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# Multispin correlation functions for Ising models*

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We calculate the multispin correlation functions away from $T_c$ of a two-dimensional Ising lattice along the diagonal. We also present the result for a four-spin correlation function in the scaling limit. These results show that with slight modifications, the operator reduction formulae of Kadanoff hold.

## I. INTRODUCTION

The multispin-correlation functions in the two-dimensional rectangular Ising models have been studied by many authors.[1-4] In particular, Kadanoff[1] has studied the $2n$-spin correlations along a line at $T=T_c$. These results suggest the operator reduction algebra.[1,5,6] In order to check these reduction formulae away from $T_c$, we shall compute a few cases of the multispin correlation functions exactly. The first calculation is for two groups of spins, when their separation is much larger than the correlation length and each group contains odd number of spins. Secondly, we shall present[7] the rigorous result for the four-spin correlation function $\langle \sigma_0 \sigma_N \sigma_{N+M} \sigma_{N+M+L}\rangle$ in the scaling limit, that is the limit when all the distances between the four spins $M$, $N$, and $L$ approach infinity, and $T$ goes to $T_c$.

## II. ODD-ODD SPIN-CORRELATION FUNCTIONS

Let $\sigma_l$ denote a spin on the $(l, l)$ sites along the diagonal of the rectangular Ising lattice, so that $\sigma_l = \sigma_{l,l}$. The spin-correlation function is an $N\times N$ Toeplitz determinant,[8-10] while the ratio $\langle \sigma_0\sigma_l\sigma_{l+1}\sigma_N\rangle/\langle\sigma_0\sigma_N\rangle$ is the $l$th diagonal element of the inverse of this finite Toeplitz matrix.[2] When $T<T_c$, $N \gg |T/T_c - 1|^{-1}$, the spin correlation is known[8] to behave as

$$\langle \sigma_0\sigma_N\rangle = \langle\sigma_0\rangle^2 \left\{1+\frac{e^{-2N|t|}}{8\pi N^2|t|^2}\left[1+O\left(\frac{1}{N|t|}\right)\right]\right\}, \quad (1)$$

where $|t|^{-1}$ is the correlation length, given by

$$t = k_0^{1/2} - k_0^{-1/2} \propto T/T_c - 1\,, \quad (2)$$

$$k_0 = (\sinh 2K_1 \sinh 2K_2)^{-1}\,. \quad (3)$$

When $l$ is much smaller than $N$, the ratio $\langle \sigma_0\sigma_l\sigma_{l+1}\sigma_N\rangle/\langle\sigma_0\sigma_N\rangle$ can still be calculated by the same technique used by Wu.[8] In particular, we find

$$\langle \sigma_0\sigma_1\sigma_2\sigma_N\rangle = \langle\sigma_0\sigma_N\rangle(1-\tfrac{1}{4}k_0^2)\{1+(4\pi N^2|t|)^{-1}e^{-2N|t|} \times[1+O(1/N|t|)]\}\,, \quad (4)$$

in which $k_0$ is defined in (3). More generally, we have for $l \ll N$,

$$\langle \sigma_0\sigma_l\sigma_{l+1}\sigma_N\rangle = \langle\sigma_0\sigma_N\rangle\left(B_l^- + \frac{b_l^- e^{-2N|t|}}{4\pi N^2|t|}+\cdots\right), \quad (5)$$

where $B_l^-$ is the ratio[11] $\langle\sigma_0\sigma_l\sigma_{l+1}\rangle/\langle\sigma_0\rangle$ for $T<T_c$, and is given by

$$B_l^- = \lim_{N\to\infty}\frac{\langle\sigma_0\sigma_l\sigma_{l+1}\sigma_N\rangle}{\langle\sigma_0\sigma_N\rangle} = \sum_{n=0}^{l}\frac{k_0^{2n}(-\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2}\,, \quad (6)$$

in which

$$(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)\,, \quad (7)$$

while

$$b_l^- = k_0^{-2l}\left(\sum_{n=0}^{l}\frac{k_0^{2n}(\frac{1}{2})_n}{n!}\right)\left(\sum_{j=0}^{l}\frac{k_0^{2j}(-\frac{1}{2})_j}{j!}\right) = \frac{k_0^{-2l}\Gamma(l+\frac{1}{2})\Gamma(l+\frac{3}{2})F(-l,\frac{1}{2};\frac{3}{2};1-k_0^2)F(-l,\frac{1}{2};-\frac{1}{2};1-k_0^2)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(l!)^2}\,, \quad (8)$$

where $F(a,b;c;z)$ denotes the hypergeometric function. The particular values of $B_l^-$ and $b_l^-$ are

$$B_1^- = 1-\tfrac{1}{4}k_0^2,\quad B_2^- = 1-\tfrac{1}{4}k_0^2-\tfrac{3}{64}k_0^4\,,$$
$$B_3^- = 1-\tfrac{1}{4}k_0^2-\tfrac{3}{64}k_0^4-\tfrac{5}{256}k_0^6\,, \quad (9)$$

$$b_1^- = k_0^{-2}-\tfrac{1}{4}k_0^2,\quad b_2^- = k_0^{-4}-\tfrac{1}{4}k_0^2-\tfrac{3}{64}k_0^4\,,$$
$$b_3^- = k_0^{-6}-\tfrac{15}{64}k_0^2-\tfrac{1}{16}k_0^4-\tfrac{5}{256}k_0^6\,. \quad (10)$$

At $T=T_c(k_0=1)$, the sum in (6) can be evaluated exactly to yield

$$(B_l^-)_c = \left(\frac{\langle\sigma_0\sigma_l\sigma_{l+1}\rangle}{\langle\sigma_0\rangle}\right)_{T\to T_c^-} = \frac{\Gamma(l+\frac{1}{2})\Gamma(l+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(l!)^2}\,. \quad (11)$$

This shows even though the expectation values $\langle\sigma_0\sigma_l\sigma_{l+1}\rangle$ and $\langle\sigma_0\rangle$ vanish at $T_c$, their ratio remains finite. Furthermore, by comparing (11) with (8) we find for $k_0 = 1$, $(b_l^-)_c = (B_l^-)_c$. Consequently, in the limit $l \ll |t|^{-1}$, $b_l^- \sim B_l^-$; the correlation function of (5) has the simpler form

$$\langle \sigma_0 \sigma_l \sigma_{l+1} \sigma_N \rangle = B_l^- \langle \sigma_0 \sigma_N \rangle \left(1 + \frac{e^{-2N|t|}}{4\pi |t| N^2} + \cdots \right). \quad (12)$$

Thus, we find that the product of three spins behaves as a single spin.

It is also easy to evaluate the multispin correlations,

$$\langle \sigma_0 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_N \rangle = \langle \sigma_0 \sigma_N \rangle \; ((1 - \tfrac{1}{4} k_0^2)^2 - \tfrac{1}{64} k_0^6) \times \left(1 + \frac{e^{-2N|t|}}{2\pi N^2 |t|} + \cdots \right), \quad (13)$$

$$\langle \sigma_0 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_{N-2} \sigma_{N-1} \sigma_N \rangle = \langle \sigma_0 \sigma_N \rangle \, ((1 - \tfrac{1}{4} k_0^2)^2 - \tfrac{1}{64} k_0^6)\,(1 - \tfrac{1}{4} k_0^2) \times \left(1 + \frac{3 e^{-2N|t|}}{4\pi N^2 |t|} + \cdots \right), \quad (14)$$

and

$$\langle \sigma_0 \sigma_l \sigma_{l+1} \sigma_{N-m-1} \sigma_{N-m} \sigma_N \rangle = \langle \sigma_0 \sigma_N \rangle = \langle \sigma_0 \sigma_N \rangle \left[ B_l^- B_m^- + \frac{(B_l^- b_m^- + B_m^- b_l^-) e^{-2N|t|}}{4\pi N^2 |t|} + \cdots \right] \quad (15)$$

for $l, m \ll N$, in which $B_j^-$, and $b_j^-$, are given by (6) and (8), respectively. These results show that the correlation function of any two group of spins, each of which contains an odd numbers of spins, has the same form.

Next, we shall show that the correlation function still has the same form when no pair of spins are neighbors of each other as in the above examples. We find that it is possible (but more tedious) to compute $\langle \sigma_0 \sigma_l \sigma_{l+m} \sigma_N \rangle$ for $l + m \ll |t|^{-1}$, and

$$\langle \sigma_0 \sigma_l \sigma_{l+m} \sigma_N \rangle = D_l^m \langle \sigma_0 \sigma_N \rangle \left(1 + \frac{m e^{-2N|t|}}{4\pi N^2 |t|} + \cdots \right), \quad (16)$$

where $D_l^m$ denotes the ratio $\langle \sigma_0 \sigma_l \sigma_{l+m} \rangle / \langle \sigma_0 \rangle$ for $T < T_c$. It can be expressed as an $m \times m$ determinant

$$D_l^m = |d_{i,j}|_{l \le i,\, j \le l+m-1} \quad (17)$$

whose elements are

$$d_{i,j} = \sum_{n=0}^{\min[i,j]} \frac{k_0^{i+j-2n} (-\frac{1}{2})_{i-n} (\frac{1}{2})_{j-n}}{(i-n)!\,(j-n)!}. \quad (18)$$

This again shows that the correlation function $\langle \sigma_0 \sigma_l \sigma_{l+m} \sigma_N \rangle$ has its leading term proportional to the spin-spin correlation $\langle \sigma_0 \sigma_N \rangle$.

When $T > T_c$, the spontaneous magnetization is zero; this means that the Toeplitz determinant $\langle \sigma_0 \sigma_N \rangle$ vanishes in the limit $N \to \infty$. This in turn means the semi-infinite Toeplitz matrix is not invertible,[12,13] and here we would need to calculate its inverse elements $\langle \sigma_0 \sigma_l \sigma_{l+1} \rangle / \langle \sigma_0 \rangle$!. This difficulty is overcome by using the method of Wu,[8] and we find

$$\langle \sigma_0 \sigma_l \sigma_{l+1} \sigma_N \rangle = B_l^+ \langle \sigma_0 \sigma_N \rangle + b_l^+ \langle w_0 \sigma_N \rangle + O\left(\frac{t^{1/4} e^{-N|t|}}{N^2}\right) \quad (19)$$

for $T > T_c$, where[8]

$$\begin{aligned} \langle \sigma_0 \sigma_N \rangle &= (1 - k_0^{-2})^{1/4} k_0^{-N} [\Gamma(N - \tfrac{1}{2}) / \Gamma(\tfrac{1}{2}) \Gamma(N)] \\ &\quad \times F(\tfrac{1}{2}, N - \tfrac{1}{2}; N; k_0^{-2}) + O(e^{-3N|t|}) \\ &\sim (2|t|)^{-1/4} e^{-N|t|} (\pi N)^{-1/2} \\ &\quad \times [1 + (8N|t|)^{-1} + \cdots], \end{aligned} \quad (20)$$

while

$$\begin{aligned} \langle w_0 \sigma_N \rangle &\sim (1 - k_0^{-2})^{1/4} k_0^{-N} [\Gamma(N - \tfrac{3}{2}) / 2\Gamma(\tfrac{1}{2}) \Gamma(N)] \\ &\quad \times F(\tfrac{1}{2}, N - \tfrac{3}{2}; N, k_0^{-2}) \\ &\sim (2N)^{-1} (2|t|)^{-1/4} e^{-N|t|} (\pi N)^{-1/2} \\ &\quad \times \left(1 + \frac{3}{8N|t|} + \cdots \right), \end{aligned} \quad (21)$$

in which the constant $B_l^+$ and $b_l^+$ are given as

$$\begin{aligned} B_l^+ &= \lim_{N \to \infty} \frac{\langle \sigma_0 \sigma_l \sigma_{l+1} \sigma_N \rangle}{\langle \sigma_0 \sigma_N \rangle} \\ &= (l!)^{-1} (\tfrac{1}{2})_l \sum_{n=0}^{l+1} \frac{k_0^{1-2n} (-\frac{1}{2})_n}{n!} \\ &\quad - \sum_{n=0}^{l} \frac{k_0^{-1-2n} (\frac{1}{2})_n (-\frac{1}{2})_{n+1}}{n!(n+1)!}, \end{aligned} \quad (22)$$

$$b_l^- = (l!)^{-1} (\tfrac{1}{2})_l \sum_{n=0}^{l} \frac{k_0^{1-2n} (-\frac{1}{2})_n (l+1-n)}{n!}, \quad (23)$$

with the particular values

$$\begin{aligned} B_1^+ &= \tfrac{1}{2} k_0 + \tfrac{1}{4} k_0^{-1}, \quad B_2^+ = \tfrac{3}{8} k_0 + \tfrac{5}{16} k_0^{-1} + \tfrac{1}{64} k_0^{-3}, \\ B_3^+ &= \tfrac{5}{16} k_0 + \tfrac{11}{32} k_0^{-1} + \tfrac{3}{128} k_0^{-3} + \tfrac{1}{256} k_0^{-5}, \end{aligned} \quad (24)$$

$$\begin{aligned} b_1^+ &= \tfrac{1}{2}(2k_0 - k_0^{-1}), \quad b_2^+ = \tfrac{3}{8}(3k_0 - k_0^{-1} - \tfrac{1}{8} k_0^{-3}), \\ b_3^+ &= \tfrac{5}{16}(4k_0 - \tfrac{3}{2} k_0^{-1} - \tfrac{1}{4} k_0^{-3} - \tfrac{1}{16} k_0^{-5}). \end{aligned} \quad (25)$$

Again, at $T = T_c (k_0 = 1)$, we can evaluate the sums in (22) to obtain

$$(B_l^+)_c = \left( \frac{\langle \sigma_0 \sigma_l \sigma_{l+1} \rangle}{\langle \sigma_0 \rangle} \right)_{T \to T_c^+} = \frac{\Gamma(l + \frac{1}{2}) \Gamma(l + \frac{3}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) (l!)^2}. \quad (26)$$

On comparing this equation with (11), we find $(B_l^+)_c = (B_l^-)_c$. This shows that the expression in

(22) for $\langle \sigma_0\sigma_l\sigma_{l+1}\rangle/\langle\sigma_0\rangle$, which is an inverse element of a noninvertible Toeplitz matrix, is correct. The sums in (23) for $k_0=1$ can also be evaluated exactly yielding

$$(b_l^+)_c=(B_l^+)_c \quad \text{or} \quad b_l^+=B_l^++O(tl) \ . \tag{27}$$

When $l, m\ll N$ and $|t|^{-1}\ll N$, we have

$$\langle \sigma_0\sigma_l\sigma_{l+1}\sigma_{N-m}\sigma_{N-m}\sigma_N\rangle \sim B_l^+B_M^+\langle\sigma_0\sigma_N\rangle + (B_l^+b_m^++B_m^+b_l^+)\langle\sigma_0 w_N\rangle \tag{28}$$

for $T>T_c$ with constants $B_j^+$ and $b_j^+$ defined in (22) and (23).

The equations (5), (19) for $\langle\sigma_0\sigma_l\sigma_{l+1}\sigma_N\rangle$ and (15), (28) for $\langle\sigma_0\sigma_l\sigma_{l+1}\sigma_{N-m-1}\sigma_{N-m}\sigma_N\rangle$ are still valid expressions even for $l, m\to\infty$, $|t|\to 0$, with $|t|l$, $|t|m\sim 0(1)$. In this limit we find

$$B_l^-=\left(\frac{\langle\sigma_0\sigma_l\sigma_{l+1}\rangle}{\langle\sigma\rangle}\right)_{T<T_c} \sim\langle\sigma_0\sigma_1\rangle+(|t|\pi^{-1}) \times(2|t|l)^{-2}e^{-2|t|l}[1-(|t|l)^{-1}+\cdots] \tag{29}$$

and

$$B_l^+=\left(\frac{\langle\sigma_0\sigma_l\sigma_{l+1}\rangle}{\langle\sigma\rangle}\right)_{T>T_c} \sim\langle\sigma_0\sigma_1\rangle+\left(\frac{2|t|}{l\pi}\right)^{1/2} +\frac{|t|}{\pi}(2|t|l)^{-3}e^{-2|t|l}\left(\frac{1}{2}-\frac{9}{8(|t|l)}+\cdots\right), \tag{30}$$

which together describes the behavior of the spin-energy correlation.

These results show that the operator reduction hypothesis holds away from $T_c$. More specifically, we find that the product of any odd number of spins can be written

$$\prod_{i=0}^{2n-1}\sigma_i\sim B\sigma_r+b\Omega_r, \tag{31}$$

where $\Omega_r$ denotes a less singular operator. From the equations

$$\frac{\langle\sigma_0\sigma_N\rangle}{\langle\sigma_0\sigma_{N-n}\rangle}\sim 1+\frac{ne^{-2N|t|}}{4\pi^2N^2|t|}+\cdots \quad \text{for } T<T_c \tag{32}$$

and

$$\langle\sigma_0\sigma_{N-n}\rangle\sim\langle\sigma_0\sigma_N\rangle-n\langle\sigma_0 w_N\rangle+\cdots \quad \text{for } T<T_c, \tag{33}$$

with $n\ll|t|^{-1}$, we can see that the leading term in the multispin-correlation function, given by (4), (12)–(16), (19), or (28), is independent of the choice of $r$ except $|r-i|<|t|^{-1}$. The constant $B$ is determined by the ratio

$$B=\left\langle\prod_{i=0}^{2n-1}\sigma_i\right\rangle\Big/\langle\sigma\rangle, \tag{34}$$

while the correlation $\langle\Omega_0\sigma_R\rangle$ can be deduced from these equations as

$$\langle\Omega_0\sigma_R\rangle\propto\frac{\langle\sigma\rangle^2e^{-2N|t|}}{N^2|t|}\propto|t|(\langle\sigma_0\sigma_R\rangle-\langle\sigma\rangle^2) \quad \text{for } T<T_c \tag{35}$$

and

$$\langle\Omega_0\sigma_R\rangle\propto\frac{|t|^{-1/4}e^{-N|t|}}{N^{3/2}}\propto N^{-1}\langle\sigma_0\sigma_R\rangle \quad \text{for } T>T_c. \tag{36}$$

However, it is worthwhile to note that $\langle\Omega_0\sigma_R\rangle \neq\langle\Omega_R\sigma_0\rangle$. It is because from (12), we have

$$\langle\sigma_0\sigma_2\sigma_3\sigma_N\rangle\sim B_2\langle\sigma_0\sigma_N\rangle(1+e^{-2N|t|}/4\pi N^2|t|+\cdots), \tag{37}$$

while (16) yields

$$\langle\sigma_{-N+3}\sigma_0\sigma_2\sigma_3\rangle=\langle\sigma_0\sigma_1\sigma_3\sigma_N\rangle =B_2\langle\sigma_0\sigma_N\rangle(1+2e^{-2N|t|}/4\pi N^2|t|+\cdots). \tag{38}$$

## III. FOUR-SPIN CORRELATION FUNCTION

Now, we shall present the result of our analysis of the four-spin correlation function $\langle\sigma_0\sigma_N\sigma_{N+M}\sigma_{N+M+L}\rangle$ in the scaling limit. Let us put

$$x_1=tN,\quad x_2=tL,\quad X=|t|(M+\tfrac{1}{2}N+\tfrac{1}{2}L), \tag{39}$$

with $t$ defined in (2). We find the four-point correlation function can be expressed in terms of these scaling variables

$$\frac{\langle\sigma_0\sigma_N\sigma_{N+M}\sigma_{N+M+L}\rangle}{\langle\sigma_0\sigma_N\rangle\langle\sigma_0\sigma_L\rangle}=F_4(x_1,x_2,X) +O((N+L)t^2\ln|t|,(N+L)/M^2), \tag{40}$$

where $F_4(x_1,x_2,X)$ is the scaling function, which has the asymptotic expansion

$$\begin{aligned}F_4(x_1,x_2,X)=1&+\tfrac{1}{4}x_1x_2[1-\tfrac{1}{2}x_1\Lambda_1-\tfrac{1}{2}x_2\Lambda_2+\tfrac{1}{4}x_1^2\Lambda_1^2+\tfrac{1}{4}x_2^2\Lambda_2^2+\tfrac{1}{4}x_1x_2\Lambda_1\Lambda_2-\tfrac{1}{16}(x_1^2+x_2^2)](K_1^2-K_0^2)\\&+\tfrac{1}{16}x_1x_2(x_1+x_2)(1-\tfrac{1}{2}x_1\Lambda_1-\tfrac{1}{2}x_2\Lambda_2)(K_1^2-K_0K_2)+\tfrac{1}{128}x_1^2x_2^2(K_1^2+K_1K_3-K_2^2-K_0^2)+\tfrac{1}{128}x_1x_2(x_1^2+x_2^2)\\&\times(K_1K_3+3K_1^2+2K_0^2-2K_0K_2)+O(x_1^5\ln x_1K_n(X)K_{4-n}(X)),\end{aligned} \tag{41}$$

where $K_n = K_n(X)$ are the modified Bessel function[5], while

$$\Lambda_i = \gamma + \ln|x_i| - 3\ln 2, \quad i = 1, 2. \tag{42}$$

On using the results[14]

$$\langle\sigma_0\sigma_N\rangle/\langle\sigma_0\sigma_N\rangle^c = 1 + \tfrac{1}{2}x_1\Lambda_1 + \tfrac{1}{16}x_1^2 + O(x_1^3) \tag{43}$$

and

$$\langle\sigma_0\sigma_N\rangle/\langle\sigma_0\sigma_L\rangle^c = 1 + \tfrac{1}{2}x_2\Lambda_2 + \tfrac{1}{16}x_2^2 + O(x_2^3), \tag{44}$$

we find the scaling function $f_4$ defined by

$$f_4(x_1x_2X) = \frac{\langle\sigma_0\sigma_N\sigma_{N+M}\sigma_{N+M+L}\rangle - \langle\sigma_0\sigma_N\rangle\langle\sigma_0\sigma_L\rangle}{\langle\sigma_0\sigma_N\rangle^c\langle\sigma_0\sigma_L\rangle^c} \tag{45}$$

has the incredibly simple form

$$\begin{aligned} f_4(x_1, x_2, X) \sim{} & \tfrac{1}{4}x_1x_2(K_1^2 - K_0^2) \\ & + \tfrac{1}{16}x_1x_2(x_1+x_2)(K_1^2 - K_0K_2) \\ & + \tfrac{1}{128}x_1^2x_2^2(K_1^2 + K_1K_3 - K_0^2 - K_2^2) \\ & + \tfrac{1}{128}x_1x_2(x_1^2+x_2^2) \\ & \times(K_1K_3 + 3K_1^2 - 2K_0^2 - 2K_0K_2). \end{aligned} \tag{46}$$

All the terms involving $\ln x_1$ and $\ln x_2$ disappears in this expression. At $T = T_c$, we find

$$f_4^c = \frac{NL}{4R^2} + \frac{NL(N+L)^2}{16R^4} - \frac{3N^2L^2}{32R^4} + O\left(\left(\frac{NL}{R^2}\right)^3\right), \tag{47}$$

with $R = M + \frac{1}{2}N + \frac{1}{2}L$, which agrees with the results of Kadanoff.[1] Since the energy-energy correlation function behaves as

$$\langle\epsilon_0\epsilon_R\rangle = \pi^{-2}t^2[K_1^2(R|t|) - K_0^2(R|t|)], \tag{48}$$

where

$$\epsilon_R = \sigma_R\sigma_{R+1} - \langle\sigma_0\sigma_1\rangle, \tag{49}$$

we can see from (45) and (46), that the reduction formula for $T \neq T_c$ must be modified as

$$\sigma_0\sigma_N = \langle\sigma_0\sigma_N\rangle + (\langle\sigma_0\sigma_N\rangle^c/\langle\sigma_0\sigma_1\rangle^c)N\epsilon_{N/2} + O(N^2t_{1,1}), \tag{50}$$

where the superscript denotes values at $T = T_c$, with $\langle\sigma_0\sigma_1\rangle^c = 2/\pi$.

Even though the final result is so simple, the mathematical difficulties involved in this problem are enormous. We have used the theorems on the generalized hypergeometric functions extensively, which enable us to estimate the errors accurately. This analysis shall be published elsewhere.[7]

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