Existence of phase transitions near the displacive limit of a classical n-component lattice model

S. Sarbach*
IBM Zurich Research Laboratory, 8803 Rüschlikon-ZH, Switzerland
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Using a new method proposed by Fröhlich, Simon, and Spencer, we prove the existence of a phase transition near the displacive limit of a classical n-component displacement model on a d-dimensional ($d \geq 3$) lattice. In certain cases, the proof can be extended for $d = 2$ and $n = 1$. Moreover, we derive exact lower bounds for the critical temperature of the spin-s ($s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$) extension of the Blume-Capel model.

I. INTRODUCTION

In recent years, there has been considerable interest in systems undergoing structural phase transitions.[1-4] One of the most salient features of such systems is the existence of the so-called "displacive limit,"[1,2,5-8] where the critical temperature, as a function of the parameters in the Hamiltonian, is expected to vanish continuously. Beyond the displacive limit there is no phase transition. Slightly below the displacive limit, in the displacive regime, there is experimental evidence for a finite critical temperature. Good examples of real systems undergoing a phase transition in the displacive regime are $SrTiO_3$ and $LaAlO_3$.[9] Moreover, molecular-dynamics investigations have demonstrated the existence of a phase transition at finite temperature close to the displacive limit.[1] Far from the displacive limit, in the so-called "order-disorder regime,"[1] the existence of a phase transition has been proven rigorously in a two- or more-dimensional classical one-component displacement model. This was achieved using correlation inequalities[10,11] or a modified version of the well-known Peierls argument.[12] Nevertheless, these arguments fail when the displacive limit is approached, and so far, there is no rigorous result demonstrating the existence of a finite critical temperature in the neighborhood of the displacive limit.

The aim of this paper is to use a recent and powerful method proposed by Fröhlich, Simon, and Spencer[13,14] (FSS) to prove the existence of a phase transition in the displacive regime of a classical n-component displacement model. To explain the method let $\vec{s}_j$ denote the n-component displacement of an atom j from its reference position $\vec{j}$. The central idea behind the FSS strategy is embodied in an *a priori* bound on the low-momentum singularity of the two-point correlation function. Specifically, let $F(\vec{k})$ be the Fourier transform of the two-point function $\langle \vec{s}_i \cdot \vec{s}_j \rangle$; this is a positive distribution of the form

$$F(\vec{k}) = c\delta(\vec{k}) + g(\vec{k}) , \tag{1.1}$$

where, for the symmetric models considered below, c is the long-range order parameter

$$c = \lim_{|\vec{i}-\vec{j}| \to \infty} \langle \vec{s}_i \cdot \vec{s}_j \rangle . \tag{1.2}$$

Evidently $c \neq 0$ implies the existence of long-range order and, hence, of a phase transition. In order to prove the existence of the transition ($c > 0$) one needs: (i) an upper bound on $g(\vec{k})$ (step A of FSS), and (ii) a lower bound on $\langle |\vec{s}_j|^2 \rangle$ (step B). In all the cases considered hereafter the upper bound on $g(\vec{k})$ derived by FFS[14] is still valid and reads

$$0 \leq g(\vec{k}) \leq (2\beta J)^{-1} (2\pi)^{-d} \left(\sum_{i=1}^{d} (1 - \cos k_i) \right)^{-1} , \tag{1.3}$$

where $\beta^{-1} = k_B T$ is the temperature, $J \geq 0$ is the interaction strength, and d denotes the lattice dimensionality. On integrating (1.1) over $\vec{k}$ and using (1.3) we obtain

$$c \geq \langle |\vec{s}_j|^2 \rangle - (2\beta J)^{-1} q(d) , \tag{1.4}$$

where $0 < q(d) < \infty$ for $d \geq 3$. [One may note that in one and two dimensions the integral over the right-hand side of (1.3) is divergent.] To establish the existence of a phase transition it remains then to establish a nonzero lower bound on $\langle |\vec{s}_j|^2 \rangle$.

The FSS method, although valid for all n, is evidently restricted to lattices of dimension $d \geq 3$. Nevertheless, we will show how to use the method to prove the existence of a phase transition in the displacive regime of a two-dimensional ($d = 2$), one-component ($n = 1$) classical lattice system. However, for $d = 2$ and $n > 1$ no spontaneous magnetization or long-range order can exist in the isotropic model considered below.[15-17]

The main result for d-dimensional ($d \geq 3$) lattice systems of n-component displacements ($1 \leq n \leq \infty$) is proven in Sec. II. In Sec. III, the case $n = 1$, $d = 2$ is considered. Section IV deals with a discrete-spin model, namely, the spin-S extension of the Blume-Capel model.[18,19] For this model the FSS method enables us to derive lower bounds for the critical temperature.

II. n-COMPONENT DISPLACEMENT MODEL IN THREE OR MORE DIMENSIONS

Let $\mathfrak{z}^d$, $d \in \mathfrak{N}$, be an infinite lattice, and Λ a finite sublattice in $\mathfrak{z}^d$ with $|\Lambda| = N$ ($|\Lambda|$ is the number of lattice points in Λ). At each lattice site j there is an n-component displacement (spin) $\vec{s}_j = \{s_{j1}, \ldots, s_{jn}\} \in \mathfrak{R}^n$ $(1 \leq n \leq \infty)$. The Hamiltonian of our classical system is given by

$$H_N^{(n)}(\{\vec{s}\}) = \sum_{j=1}^{N} nV(n^{-1}|\vec{s}_j|^2) + \tfrac{1}{2} J \sum_{\langle i,j \rangle} |\vec{s}_i - \vec{s}_j|^2 , \tag{2.1}$$

where J is positive, $\langle i, j\rangle$ denotes nearest-neighbor lattice points, and $V(z^2)$ has the following properties:

$$V(z^2) \geq V_0 > -\infty \quad \text{for all } z \in \mathfrak{R} , \tag{2.2}$$

$$\int_{\mathfrak{R}^n} \exp[-nV(n^{-1}|\vec{s}|^2) + a|\vec{s}|^2]\, d^n s < \infty \tag{2.3}$$

for all positive a. In (2.1) we neglect the kinetic energy which plays no role in classical equilibrium statistical mechanics. Such a Hamiltonian describes a system of N-coupled anharmonic oscillators when $\vec{s}_j$ is interpreted as an n-component displacement around the reference position j.[5,7,8] If $\vec{s}_j$ is interpreted as an n-component classical spin, the Hamiltonian (2.1) describes a ferromagnetic system.[20] The Ising model ($n=1$), the x-y model ($n=2$) and the classical Heisenberg model ($n=3$) are special cases of (2.1).

Corresponding to the Hamiltonian (2.1) we define the partition function

$$Z_N^{(n)}(\beta) = \int_{\mathfrak{R}^{Nn}} \exp[-\beta H_N^{(n)}(\{\vec{s}\})] \times \prod_{j=1}^{N} \exp(-\epsilon|\vec{s}_j|^2) d^n s_j , \tag{2.4}$$

where $\epsilon > 0$ so that $Z_N^{(n)}(\beta = 0)$ remains finite, and the "free energy" per site and component,

$$F_N^{(n)}(\beta) = (Nn)^{-1} \ln Z_N^{(n)}(\beta) . \tag{2.5}$$

The main result in this section is the following:

Theorem 2.1: Let $d \geq 3$ *and* $z_0^2 \in \mathfrak{R}_+$ *the smallest value of* z^2 *such that* $V(z^2) = V_0$ *for all* $z^2 \in \mathfrak{R}_+$. *Then, if* $z_0^2 > 0$, *there is a phase transition with* $T_c > 0$ *in the model defined by (2.1) for any integer* n $(1 \leq n \leq \infty)$.

Remarks: (i) By phase transition we mean here existence of long-range order. We refer to FSS[14] for the discussion of the equivalence between long-range order and spontaneous magnetization.

(ii) It is easy to verify that when $z_0^2 > 0$ the ground state is degenerate.

Proof of Theorem 2.1: To prove Theorem 2.1 we need the following:

Lemma 2.2: For the model defined by Hamiltonian (2.1), for any $n(1 \leq n \leq \infty)$ *and for any* $d \in \mathfrak{N}$;

$$\lim_{\beta\to\infty} n^{-1}\langle |\vec{s}_i|^2\rangle \geq z_0^2 . \tag{2.6}$$

Then, using (1.3), and the approach explained in the Introduction, theorem 2.1 is a straightforward application of the FSS method.[13,14]

Proof of the Lemma: (see FSS[14] Lemma 3.2). For all $J \geq 0$ we can write

$$\begin{aligned} Z_N^{(n)}(\beta) &\leq \exp[-\beta NnV(z_0^2)] \times \int_{\mathfrak{R}^{Nn}} \prod_{j=1}^{N} \exp(-\epsilon|\vec{s}_j|^2) d^n s_j \\ &= \exp[-\beta NnV(z_0^2)](\pi\epsilon^{-1})^{Nn/2} . \end{aligned} \tag{2.7}$$

To get a lower bound for the partition function, we restrict the domain of integration in the partition function. Let, namely, $D \subset \mathfrak{R}^n$ be the following set:

$$D = \{\vec{s} \in \mathfrak{R}^n | z_0 - a \leq s_\alpha \leq z_0 + a; \quad 1 \leq \alpha \leq n\} \tag{2.8}$$

and

$$D^N = D \times D \times \ldots \times D \quad (N \text{ factors}) . \tag{2.9}$$

Then,

$$|\vec{s}_i - \vec{s}_j|^2 = \sum_{\alpha=1}^{n} (s_{i\alpha} - s_{j\alpha})^2 \leq 4na^2 \tag{2.10}$$

for all $\vec{s}_i, \vec{s}_j \in D$, so that

$$\sum_{\langle i,j\rangle} |\vec{s}_i - \vec{s}_j|^2 = 4\,dN na^2 \tag{2.11}$$

for all $\vec{s}_i \in D, i = 1, \ldots, N$. (Without loss of generality we consider here simple cubic or hypercubic lattices with $2-d$ nearest neighbors.) Moreover,

$$n^{-1}|\vec{s}_i|^2 \leq (z_0 + a)^2 \quad \text{for all } \vec{s}_i \in D . \tag{2.12}$$

Now, when a is small enough and if $\vec{s}_i \in D$, one has

$$V(n^{-1}|\vec{s}_i|^2) \leq V((z_0 + a)^2) = V(z_0^2) + g(a) , \tag{2.13}$$

where

$$g(a) = \sum_{k=1}^{\infty} \frac{(2z_0 a + a^2)^k}{k!} \frac{\partial^k}{\partial(z^2)^k} V(z_0^2) , \tag{2.14}$$

so that

$$\lim_{a\to 0} g(a) = 0 . \tag{2.15}$$

Therefore, we get the following lower bound for the partition function:

$$\begin{aligned} Z_N^{(n)}(\beta) \geq (2a)^{Nn} \exp[&-2\,\beta d JNna^2 \\ &- \beta NnV(z_0^2) - \beta Nn g(a) \\ &- \epsilon Nn(z_0 + a)^2] . \end{aligned} \tag{2.16}$$

From (2.7) and (2.16) we get for the free energy

$$\beta^{-1}\ln 2a - 2a^2 dJ - g(a) - \beta^{-1}\epsilon(z_0+a)^2 - V(z_0^2) \leq \beta^{-1}F_N^{(n)}(\beta) \leq -V(z_0^2) - \tfrac{1}{2}\beta^{-1}\ln\epsilon + \tfrac{1}{2}\beta^{-1}\ln\pi \tag{2.17}$$

for all positive a and ϵ. Letting *first* $\beta\to\infty$ and *then* a and $\epsilon\to 0$, we get

$$\lim_{\beta\to\infty}\beta^{-1}F_N^{(n)}(\beta) = -V(z_0^2). \tag{2.18}$$

Now, $F_N^{(n)}(\beta)$ is a convex function of β and the following relation is true:

$$\frac{F_N^{(n)}(\beta) - F_N^{(n)}(\frac{1}{2}\beta)}{\frac{1}{2}\beta} \leq \frac{dF_N^{(n)}(\beta)}{d\beta} \leq \frac{F_N^{(n)}(2\beta) - F_N^{(n)}(\beta)}{\beta}. \tag{2.19}$$

Using (2.18) we therefore obtain the result

$$\lim_{\beta\to\infty}\frac{dF_N^{(n)}(\beta)}{d\beta} = -V(z_0^2), \tag{2.20}$$

so that

$$\begin{aligned} -V(z_0^2) &= \lim_{\beta\to\infty}\frac{dF_N^{(n)}(\beta)}{d\beta} \\ &= \lim_{\beta\to\infty}[-\langle V(n^{-1}|\vec{s}_i|^2)\rangle - dJn^{-1}\langle|\vec{s}_i|^2\rangle + dJn^{-1}\langle\vec{s}_0\cdot\vec{s}_1\rangle] \\ &\leq -\lim_{\beta\to\infty}\langle V(n^{-1}|\vec{s}_i|^2\rangle. \end{aligned} \tag{2.21}$$

In the latter step we used the Schwarz inequality

$$\langle\vec{s}_0\cdot\vec{s}_1\rangle \leq \langle|\vec{s}_0|^2\rangle. \tag{2.22}$$

Therefore,

$$\lim_{\beta\to\infty}\langle V(n^{-1}|\vec{s}_i|^2)\rangle \leq V(z_0^2). \tag{2.23}$$

On the other hand, by hypothesis,

$$\langle V(n^{-1}|\vec{s}_i|^2)\rangle \geq V(z_0^2), \tag{2.24}$$

so that finally,

$$\lim_{\beta\to\infty}\langle V(n^{-1}|\vec{s}_i|^2)\rangle = V(z_0^2). \tag{2.25}$$

If the value z_0^2 which minimizes $V(z^2)$ is uniquely determined, (2.25) requires that

$$\lim_{\beta\to\infty} n^{-1}\langle|\vec{s}_i|^2\rangle = z_0^2. \tag{2.26}$$

When there is more than one value of z^2 such that $V(z^2)$ is minimum, and if z_0^2 is the smallest one, then

$$\lim_{\beta\to\infty} n^{-1}\langle|\vec{s}_i|^2\rangle \geq z_0^2. \tag{2.27}$$

This completes the proof of the lemma.

Consider, as an example, the case[1,5,8] where

$$V(z^2) = (B/2p)z^{2p} + (\tfrac{1}{2}\Delta)z^2, \tag{2.28}$$

where B is positive, Δ a real number, and $p\in\mathfrak{R}$, $p>1$ (p can be ∞). In this case, the displacive limit is defined by $\Delta=0$,[1,5,6] and phase transitions are expected when $\Delta<0$. For Δ negative, $V(z^2)$ reaches its minimum value for

$$z_0^2 = (-\Delta B^{-1})^{1/(p-1)} > 0. \tag{2.29}$$

Theorem 2.1 states that a phase transition exists for all negative values of Δ.

III. ONE-COMPONENT MODEL IN TWO DIMENSIONS

It is well known that in two dimensions the isotropic model defined by (2.1) has no spontaneous magnetization at any finite temperature as long as the number of components n is larger than 1.[15-17] We therefore consider the case $d=2, n=1$. For simplicity look at the following function:

$$V(z^2) = (\tfrac{1}{4}B)z^4 + (\tfrac{1}{2}\Delta)z^2;\quad B>0;\quad \Delta\in\mathfrak{R}. \tag{3.1}$$

In two dimensions, the FSS method fails.[14] Nevertheless, we can prove the following:

Theorem 3.1. Let $d=2$, $n=1$ and $V(z^2)$ given by (3.1). Let J be small enough positive. Then, the model defined by Hamiltonian (2.1) exhibits spontaneous magnetization at finite temperature for all $\Delta<0$.

Proof. Note first that the Lee-Yang theorem[21] is true for $n=1$, $d=2$, and $V(z)$ given by (3.1).[22] Therefore a lower bound on the long-range order parameter is also a lower bound on the spontaneous magnetization. Now, let $J_{ij}=J$ when i and j are nearest-neighbor sites and $J_{ij}=0$ otherwise. The magnetization per site corresponding to (2.1) is given by

$$m(\beta,B,\Delta,J_{ij}) = Z_N^{-1}\int_{\mathfrak{R}^N} s_k\exp\left(-\frac{\beta}{2}\sum_{i,j}J_{ij}(s_i-s_j)^2\right)\prod_{j=1}^N\exp\left(-\frac{\beta B}{4}s_j^4 - \frac{\beta\Delta}{2}s_j^2\right)ds_j. \tag{3.2}$$

Putting $s_i=\lambda^{-1}x_i$, $1\leq i\leq N$, we can rewrite (3.2) as follows:

$$\begin{aligned} m(\beta,B,\Delta,J_{ij}) &= Z_N^{-1}\int_{\mathfrak{R}^N}\lambda^{-1}x_k\exp\left(-\frac{\beta}{2}\lambda^{-2}\sum_{i,j}J_{ij}(x_i-x_j)^2\right)\prod_{j=1}^N\exp\left(-\frac{\beta B\lambda^{-4}}{4}x_j^4 - \frac{\beta\Delta\lambda^{-2}}{2}x_j^2\right)dx_j \\ &= \lambda^{-1}m(\lambda^{-4}\beta, B, \lambda^2\Delta, \lambda^2 J_{ij}). \end{aligned} \tag{3.3}$$

Now, consider the derivative of m with respect to J_{ij}:

$$\frac{\partial}{\partial J_{ij}} m(\beta,B,\Delta,J_{ij}) \equiv f_{ij}(\beta,B,\Delta,J_{ij}) = -\tfrac{1}{2}\beta[\langle x_k(x_i-x_j)^2\rangle - \langle x_k\rangle\langle(x_i-x_j)^2\rangle]. \quad (3.4)$$

Obviously, $f_{ij}=f_{kl}=f$ for all pairs of nearest neighbors. Now fix β, B, Δ. If f is a positive function of J_{ij}, for all J_{ij}, we choose $\lambda>1$ so that

$$m(\beta,B,\Delta;\; J_{ij}) \geq m(\beta,B,\Delta;\; \lambda^{-2}J_{ij}) = \lambda^{-1}m(\lambda^{-4}\beta,B,\lambda^2\Delta;J_{ij}). \quad (3.5)$$

In the latter step we used (3.3). Now (3.5) means that the magnetization at a small value of Δ is bounded below by the magnetization divided by λ, at a larger value of Δ. For $\lambda^{-4}\beta$ and $\lambda^2\Delta$ large enough, Nelson[10] and Kunz and Payandeh[11,12] showed that the magnetization (3.2) is positive and in that case the theorem is proven.

If at a fixed value of β, B and Δ, f is not positive for all J_{ij}, then there are two possibilities:

(i) There is an interval $[0,\tilde{J}], \tilde{J}>0$ such that $f(\beta,B,\Delta;J_{ij})\geq 0$, when the J_{ij} are in $[0,\tilde{J}]$. In that case the argument above can be repeated and the theorem is proved for $J\leq\tilde{J}$, and (ii) $f(\beta,B,\Delta;J_{ij})\leq 0$ in an interval $[0,\overline{J}],\overline{J}>0$. This means that the magnetization of the two-dimensional model is bounded below by the magnetization of the three-dimensional model. By Theorem 2.1 and by the Lee-Yang theorem, Theorem 3.1 is proven. Proof of Theorem 3.1 is therefore completed.

IV. LOWER BOUNDS FOR THE CRITICAL TEMPERATURE OF THE SPIN-s BLUME-CAPEL MODEL

As a simple application of the FSS method[13,14] we shall derive exact lower bounds for the spin-s ($s=\frac{1}{2},1,\frac{3}{2},\ldots$) extension of the Blume-Capel model[18,19] in three or more dimensions. The discrete-spin Hamiltonian is defined as follows:

$$H(\{\mu\}) = -J\sum_{\langle i,j\rangle}\mu_i\mu_j + \Delta\sum_{j=1}^{N}\mu_j^2, \quad (4.1)$$

where μ_k ($1\leq k\leq N$) may assume one of the $2s+1$ values 1, $s^{-1}(s-1)$, $s^{-1}(s-2)$, $\ldots$, $s^{-1}(1-s)$, -1. J is positive and Δ is a real number. If $s=\frac{1}{2}$, (4.1) is the conventional Ising Hamiltonian. Setting

$$\alpha = dJ-\Delta, \quad (4.2)$$

the partition function corresponding to (4.1) reads

$$Z_N^{(s)}(\beta) = \sum_{\{\mu\}}\exp\left(\beta\alpha\sum_{j=1}^{N}\mu_j^2 - \tfrac{1}{2}\beta J\sum_{\langle i,j\rangle}(\mu_i-\mu_j)^2\right), \quad (4.3)$$

where $\{\mu\}$ is the set of all configurations of the μ_k ($1\leq k\leq N$). The analogy with the displacive case is obvious. Here we shall prove that a phase transition exists whenever $\alpha>0$. We first note that

$$Z_N^{(s)}(\beta=0) = (1+2s)^N, \quad (4.4)$$

and that

$$Z_N^{(s)}(\beta) \geq \sum_{\substack{\mu_k=1\\ \text{all } k}}\exp\left(\beta\alpha\sum_{j=1}^{N}\mu_j^2 - \tfrac{1}{2}\beta J\sum_{\langle i,j\rangle}(\mu_i-\mu_j)^2\right) = e^{\beta\alpha N}. \quad (4.5)$$

Moreover, the "free energy" $F_N(\beta)=N^{-1}\ln Z_N^{(s)}(\beta)$ is a convex function of β, so that using (4.4), (4.5), and the Schwarz inequality we get

$$\alpha\langle\mu_i^2\rangle \geq (\alpha-dJ)\langle\mu_i^2\rangle + dJ\langle\mu_0\mu_1\rangle = \frac{dF_N(\beta)}{d\beta} \geq \frac{F_N(\beta)-F_N(0)}{\beta} \geq \alpha - \beta^{-1}\ln(1+2s). \quad (4.6)$$

Therefore, for $\alpha\geq 0$,

$$\langle\mu_i^2\rangle \geq 1-(\beta\alpha)^{-1}\ln(1+2s). \quad (4.7)$$

Now, by FSS,[14] the long-range order parameter c is bounded below by

$$c \geq \langle\mu_i^2\rangle - (2\beta J)^{-1}q(d), \quad (4.8)$$

where

$$q(d) = (2\pi)^{-d}\int_0^{2\pi} d^d\omega\left(d-\sum_{j=1}^{d}\cos\omega_j\right)^{-1}, \quad (4.9)$$

$q(d)$ is finite[23,24] for $d\geq 3$. For instance,[23]

$$q(3) = 0.505\,462\,019\,7\ldots. \quad (4.10)$$

Setting

$$t = \alpha J^{-1}, \quad (4.11)$$

and using (4.7) and (4.8), we obtain the following lower bound for the critical temperature of the model defined by (4.1):

$$kT_c \geq k\tilde{T}_c(t;s) = 2J[q(d)+2t^{-1}\ln(1+2s)]^{-1}. \quad (4.12)$$

Note that $k\tilde{T}_c(t;s)$ is a linear function of t when t is small.

For the spin-s Ising model, that is, when $\Delta=0$ ($t=d$), we even derive a best lower bound for kT_c. In fact, when $\Delta=0$

$$\frac{\partial}{\partial\beta}\langle\mu_k^2\rangle = J\sum_{\langle i,j\rangle}(\langle\mu_k^2\mu_i\mu_j\rangle - \langle\mu_k^2\rangle\langle\mu_i\mu_j\rangle) \geq 0, \quad (4.13)$$

where the last inequality is a consequence of the second Griffith's inequality.[25,26] Therefore,

$$\langle\mu_k^2\rangle_\beta \geq \langle\mu_k^2\rangle_{\beta=0} \text{ for } all\ \beta\geq 0. \quad (4.14)$$

Now,

$$\langle \mu_k^2 \rangle_{\beta=0} = (1+2s)^{-N} \sum_{\{\mu\}} \mu_k^2 = \tfrac{1}{3}(1+s^{-1}) . \tag{4.15}$$

From FSS,[14] the long-range order parameter c is thus bounded below by

$$c(\beta) \geq \langle \mu_k^2 \rangle_\beta - (2\beta J)^{-1} q(d) = \tfrac{1}{3}(1+s^{-1}) + (2\beta J)^{-1} q(d) . \tag{4.16}$$

We then obtain the following lower bound for the critical temperature of the spin-s Ising model ($t=d$):

$$kT_c(s) \geq kT_c^*(s) = 2(1+s^{-1})[3q(d)]^{-1} J . \tag{4.17}$$

It must be emphasized that

$$\lim_{t\to\infty} k\tilde{T}_c(t;s) = \frac{2J}{q(d)} \text{ for } all\ s \tag{4.18}$$

and

$$\lim_{s\to\infty} kT^*(s) = \frac{2J}{3q(d)} . \tag{4.19}$$

However,

$$\lim_{s\to\infty} k\tilde{T}(t;s) = 0 \text{ for } all\ t . \tag{4.20}$$

Therefore, (4.12) is not good enough to prove the existence of a phase transition for small $t=\alpha J^{-1}$ in the limit where $s\to\infty$. Nevertheless, using the technic developed in the proof of Lemma 2.2 it is possible to obtain a lower bound for the critical temperature of the Blume-Capel model when $s\to\infty$ even at small t. When $s\to\infty$ the partition function of the Blume-Capel model can be written[25]

$$Z_N(\beta) = \int_{-1}^{1} \cdots \int \exp\left(\beta\alpha \sum_{j=1}^{N} x_j^2 - \tfrac{1}{2}\beta J \sum_{\langle i,j\rangle} (x_i - x_j)^2\right) \times \prod_{j=1}^{N} dx_j , \tag{4.21}$$

where $x_k \in \mathfrak{R}$, $1\leq k\leq N$. Then

$$Z_N(\beta=0) = 2^N \tag{4.22}$$

and

$$Z_N(\beta) \geq \int_{a}^{1} \cdots \int \exp\left(\beta\alpha \sum_{j=1}^{N} x_j^2 - \tfrac{1}{2}\beta J \sum (x_i - x_j)^2\right) \times \prod_{j=1}^{N} dx_j \geq (1-a)^N \exp[\beta\alpha a^2 N - \tfrac{1}{2}\beta d J (1-a)^2 N] \tag{4.23}$$

for all a, such that $0\leq a\leq 1$. Therefore,

$$\alpha\langle x_i^2\rangle \geq \frac{dF_N(\beta)}{d\beta} \geq \frac{F_N(\beta)-F_N(0)}{\beta} \geq \alpha a^2 - \tfrac{1}{2} d J (1-a)^2 + \beta^{-1} \ln[\tfrac{1}{2}(1-a)] \tag{4.24}$$

for all a such that $0\leq a\leq 1$. Applying once again the FSS result[14] for the long-range order parameter, we get

$$c \geq a^2 - \tfrac{1}{2} d t^{-1}(1-a)^2 + \beta\alpha^{-1}\{\ln[\tfrac{1}{2}(1-a)] + \tfrac{1}{2} t\, q(d)\} , \tag{4.25}$$

so that the critical temperature is bounded below by

$$k\tilde{T}_c(t;\infty;a) J^{-1} = \frac{2a^2 - d t^{-1}(1-a)^2}{q(d) - 2t^{-1}\ln[\tfrac{1}{2}(1-a)]} \tag{4.26}$$

for all a, such that $0\leq a\leq 1$. In the limit where *first* $\alpha\to\infty$ ($t^{-1}\to 0$) and *then* $a\to 1$ we recover result (4.18). When α is finite we have to find the value of a which gives the largest value of $k\tilde{T}_c(\alpha;\infty)$ in (4.26). This is easy to perform numerically. The behavior of $k\tilde{T}_c(t;\infty) \equiv \max_a k\tilde{T}_c(t;\infty;a)$ is given in Fig. 1.

Remark: For the spin-s Ising model ($t=d$) Griffiths[25] obtained the following lower bound:

$$kT_c(s) \geq \tfrac{1}{4} kT_c(\tfrac{1}{2}) . \tag{4.27}$$

For $d=3$, high-temperature series expansions give

$$\tfrac{1}{4} kT_c(\tfrac{1}{2}) \sim 1.1277J . \tag{4.28}$$

Here, we find

$$k\tilde{T}_c(t=3;s) > 1.1277J \quad (s<\tfrac{5}{2}) , \tag{4.29}$$

$$k\tilde{T}_c(t=3;\infty) = 0.6446J . \tag{4.30}$$

Moreover,

$$kT^*(s) = 1.31893(1+s^{-1})J \text{ for } all\ s . \tag{4.31}$$

V. CONCLUSION

We have proven the existence of phase transitions near the displacive limit of a variety of classical, three- or more-dimensional lattice models. In certain cases, we were even able to derive exact lower bounds for the critical temperature. All these results are very simple applications of

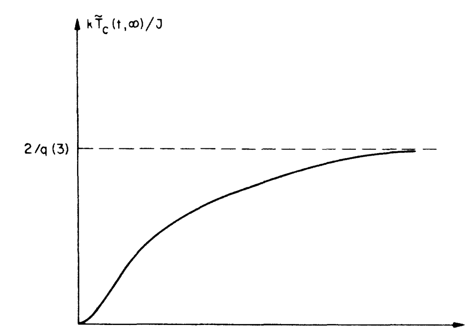

FIG. 1. Behavior of $k\tilde{T}_c(t,\infty) = \max_a k\tilde{T}_c(t;\infty,a)$ as a function of $t=d-\Delta/J$. For small t, $k\tilde{T}_c(t;\infty) \sim g^{-1}(t)$, where $g(T_c) \sim T_c^{-1}\ln T_c$.

the recent method of Fröhlich, Simon, and Spencer. Some results can be extended in two dimensions. Obviously, the FSS method is very powerful and there is no doubt that further results, in particular for quantum-mechanical lattice models, can be derived.

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*Present address: Dept. of Chemistry, Cornell University, Ithaca, N. Y. 14853.

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