Decimation method for two-dimensional Ising exponents*

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We have applied a decimation (dedecoration) method to a quantum system in order to obtain a critical exponent of the two-dimensional Ising model. Because the renormalization-group transformation is not exact, we have used a perturbation similar to the one used by Niemeijer and Van Leeuwen. The calculations were done to third order. Contrary to the 3-spin-cell method, the second order generates only next-nearest interaction, while the third order generates the next-next-nearest neighbor, etc. The numerical results are compared and analyzed. There are oscillations in our calculation, and the perturbation seems to be at least asymptotic.

I. INTRODUCTION

Recently, Friedman[1] has applied the Niemeijer and Van Leeuwen[2] method to a quantum system in order to obtain critical exponents for a three-dimensional Ising model. His work was based upon the assumption that the X-Y-type model with a transverse magnetic field has the same critical exponents as one-higher-dimensional Ising model.[3-5] This assumption was rigorously proved by Suzuki[3] for the two-dimensional Ising model. Suzuki also has a relation which gives the Curie point of the two-dimension Ising model from his one-dimensional quantum model. On the other hand, Pfeuty[6] has solved a simpler version of Suzuki's model exactly and showed that the exponents indeed match. In Pfeuty's model, the Curie point of the Ising model cannot be obtained. So far, from the previous work, it appears that the ground-state vector of a quantum model is related to the eigenvector of the largest eigenvalue of a classical Ising model of one higher dimension.[5] This phenomenon stems from the correspondence of the inverse temperature of a quantum model to the number of spin degrees of freedom of the Ising model and that of the transverse field of the quantum model to the temperature of the Ising model. This last correspondence is readily understood to be true because these parameters are disorder parameters. The numerical[1,4] results obtained indicate that these correspondences are probably true.

The aim of our work is to compare the renormalization-group method for a quantum system with the *same* method directly applied to the Ising model. We compare the numerical results. Moreover, since one dimension is simple in geometry, one can change the size of a cell and the order of the perturbation with relative ease. Therefore, we hope to shed some light on the very important question of the contributions of cell size and the order of the perturbation for a given accuracy of the numerical results. One might hope to get some idea of the behavior of the perturbation theory, as well.

In this first installment, we present only the decimation method of Pfeuty's model. In the sequel, we will give the calculation of the Curie point from Suzuki's model, and a three-spin cell calculation of Pfeuty's model.

II. DECIMATION PROCEDURE AND CALCULATION

Pfeuty[6] considered the following Hamiltonian:

$$\mathcal{H} = -J \sum_{i=0}^{N} \sigma_i^z \sigma_{i+1}^z - H \sum_{i=0}^{N} \sigma_i^x , \tag{2.1}$$

where σ_i^z, σ_i^x are Pauli matrices. In our definition, we leave out the usual one-half factors. Since all the thermodynamics resides in the ground-state vector, we will consider, as Friedman did, the renormalization-group transformation of the diagonal elements of the ground-state projection operator $|0\rangle_{V\,V}\langle 0|$. $|0\rangle_V$ is the ground state of the exact problem. The decimation[7] transformation can be thought of as a two-spin cell transformation, where in each cell, one of the spins is summed. Here, we will sum every other spin along a chain; the cells are $(0,1)$, $(2,3),\ldots,$ $(N-1,N)$ and, therefore, we take N to be odd. The summation over odd spins amounts to taking the trace of every other 2×2 matrix in the cross product expression of $|0\rangle_{V\,V}\langle 0|$. If $|0\rangle_{V\,V}\langle 0|$ is expressed as a cross product of 4×4 matrices, then we should sum the 2×2 matrices located along the diagonal of each 4×4 matrix. We will denote the decimation operation on $|0\rangle_{V\,V}\langle 0|$ as $\mathrm{Tr}'\,|0\rangle_{V\,V}\langle 0|$, and we will *freely use diagonally equivalent matrices*, i.e., the matrices with the same diagonal elements, when we express $|0\rangle_{V\,V}\langle 0|$.

Formally, the renormalization-group transfor-

mation is

$$|0'\rangle_{V',V'}\langle 0'| = \mathrm{Tr}'\,|0\rangle_{V V}\langle 0|\,, \tag{2.2}$$

where $|0'\rangle_{V'}$ is the ground-state vector of the newly transformed Hamiltonian $\mathcal{H}'$. In fact, in order that Eq. (2.2) be an exact symmetry of the Hamiltonian, we would need infinitely many terms (nearest neighbors, next-nearest, etc.) in $\mathcal{H}$. In practice, we do a perturbation calculation, and add a few terms each stage. It is found[2,8] that the perturbation series seems to be asymptotic if the Hamiltonian is perturbed around intracell interactions. In our problem, as Friedman pointed out, $\mathcal{H}'$ must be also perturbed around the *old* intracell interactions in order that $\mathcal{H}'$ shall remain form invariant, i.e., has the same form as $\mathcal{H}$, with the addition of a few terms. After the renormalization-group transformation is carried out, only one spin is left in each old cell so the $\mathcal{H}_0'$ consists of the single spin terms.

Thus, the unperturbed piece $\mathcal{H}_0$ of $\mathcal{H}$ is

$$\mathcal{H}_0 = -J\sum_{i=\text{ even}}\sigma_i^z\sigma_{i+1}^z - H\sum_{i=0}^{N}\sigma_i^x \tag{2.3}$$

and

$$V = -J\sum_{i=\text{odd}}\sigma_i^z\sigma_{i+1}^z\,. \tag{2.4}$$

$\mathcal{H}_0$ can be written as

$$\mathcal{H}_0 = -\,[J\sigma_0^z\sigma_1^z + H(\sigma_0^x + \sigma_1^x)]$$
$$\vdots$$
$$-\,[J\sigma_{N-1}^z\sigma_N^z + H(\sigma_{N-1}^x + \sigma_N^x)]\,. \tag{2.5}$$

Since all the terms commute, the eigenvectors are easily found to be of the form $v_{i_0}\otimes\cdots\otimes v_{i_n}$, where $i_k = 1,2,3,4$,

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix},\quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\quad v_3 = \frac{1}{C_-}\begin{pmatrix}1\\-\delta-(\delta^2+1)^{1/2}\\-\delta-(\delta^2+1)^{1/2}\\1\end{pmatrix}\quad v_4 = \frac{1}{C_+}\begin{pmatrix}1\\-\delta+(\delta^2+1)^{1/2}\\-\delta+(\delta^2+1)^{1/2}\\1\end{pmatrix}$$

where

$$C_- = 2\{1+[\delta+(\delta^2+1)^{1/2}]^2\}^{1/2}\,,$$
$$C_+ = 2\{1+[-\delta+(\delta^2+1)^{1/2}]^2\}^{1/2}\,, \tag{2.6}$$
$$\delta = J/2H\,,$$

and the respective eigenvalues are

$$\lambda_1 = J,\quad \lambda_2 = -J,\quad \lambda_3 = (J^2+4H^2)^{1/2},$$
$$\lambda_4 = -(J^2+4H^2)^{1/2}.$$

The ground-state vector is $v_4\otimes\cdots\otimes v_4 \equiv |0\rangle$ with energy $-[(N+1)/2)](J^2+4H^2)^{1/2}$. Here $|0\rangle\langle 0|$ is a cross product of the 4×4 matrices. Since the ground state is nondegenerate, the following perturbation is valid[9] (up until the third order):

$$\begin{aligned}
|0\rangle_{V V}\langle 0| = |0\rangle\langle 0| &+ \left(\sum_{n\neq 0}\frac{|0\rangle\langle 0|V|n\rangle\langle n|}{E_0-E_n} + T\right) - \left(\sum_{n\neq 0}\frac{|0\rangle\langle 0|V|0\rangle\langle 0|V|n\rangle\langle n|}{(E_0-E_n)^2} + T\right)\\
&- \sum_{n\neq 0}\frac{|0\rangle\langle 0|V|n\rangle\langle n|V|0\rangle\langle 0|}{(E_0-E_n)^2} + \left(\sum_{\substack{n\neq 0\\ m\neq 0}}\frac{|0\rangle\langle 0|V|m\rangle\langle m|V|n\rangle\langle n|}{(E_0-E_n)(E_0-E_m)} + T\right)\\
&+ \sum_{\substack{n\neq 0\\ m\neq 0}}\frac{|m\rangle\langle m|V|0\rangle\langle 0|V|n\rangle\langle n|}{(E_0-E_n)(E_0-E_m)} + \left(\sum_{\substack{n\neq 0\\ m\neq 0\\ l\neq 0}}\frac{|0\rangle\langle 0|V|n\rangle\langle n|V|m\rangle\langle m|V|l\rangle\langle l|}{(E_0-E_n)(E_0-E_m)(E_0-E_l)} + T\right)\\
&+ \left(\sum_{\substack{n\neq 0\\ m\neq 0\\ l\neq 0}}\frac{|l\rangle\langle l|V|0\rangle\langle 0|V|n\rangle\langle n|V|m\rangle\langle m|}{(E_0-E_n)(E_0-E_m)(E_0-E_l)} + T\right) - \left(\sum_{\substack{n\neq 0\\ m\neq 0}}\frac{|0\rangle\langle 0|V|n\rangle\langle n|V|0\rangle\langle 0|V|m\rangle\langle m|}{(E_0-E_n)(E_0-E_m)^2} + T\right)\\
&- \sum_{\substack{n\neq 0\\ m\neq 0}}\frac{|0\rangle\langle 0|V|n\rangle\langle n|V|m\rangle\langle m|V|0\rangle\langle 0|}{(E_0-E_n)(E_0-E_m)^2} + O(V^4)\,,
\end{aligned} \tag{2.7}$$

where T stands for transpose. From Eq. (2.4),

$$V=\left[\begin{pmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & & J \end{pmatrix}\otimes\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\right]\otimes 1\otimes\cdots\otimes 1 + P\,, \tag{2.8}$$

where 1 is 4×4 unit matrix, and P means permutations. Similarly,

$$\mathcal{H}_0' = -H'\sum_i \sigma_i^x\,,\quad V' = -J'\sum_i \sigma_i^z\sigma_{i+1}^z\,, \tag{2.9}$$

with the eigenvectors and eigenvalues

$$\omega_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ \lambda_1=-H'\,,\quad \omega_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},\quad \lambda_2=H'\,.$$

In Eq. (2.9) one sees the "single-spin" term explicitly. The ground-state vector is

$$|0'\rangle=\omega_1\otimes\cdots\otimes\omega_1\,.$$

Therefore, for zeroth order, both sides of Eq. (2.2) yield $(\frac{1}{2})^{(N+1)/2}1\otimes\cdots\otimes 1$, where 1 is 2×2 unit matrix. Note that if $\mathcal{H}'$ were perturbed any other way, Eq. (2.2) is not satisfied even for zeroth order.

A. First-order calculation

It is easy to see that $\langle 0|V|0\rangle = 0$ and, therefore, the third term of Eq. (2.7) does not contribute. The typical terms which contribute to the right of (2.2) are

$$\frac{|0\rangle\langle 0|V|1144\cdots 4\rangle\langle 1144\cdots 4|}{2(\lambda_4-\lambda_1)} \tag{2.10}$$

and

$$\frac{|0\rangle\langle 0|V|2244\cdots 4\rangle\langle 2244\cdots 4|}{2(\lambda_4-\lambda_2)}\,,$$

where $|1144\cdots 4\rangle \equiv v_1\otimes v_1\otimes v_4\cdots\otimes v_4$, etc. These terms correspond to Niemeijer and Van Leeuwen's triangular lattice expansion.

As the result, we get

$$\begin{aligned}&(\tfrac{1}{2})^{(N+1)/2}1\otimes\cdots\otimes 1+(\tfrac{1}{2})^{(N+1)/2}\delta'[(\sigma_z\otimes\sigma_z)\otimes 1\cdots\otimes 1+P]\\ &=(\tfrac{1}{2})^{(N+1)/2}1\otimes\cdots\otimes 1-(\tfrac{1}{2})^{(N-3)/2}\,\delta\,\frac{[\delta-(\delta^2+1)^{1/2}]^4}{C_+^4[\delta+(\delta^2+1)^{1/2}]}[(\sigma_z\otimes\sigma_z)\otimes 1\cdots\otimes 1+P]\\ &\quad-(\tfrac{1}{2})^{(N-3)/2}\,\frac{+\delta}{C_+^4[\delta-(\delta^2+1)^{1/2}]}[(\sigma_z\otimes\sigma_z)\otimes\cdots\otimes+P]\,,\end{aligned} \tag{2.11}$$

where, as in Eq. (2.8), P means permutations of the part within parentheses along the chain. There are $(N+1)/2$ terms in each bracket. The recursion relation is obtained from Eq. (2.11):

$$\delta' = \delta^2[2-1/2(\delta^2+1)]\,. \tag{2.12}$$

The numerical solution to (2.12) yields

$$\delta^* = 0.61124\,,\quad \lambda = 2.121034 \tag{2.13}$$

for the fixed point and the eigenvalue of Wilson's M matrix.[10] These numbers are to be compared to $\delta^* = 0.5$ (Ref. 6) and $\lambda = 2$ (Ref. 10) of the exact solutions.

B. Second order

Following Niemeijer and Van Leeuwen, the second-order contribution of the nearest-neighbor interaction is to be computed first. The fifth and sixth terms in Eq. (2.7) generate next-nearest interaction through typical terms like

$$\frac{|0\rangle\langle 0|V|1144\cdots 4\rangle\langle 1144\cdots 4|V|1414\cdots 4\rangle\langle 1414\cdots 4|}{4(\lambda_4-\lambda_1)^2}$$

and

$$\frac{|114\cdots 4\rangle\langle 114\cdots 4|V|0\rangle\langle 0|V|4114\cdots 4\rangle\langle 4114\cdots 4|}{4(\lambda_4-\lambda_1)^2}\,. \tag{2.14}$$

Therefore, the next-nearest-neighbor interaction should be included in the first order calculation. The first-order contribution to the next-nearest-neighbor interaction of the right-hand side of Eq. (2.2) is only the next-nearest neighbor. Defining L to be the new interaction parameter and $\gamma = L/2H$,

$$
\begin{aligned}
&(\tfrac{1}{2})^{(N+1)/2}\,1\otimes\cdots\otimes 1+(\tfrac{1}{2})^{(N+1)/2}\,\delta'[(\sigma_z\otimes\sigma_z)\otimes 1\otimes\cdots\otimes 1+P]+(\tfrac{1}{2})^{(N+1)/2}\,(\gamma'+\tfrac{3}{2}\delta'^2)\\
&\quad\times[(\sigma_z\otimes 1\otimes\sigma_z)\otimes 1\cdots\otimes 1+P]+(\tfrac{1}{2})^{(N+1)/2}\,\delta'^2[(\sigma_z\otimes\sigma_z)\otimes 1\otimes\cdots\otimes(\sigma_z\otimes\sigma_z)\otimes 1+NP]\\
&\quad=(\tfrac{1}{2})^{(N+1)/2}\,1\otimes\cdots\otimes 1-(\tfrac{1}{2})^{(N-3)/2}\,\frac{[-\delta+(\delta^2+1)^{1/2}]^4(\delta-2\gamma)}{C^4[\delta+(\delta^2+1)^{1/2}]}[(\sigma_z\otimes\sigma_z)\otimes 1\cdots 1+P]-(\tfrac{1}{2})^{(N-3)/3}\,\frac{\delta+2\gamma}{C_+^4[\delta-(\delta^2+1)^{1/2}]}\\
&\quad\times[(\sigma_z\otimes\sigma_z)\otimes 1\cdots\otimes 1+P]+(\tfrac{1}{2})^{(N+1)/2}\,\frac{12\delta^2}{C_+^6}\Big([-\delta+(\delta^2+1)^{1/2}]^8+[\delta+(\delta^2+1)^{1/2}]^2\\
&\qquad\qquad-\frac{2}{3}\,\frac{[-\delta+(\delta^2+1)^{1/2}]^5+[-\delta+(\delta^2+1)^{1/2}]}{(\delta^2+1)^{1/2}}\\
&\qquad\qquad-\frac{[-\delta+(\delta^2+1)^{1/2}]^4+[-\delta+(\delta^2+1)^{1/2}]^2}{3(\delta^2+1)}\Big)\\
&\quad\times[(\sigma_z\otimes\sigma_z)\otimes 1\cdots\otimes 1+P]+(\tfrac{1}{2})^{(N-3)/2}\,\frac{4\delta^2}{C_+^8}\Big(\frac{[\delta-(\delta^2+1)^{1/2}]^8}{[\delta-(\delta^2+1)^{1/2}]^2}+\frac{1}{[\delta-(\delta^2+1)^{1/2}]^2}-[-\delta+(\delta^2+1)^{1/2}]^4\\
&\qquad\qquad+\frac{1}{2}\,\frac{[\delta-(\delta^2+1)^{1/2}]^3}{(\delta^2+)^{1/2}}-\frac{1}{2}\,\frac{[\delta-(\delta^2+1)^{1/2}]^4}{(\delta^2+1)^{1/2}[\delta+(\delta^2+1)^{1/2}]}\Big)\\
&\qquad\qquad\times[(\sigma_z\otimes\sigma_z)\otimes 1\cdots 1\otimes\sigma_z\otimes\sigma_z\otimes 1\cdots 1+N]\,,
\end{aligned}
\tag{2.15}
$$

where NP means nonoverlapping permutations of the two.

Simplifying Eq. (2.15), we obtain

$$
\begin{aligned}
\delta'&=\delta^2\Big(2-\frac{1}{2(\delta^2+1)}\Big)+4\gamma(\delta^2+1)^{1/2}-\frac{3\gamma}{(\delta^2+1)^{1/2}}\,,\\
\gamma'+\frac{3}{2}\,\delta'^2&=\frac{3}{16}\,\frac{\delta^2}{(\delta^2+1)^{3/2}}\Big([-\delta+(\delta^2+1)^{1/2}]^5\\
&\qquad+[\delta+(\delta^2+1)^{1/2}]^5\\
&\qquad-\frac{8}{3}(\delta^2+1)^{1/2}+\frac{2}{3}\,\frac{1}{(\delta^2+1)^{1/2}}\Big)\,,\\
\delta'^2&=\delta^4\Big(2-\frac{1}{2(\delta^2+1)}\Big)^2\,.
\end{aligned}
\tag{2.16}
$$

The last equation of (2.16) does not lead to a new recursion relation because it is obtained from the first one, if we neglect γ^2, $\delta\gamma$ which is in accord with this order of perturbation. In the same spirit, one should use the last equation when we express γ' in terms of δ.

Thus, the recursion relations read

$$
\begin{aligned}
\delta'&=\delta^2\Big(2-\frac{1}{2(\delta^2+1)}\Big)+4\gamma(\delta^2+1)^{1/2}-\frac{3\gamma}{(\delta^2+1)^{1/2}}\,,\\
\gamma'&=\frac{3}{16}\,\frac{\delta^2}{(\delta^2+1)^{3/2}}\Big([-\delta+(\delta^2+1)^{1/2}]^5+[\delta+(\delta^2+1)^{1/2}]^5\\
&\qquad-\frac{8}{3}(\delta^2+1)^{1/2}+\frac{2}{3}\,\frac{1}{(\delta^2+1)^{1/2}}\Big)\\
&\quad-\frac{3}{2}\,\delta^4\Big(2-\frac{1}{2(\delta^2+1)}\Big)^2\,.
\end{aligned}
\tag{2.17}
$$

The numerical results of Eq. (2.17) are

$$
\delta^*=0.50102\,,\quad \gamma^*=0.055421\,,\quad \lambda=2.18278\,. \tag{2.18}
$$

We see that δ^* is highly accurate, while λ has drifted off a little from the exact value.

C. Third order

From this order on, everything is similar to the second order. If one calculates the third order of the nearest-neighbor interaction, one finds the next-next-nearest-neighbor interaction is generated. The contributions come from the seventh and eighth terms in Eq. (2.7). The typical contributing terms are

$$
\frac{|0\rangle\langle 14414\cdots 4|\langle 0|V|1144\cdots 4\rangle\langle 1144\cdots 4|V|1414\cdots 4\rangle\langle 1414\cdots|V|1441\cdots\rangle}{8(\lambda_4-\lambda_1)^3}
$$

and

$$
\frac{|114\cdots 4\rangle\langle 41414\cdots|\langle 114\cdots|V|0\rangle\langle 0|V|4114\cdots\rangle\langle 4114\cdots|V|41414\cdots\rangle}{8(\lambda_4-\lambda_1)^3}\,.
$$

If one tries to include the next-nearest interaction to the second order, one finds the left-hand side of Eq. (2.2) generates the next-next-next-nearest interaction. Therefore, in this case, one has to include

the next-next-next-nearest interaction in the first order. Although this is fairly straightforward to do, we will not do it here. The recursion relations obtained are

$$\delta' = \delta(\delta+\alpha)\left(2-\frac{1}{2(\delta^2+1)}\right)+4\gamma(\delta^2+1)^{1/2}-\frac{3\gamma}{(\delta^2+1)^{1/2}},$$

$$\gamma' = \alpha\delta\left(2-\frac{1}{2(\delta^2+1)}\right)+\frac{3\delta^2}{16(\delta^2+1)^{3/2}}\left([-\delta+(\delta^2+1)^{1/2}]^5+[\delta+(\delta^2+1)^{1/2}]^5-\frac{8}{3}(\delta^2+1)^{1/2}+\frac{2}{3}\frac{1}{(\delta^2+1)^{1/2}}\right)$$

$$-\frac{3}{2}\delta^4\left(2-\frac{1}{2(\delta^2+1)}\right)^2,$$

$$\alpha' = \frac{5}{32}\frac{\delta^3}{(\delta^2+1)^2}\left([\delta+(\delta^2+1)^{1/2}]^7-[-\delta+(\delta^2+1)^{1/2}]^7-\frac{96}{5}\delta(\delta^2+1)-\frac{2}{5}\frac{\delta}{\delta^2+1}+\frac{36}{5}\delta\right)-\frac{5}{2}\delta^6\left(2-\frac{1}{2(\delta^2+1)}\right)^3, \tag{2.19}$$

where $\alpha = M/2H$ and M is the next-next-neighbor interaction strength. The numerical results are

$$\delta^* = 0.43675\,,\quad \gamma^* = 0.06436\,,$$
$$\alpha^* = 0.04544\,,\quad \lambda = 2.07508\,. \tag{2.20}$$

δ^* now deviates, while λ is accurate. The magnitudes of γ^* and α^* are close, and it is well that our perturbation counted them in the same order.

III. DISCUSSION AND CONCLUSION

Oscillations are apparent in our results, but since this also happens using larger cells[11] (although to a lesser degree), we think that the two-spin cell procedure is acceptable. In the midst of this oscillation, the perturbation series seems to be at least asymptotic. For instance, the first and third orders give accurate values for λ, while δ^* is not as accurate, but the third order is closer to the exact value. It is probable that odd orders may converge smoothly without oscillation in both δ^* and λ, and similarly for even orders, up to a certain optimum order.

For the prediction of the critical index ν, the accuracy of (2.20) is comparable to the value obtained by Niemeijer and Van Leeuwen[2] for their three-spin cell up to the next-next-neighbor approximation and also to the value obtained by Nauenberg and Nienhuis[12] for four-cell clusters. However, it is not as good as the nine-spin cell calculation of Hsu, Niemeijer, and Gunton.[8]

Our result seems to confirm the common belief that the accuracy of the cumulant expansion of Niemeijer and Van Leeuwen depends on the size of the cell. Our result also seems to indicate that the two-spin cell calculation of this type is "numerically" close to an approximate four-spin cell calculation in two dimension.

It would appear from these results that the accuracy of an approximation scheme is more directly tied to the number of neighboring cells included, than to the order of the perturbation. Our next calculation of the three-spin cell system should, according to this conjecture, be comparable in accuracy to a nine-spin cell calculation in two dimensions.

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