below T_c. Besides the streaming mode solutions χ_q as found above T_c, we must also consider the eigenvector for the component-1 bosons,

$$\chi_b = A_b^{-1/2}\begin{pmatrix}1\\0\end{pmatrix},$$

which is no longer in the null space of K. This new vector contributes to the viscosity piece of $\tau_\pm$ [see Eq. (3.23)] as shown by MS, but is not the most divergent singularity. Here we are interested in the dominant contributions to $\tau_\pm$ which comes from the streaming modes χ_q.

Since m is large, the $\perp$-component bosons give the dominant contribution to $\tau_\pm$. The energy eigenvalue is now given by

$$\omega_\perp(q) \sim Cq^2 + C'(N_0/N)q^{4-d} \tag{3.25}$$

for small q. C and C' are constants and the second term comes from the added piece in χ below T_c. It is now straightforward to obtain the singular pieces of $\Gamma_{1,2}$. From (3.23) we see that $\tau_\pm$ are made of viscosity and thermal conductivity contributions. As the latter is most singular, we shall investigate it first. The eigenvector $\chi_{q\perp}$ corresponding to $\omega_\perp(q)$ is similar to (2.54) and we find the matrix element

$$\langle p_x p^2, \chi_q\rangle \sim \mathrm{const} q_x/\sinh(\tfrac{1}{4}\beta q^2). \tag{3.26}$$

The thermal conductivity is thus

$$\kappa \sim \frac{\mathrm{const}}{u^2}\int \frac{d^dq}{(2\pi)^d}\frac{q_x^2}{\sinh^2(\frac{1}{4}\beta q^2)(Cq^2+C'(N_0/N)q^{4-d})}. \tag{3.27}$$

Since the spectrum $\omega_\perp(q)$ is continuous down to zero and $r_1, r_\perp = 0$ on the coexistence curve, there is no lower cutoff on the integral (3.27) as above T_c. For $d > 3$ the integral is convergent and goes like $\mathrm{const}(N_0/N)^{(d-4)/(d-2)}$. However for $d \le 3$, the integral has an infrared divergence and

$$\kappa \sim \mathrm{const}\,\xi_\perp^{4-d}, \quad d>3, \qquad \kappa \to \infty, \quad d \le 3, \tag{3.28}$$

where $\xi_\perp \sim (N_0/N)^{-1/(d-2)}$. The result for $d > 3$ agrees with scaling whereas the infinite result for $d \le 3$ is unexpected. In the lower dimensions, the mean-free path becomes infinite and as a result κ goes to infinity. Above T_c, this infrared divergence is eliminated by the chemical potential r which is a lower cutoff on the integral. The correlations decay like a power law below T_c and this is probably related to the divergence in (3.28) for small d. It is expected but not proved that higher-order interactions, will reduce the dynamic exponent z to a value less than 2 and the integral will converge. This is expected to occur for helium, $m = 2$ and $z \sim \frac{3}{2}$, since known experimental results[9] give well-defined thermal conductivity and sound attenuations below T_c. The sound attenuations for $d > 3$ are

$$\Gamma_{1,2} \sim \mathrm{const}_{1,2}\,\xi_\perp^{4-d}. \tag{3.29}$$

The infinite result for κ and $\Gamma_{1,2}$ for $d \le 3$ means that in the $m \to \infty$ limit the sound velocity below T_c is not defined. This is probably an artifact of the large-m limit and finite results for $\Gamma_{1,2}$ should be obtained in more realistic models. MS did not have this difficulty as their result for the thermal conductivity was not singular and did not contain these infrared divergences.

The viscosity, does not have this difficulty as the integrals are convergent for all dimensions and results for η agree with scalings.

IV. DISCUSSION

The leading singular behavior of the kinetic transport coefficients have been calculated for the spherical model in the large m limit. While this model does not correspond to any physical system, the present paper and that of Ma and

Senbetu[1] and Sak[2] indicate the mathematical complexity in deriving and solving microscopic equations for critical dynamics. The procedure follows closely the abundance of work on Boltzmann's equation and is the only known method of obtaining a microscopic proof for dynamics scaling. Unfortunately the present analysis does not give z beyond the lowest order and thus does not really test dynamic scaling.

A word on further calculations using this procedure is in order. One would like to do a similar calculation in the ϵ expansion for small values of m (e.g., helium) but this turns out to be quite difficult. The lowest-order calculations, with K given to order ϵ^2 is a straightforward generalization of the present work and gives $z = 2 + O(\epsilon)$, as expected. To the next order, however, many of the assumptions made in the deviation of the kinetic equation (see MS, Appendix A) are no longer valid and it is not clear how to write down the kinetic equation let alone solve it. At this time there exists no consistent calculation of the kinetic equation to order ϵ^3, and as a result, a microscopic derivation of dynamic scaling is not available.

ACKNOWLEDGMENT

The author would like to thank J. Sak for many helpful discussions and a critical reading of the manuscript.

*Supported in part by the NSF Grant Nos. DMR 72-03230-AO1 and DMR 75-13350.

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