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Dynamics of spins interacting with quenched random impurities

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The critical dynamics of the time-dependent Ginzburg-Landau model for a system with quenched random impurities and nonconserved order parameter is studied in the framework of the ϵ expansion. In contrast to the situation in pure systems, the dynamic critical exponent z deviates from its conventional value at first order in $\epsilon \equiv 4-d$. The impurities cause an enhancement of the shape function $f_x(\nu)$ at small frequencies ν; $f_x(\nu=0)$ diverges as $T \rightarrow T_c$. Below T_c the equation of state, static susceptibility χ, and dynamic response function G, are studied. A new, purely static correlated function, $C^{(s)}$, whose existence is unique to the random system is introduced. The coexistence curve singularities of $C^{(s)}$, χ, and G in systems with continuously broken symmetry are explored. The connection of the quenched-impurity model with "model C" of Halperin, Hohenberg, and Ma is discussed.

I. INTRODUCTION

The question of the influence of quenched impurities on phase transitions in magnetic systems is a long standing one.[1] Recent progress in our understanding of the static critical behavior of disordered systems has been made with renormalization-group[2] (RG) techniques. The statics of ferromagnets with random exchange strengths has been treated using the $\epsilon = 4-d$ expansion by Lubensky,[3] Grinstein and Luther,[4] and Khmel'nitskhii,[5] and in two dimensions by Harris and Lubensky.[6] These calculations, all of which deal with "weak" randomness (or systems which are nearly pure), indicate that the phase transition is sharp and second order — qualitatively identical to that of pure systems.[7] The phenomenology of strongly disordered systems on the other hand can be quite different. Dilute model ferromagnets, systems wherein some fraction p of the spins is simply removed from the lattice, are, for example, well known to exhibit the phenomenon of percolation[8]; for p larger than the "percolation concentration" p_c there is no phase transition at finite temperature. Systems with randomly mixed ferromagnetic and antiferromagnetic interactions can undergo transitions to a spin-glass phase at sufficiently low temperature.[9] The average magnetization $[\langle S_i\rangle^2]_{\text{av}}$ vanishes identically in this phase while the spin-glass order parameter is finite.[10]

In this paper we shall be concerned with the time-dependent critical behavior of weakly random systems. The model we treat is defined by the equation of motion satisfied by the n-component spin density $\sigma_i(x,t)$,

$$\frac{\partial\sigma_i(x,t)}{\partial t} = -\Gamma\frac{\delta F}{\delta\sigma_i(x,t)} + \eta_i(x,t), \tag{1.1}$$

where Γ is a bare kinetic coefficient, η_i is a Gaussian noise source,

$$\langle\eta_i(x,t)\rangle = 0, \tag{1.2a}$$

$$\langle\eta_i(x,t)\eta_j(x',t')\rangle = 2\Gamma\delta(x-x')\delta(t-t')\delta_{i,j}, \tag{1.2b}$$

and F is the Landau-Ginzburg free-energy functional generalized to include a random term coupling quadratically to the spin

$$F = \frac{1}{2}\int d^dx\,\{[\nabla\sigma(x)]^2 + r_0\sigma^2(x) + \tfrac{1}{4}u[\sigma^2(x)]^2 + \varphi(x)\sigma^2(x)\}, \tag{1.3}$$

where

$$\sigma^2(x) = \sum_{i=1}^{n}\sigma_i^2(x),$$

$$r_0 = a(T - T_c^0),$$

and a, u are positive constants. The static random noise $\varphi(x)$ describes the quenched impurities and satisfies the configurational averages

$$[\varphi(x)]_{\text{av}} = 0, \tag{1.4a}$$

$$[\varphi(x)\varphi(x')]_{\text{av}} = \Delta\delta(x-x'). \tag{1.4b}$$

This model is the time-dependent Ginzburg-Landau model (TDGL) with a Gaussian static noise φ and a nonconserved order parameter. It is the simplest dynamical model available for treating quenched impurities.[11] Impurity effects are presumably important in more realistic dynamical models (like the isotropic ferromagnet, planar ferromagnets, and isotropic antiferromagnet) as well but the analysis presented here will not touch upon these systems.

The physical quantities of interest will be the linear response functions and the correlation functions. We define the linear response function $G(k,\omega)$ as the response to an infinitesimal magnetic field $h_i(x,t)$ introduced through the addition to the free energy

$$F \rightarrow F' = F - \sum_{i=1}^{n} \int d^d x\, \sigma_i(x) h_i(x,t). \tag{1.5}$$

In particular G is defined by

$$[\langle \sigma_i(k,\omega)\rangle]_{\rm av} = G(k,\omega) h_i(k,\omega) + O(h^2), \tag{1.6}$$

where k and ω label the space and time Fourier components, respectively. The correlation function $C(k,\omega)$ is defined by

$$2\pi\delta(\omega+\omega')\delta_{k,-k'}\delta_{i,j} C(k,\omega) = [\langle \sigma_i(k,\omega)\sigma_j(k',\omega')\rangle]_{\rm av}, \tag{1.7}$$

and can be obtained from the response function via the fluctuation-dissipation theorem[12]

$$C(k,\omega) = (2/\omega)\,{\rm Im}\, G(k,\omega). \tag{1.8}$$

We introduce G because it has a more convenient perturbation-theory expansion than does C. One can also show[13] that the static susceptibility can be obtained as

$$\chi(k) = \int \frac{d\omega}{2\pi} C(k,\omega) = \lim_{\omega\to 0} G(k,\omega). \tag{1.9}$$

According to the dynamical scaling hypothesis[14] one can find a characteristic frequency $\omega_c(k)$ such that in the scaling region

$$C(k,\omega) = [\chi(k)/\omega_c(k)]\, f_x(\nu), \tag{1.10}$$

where $x = k\xi$, $\nu = \omega/\omega_c(k)$, ξ is the correlation length, and $f_x(\nu)$ is the so-called "shape function." The characteristic frequency has the scaling property

$$\omega_c(k) = k^z \Omega(k\xi), \tag{1.11}$$

where z is the dynamical critical index.

The static or equal time correlations of our model are identical to those studied in Refs. 3–5. The fixed-point structure of the RG for the static parameters r_0, u, Δ is in consequence the same as that found in these references.[15] For each value of n there is a single stable fixed point accessible to the system. For $n > n_c \equiv 4 - 4\epsilon + O(\epsilon^2)$ the fixed point which characterizes the critical behavior of the pure system is stable.[7] In this case the impurities have no effect on the phase transition; the dynamics of the random system are identical to those studied in Ref. 11. In particular, z deviates from its "conventional" value of 2 only in $O(\epsilon^2)$. When $n < n_c$, however, the so-called "random fixed point," characterized by nonzero value of Δ, becomes stable.[16] In Sec. II we investigate the structure of the dynamical RG in the vicinity of this fixed point. We find that z deviates from 2 at $O(\epsilon)$; this is the first of several instances in which the impurities exert a strong influence on the dynamics.

In Sec. III we perform a perturbation-theory calculation of the scaling form of the response function to $O(\epsilon)$. Computing the characteristic frequency $\omega_c(k)$ and the shape function $f_x(\nu)$, we find that the impurities lead to an enhancement of $f_x(\nu)$ near $\nu = 0$; $f_x(0)$ grows like x^{z-2}, diverging as $x = k\xi$ becomes very large.

The static and dynamic properties of our model for $T < T_c$ are discussed in Sec. IV. This is, to our knowledge, the first RG discussion of the properties of a system with quenched impurities below T_c. We discuss the equation of state, the static susceptibility, and the dynamic response function, and study a new, purely static spin-spin correlation function $C^{(s)}(x)$ whose existence is peculiar to the random problem. The Nambu-Goldstone modes in the system (for $n > 1$) lead to infrared divergences in the longitudinal susceptibility and response function on the coexistence curve. We analyze these singularities in a manner analogous to that of Brézin *et al.*[17] and Mazenko[18] in studies of pure systems.

The final section is devoted to a short discussion of our results and their relationship to those of the model "C" studied by Halperin, Hohenberg, and Ma (HHM).[19]

II. RENORMALIZATION-GROUP ANALYSIS AND ϵ EXPANSION

A. General considerations

The RG analyses of the TDGL model and the static Ginzburg-Landau model have been extensively discussed in the literature.[2,11] The RG analysis of our model proceeds along identical lines, which we summarize briefly. The model (1.1)–(1.4) is defined by the set of parameters

$$\mu = (r_0, u, \Delta, \Gamma). \tag{2.1}$$

The RG is a set of transformations $\{R_b; 1 \le b < \infty\}$ on the space of μ; R_b transforms the set μ to a new set μ' via the following steps:

(i) Write Eq. (1.1) in terms of Fourier components σ_k. Solve the resulting equations for σ_q with $\Lambda/b < q < \Lambda$, substitute the solutions into the remaining equations for σ_k, $k < \Lambda/b$, and average over η_q, keeping φ fixed throughout.

(ii) In the remaining equations make the substitution

$$\sigma_k(t) \to b^{1-\eta/2}\sigma_{bk}(tb^{-z}). \quad (2.2)$$

These equations are then expressed in the form (1.1), thus enabling us to identify the "renormalized" kinetic coefficient Γ' and the renormalized coefficients $r_0'(x)$ and $u'(x)$, respectively, of the quadratic and quartic terms of F. These new coefficients are functions of r_0 and u_0 and functionals of $\varphi(x)$. Averaging over all possible configurations of the impurity field $\varphi(x)$ we define the renormalized parameters r_0', u', and Δ' as $r_0' \equiv [r_0'(x)]_{av}$, $u' \equiv [u'(x)]_{av}$, and $\delta(x-x')\Delta' \equiv [[r_0'(x) - r_0'][r_0'(x') - r_0']]_{av}$. The set $\mu' \equiv (r_0', \mu', \Delta', \Gamma')$ is thereby determined in terms of μ. The relation

$$\mu' = R_b \mu \quad (2.3)$$

defines the RG transformation R_b.

Just as the RG procedure for statics generates couplings of arbitrarily high order in F,[2] so too do steps (i) and (ii) generate equations of motion far more complicated in structure than (1.1). Many more parameters should be included in the set μ to provide a complete description of R_b. It is, however, straightforward to show (again aping the usual[2,3] arguments for statics) that the four parameters we have considered are sufficient to specify the dynamical critical behavior to $O(\epsilon)$.

B. Recursion relations to $O(\epsilon)$

The parameters r_0, u, and Δ describe static properties. Their transformation under R_b has been worked out by Lubensky.[3] To $O(\epsilon)$ the results are

$$r_0' = b^2\{r_0 + [(\tfrac{1}{2}n+1)u - \Delta]K_4[\tfrac{1}{2}\Lambda^2(1-b^{-2}) - r_0 \ln b]\}, \quad (2.4a)$$

$$u' = b^\epsilon u\{1 - [(\tfrac{1}{2}n+1)u - 6\Delta]K_4 \ln b\}, \quad (2.4b)$$

$$\Delta' = b^\epsilon \Delta\{1 - [(n+2)u - 4\Delta]K_4 \ln b\}, \quad (2.4c)$$

where $K_4 = 1/8\pi^2$. Here we have recorded only the lowest-order terms in u and Δ, and assumed the static index $\eta = O(\epsilon^2)$. Equations (2.4) lead to the familiar quenched static results. In addition to the trivial Gaussian fixed point these relations possess three nontrivial fixed points of $O(\epsilon)$, summarized in Table I. One of these, the "unphysical" or "$n=0$" fixed point is *not accessible* to the real physical system since it has a negative value of Δ^* and Δ is, according to (1.4b), an intrinsically positive quantity. The other two fixed points are the isotropic n-component fixed point which characterizes the critical behavior of the pure n-component system and the "random" fixed point. Only one of these two fixed points is stable for any given value of n; for $n > n_c = 4 - 4\epsilon + O(\epsilon^2)$ the former is stable, while the latter is stable when $n < n_c$. Note from Table I that the random fixed point becomes singular at $n=1$. This is symptomatic of the fact that the random Ising system has a static fixed point of $O(\epsilon^{1/2})$ rather than of $O(\epsilon)$.[5] One must retain cubic terms in the recursion relations (2.4b) and (2.4c) in order to see this $O(\epsilon^{1/2})$ fixed point.

The additional ingredient needed for dynamics is Γ, whose transformation can be obtained by following the above steps defining R_b[20]:

$$\Gamma'^{-1} = b^{2-z}\Gamma^{-1}(1 + K_4\Delta \ln b). \quad (2.5)$$

It follows from (2.5) that the dynamical critical exponent z is given by

$$z = 2 + K_4\Delta^* \quad (2.6)$$

at a fixed point. Thus when $n < n_c$ there are $O(\epsilon)$ corrections to the conventional theory result $z = 2 - \eta$. Since the leading corrections to the conventional result occur at $O(\epsilon^2)$ in pure systems, the presence of quenched impurities has a significant effect on the value of z; to $O(\epsilon)$, for $n < n_c$,

$$z = 2 + \frac{4-n}{8(n-1)}\epsilon + O(\epsilon^2). \quad (2.7)$$

C. Feynman-graph expansion to $O(\epsilon^2)$

At $O(\epsilon^2)$ it becomes inconvenient to compute critical exponents by means of recursion relations since, just as for static calculations in pure sys-

TABLE I. Static fixed points to $O(\epsilon)$.

Fixed point	K_4u^*	$K_4\Delta^*$	Region of stability
"$n=0$" or unphysical	0	$-\frac{1}{4}\epsilon$	all n; but *inaccessible* to true physical system
Isotropic n component	$\frac{2\epsilon}{n+8}$	0	$n > n_c \equiv 4 - 4\epsilon + O(\epsilon^2)$ (see Ref. 7)
Random	$\epsilon/2(n-1)$	$(4-n)\epsilon/8(n-1)$	$n < n_c$ (not defined for $n > n_c$)

tems,[2] one must include σ^6 terms in the Hamiltonian in order to find the correct $O(\epsilon^2)$ fixed point. To carry out the $O(\epsilon^2)$ computation we utilize Wilson's Feynman-graph-expansion technique.[21] The method can be briefly summarized as follows: Starting with the equation of motion in an external magnetic field we find, after Fourier transformation, the form

$$\sigma_i(q,\omega)=\sigma_i^0(q,\omega)+G^0(q,\omega)h_i(q,\omega)+G^0(q,\omega)L^{-d/2}\sum_{q'}\varphi_{q-q'}\sigma_i(q',\omega)$$
$$-G^0(q,\omega)(\tfrac{1}{2}u)L^{-d}\sum_{q',q''}\int\frac{d\omega\,d\omega'}{(2\pi)^2}\sum_j\sigma_j(q',\omega')\sigma_j(q'',\omega'')\sigma_i(q-q'-q'',\omega'-\omega'')\,, \tag{2.8}$$

where

$$\sigma_i^0(q,\omega)=G^0(q,\omega)(1/\Gamma)\eta_i(q,\omega) \tag{2.9}$$

and

$$G^0(q,\omega)=\frac{1}{-i\omega/\Gamma+\chi_0^{-1}(q)}, \tag{2.10}$$

with $\chi_0^{-1}(q)=q^2+r_0$. Treating $\varphi\sim O(\epsilon^{1/2})$ and $u\sim O(\epsilon)$ [in the case $n=1$ we treat $\varphi\sim O(\epsilon^{1/4})$ and $u\sim O(\epsilon^{1/2})$] we can iterate σ in powers of σ_i^0 and G^0h, keeping only one power of h. After averaging over the noise and the quenched impurity configurations we generate a graphical perturbation expansion for the impurity-averaged response function. This expansion is a double power series in u and Δ; assuming that a fixed point of the RG transformation in $4-\epsilon$ dimensions exists we expect, following Wilson,[21] that there exist particular values, u_c and Δ_c [both of $O(\epsilon)$], of u and Δ for which the perturbation series for the response function has the dynamical scaling form[14] appropriate to a system near criticality. With u and Δ set equal to u_c and Δ_c, respectively, the values of critical exponents and, in principle, the scaling function itself can be obtained as a power series in ϵ. The problem of calculating critical exponents to any order in ϵ is thus reduced to that of evaluating Feynman graphs and determining u_c and Δ_c. (Note, however, that if values u_c and Δ_c for which the response function acquires the scaling form *cannot* be found, we must conclude that dynamical scaling is violated. We shall have more to say about this eventuality later, in connection with HHM model C.)

Since the static critical exponents at the random fixed point have been computed[3-5] to $O(\epsilon^2)$, only z remains to be determined. It suffices to study the response function right at the critical temperature where, according to dynamic scaling, $G^{-1}(q=0,\omega)$ has the form $\omega^{(2-\eta)/z}$. As in the usual static calculations,[21] it is most convenient to perform "mass renormalization." We first write $r_0\equiv r+\delta r$, where r is the exact inverse static susceptibility and vanishes at the true transition temperature. We can then write the bare response function as $G_0=(-i\omega/\Gamma+k^2+r)^{-1}$. The "mass counterterm" r is then determined, order by order in perturbation theory, by the requirement that $G^{-1}(q=0,\omega=0)$ be identically equal to r. Since we are working at T_c, r can be set to zero. Writing

$$G(q,\omega)=G_0^{-1}(q,\omega)-\Sigma(q,\omega), \tag{2.11}$$

we find that Σ is given to second order in u_c and Δ_c by the diagrams in Fig. 1. A straightforward evaluation of the graphs yields

$$G^{-1}(q=0,\omega)=-\frac{i\omega}{\Gamma}\left\{1+\ln\left(-\frac{i\omega}{\Gamma}\right)\left[-\frac{K_d\Delta_c}{2}-\frac{\frac{3}{4}(n+2)\ln\frac{4}{3}}{(K_du_c)}-\tfrac{5}{4}(K_d\Delta_c)^2+\frac{n+2}{2}(K_du_c)(K_d\Delta_c)\right]\right.$$
$$\left.+\ln^2\left(-\frac{i\omega}{\Gamma}\right)\left[\frac{K_d\Delta_c\epsilon}{8}+\tfrac{5}{8}(K_d\Delta_c)^2-\frac{n+2}{8}(K_du_c)(K_d\Delta_c)\right]\right\}. \tag{2.12}$$

Since u_c and Δ_c are both of $O(\epsilon)$, this expression is of the form $(-i\omega/\Gamma)^{(2-\eta)/z}$ to $O(\epsilon^2)$ provided only that

$$\frac{1}{2}\left(\frac{K_d\Delta_c}{2}\right)^2=\frac{K_d\Delta_c\epsilon}{8}+\frac{(K_d\Delta_c)^2}{8}-\frac{n+2}{8}(K_du_c)(K_d\Delta_c). \tag{2.13}$$

A straightforward calculation exactly analogous to that performed by Wilson[21] gives u_c and Δ_c. We find that at the random fixed point,

$$K_du_c=\frac{\epsilon}{2(n-1)}\left(1+\frac{25n^2-248n+64}{128(n-1)^2}\epsilon\right), \tag{2.14a}$$

$$K_d\Delta_c=\frac{\epsilon}{8(n-1)}\left(4-n-\frac{105n^3-364n^2+992n-256}{128(n-1)^2}\epsilon\right) \tag{2.14b}$$

when $n \neq 1$, and

$$K_d u_c = 4(\tfrac{2}{159}\epsilon)^{1/2}, \tag{2.14c}$$

$$K_d \Delta_c = 3(\tfrac{2}{159}\epsilon)^{1/2} \tag{2.14d}$$

when $n=1$. It is trivial to verify that these values do satisfy condition (2.13); dynamical scaling thus holds to $O(\epsilon^2)$ with

$$z = 2 - \eta + K_d\Delta_c + \tfrac{3}{4}(n+2)(K_d u_c)^2 \ln\tfrac{4}{3} + \tfrac{3}{2}(K_d\Delta_c)^2 - \tfrac{1}{2}(n+2)(K_d u_c)(K_d\Delta_c). \tag{2.15}$$

Substituting (2.14) and the $O(\epsilon^2)$ value of η (known from statics) into (2.15) we obtain

$$z = 2 + \frac{(4-n)\epsilon}{8(n-1)} + \frac{[192(n+2)(n-1)\ln\frac{4}{3} - 69n^3 + 104n^2 - 640n + 128]\epsilon^2}{1024(n-1)^3} \tag{2.16a}$$

for $n \neq 1$ and

$$z = 2 + (\tfrac{6}{53}\epsilon)^{1/2} \tag{2.16b}$$

for $n=1$.

III. CORRELATION FUNCTION TO $O(\epsilon)$ FOR $T \geq T_c$

In this section we carry out a perturbation-theory calculation, valid in the scaling region, for the correlation function. This amounts to keeping the first-order self-energy graphs in Fig. 2 and setting u and Δ to their fixed-point values while keeping q, ω, and r finite. We obtain directly from Fig. 2 that, to $O(\epsilon)$, the self-energy defined by (2.11) is given by

$$\Sigma(q,\omega) = \Sigma_H + \Sigma_I, \tag{3.1a}$$

where

$$\Sigma_H = -\frac{u}{2}(n+2)\int \frac{d^4\bar{q}\, d\bar{\omega}}{(2\pi)^5} C^0(\bar{q},\bar{\omega}) \tag{3.1b}$$

is the usual Hartree term and

$$\Sigma_I(\omega) = \Delta \int \frac{d^4\bar{q}}{(2\pi)^4} G^0(\bar{q},\omega) \tag{3.1c}$$

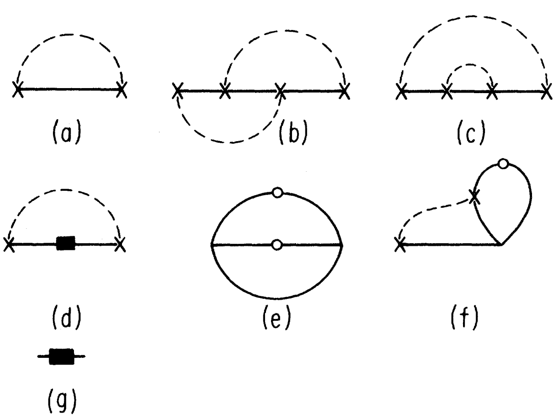

FIG. 1. Graphs contributing to the frequency-dependent self-energy $\Sigma(q,\omega)$ to $O(\epsilon^2)$. The crosses represent impurities, two crosses joined by a dotted line represent a factor of Δ, solid lines with arrows represent bare response functions, lines with open circles represent bare correlation functions, four point vertices represent factors of u, and the shaded squares represent factors of δr.

is the contribution due to impurities. Note in particular that Σ_I depends on frequency but not on wave number. It is convenient to separate out the static part of the self-energy, $\Sigma_s(q) = \Sigma(q, \omega=0)$, which is given by

$$\Sigma_s(q) = \left(\Delta - \frac{u}{2}(n+2)\right)\frac{K_4}{2}\left[\Lambda^2 - r_0 \ln\left(\frac{r_0+\Lambda^2}{r_0}\right)\right]. \tag{3.2}$$

We can then introduce the exact susceptibility

$$r = G^{-1}(0,0) = r_0 - \Sigma_s(0) \tag{3.3}$$

and consistently eliminate r_0 in favor of r to $O(\epsilon)$. We then have

$$G^{-1}(q,\omega) = q^2 + r - \Sigma(q,\omega) + \Sigma(0,0) - i\omega/\Gamma, \tag{3.4}$$

where

$$\Sigma(q,\omega) - \Sigma(0,0) = \Delta(i\omega/\Gamma)Q(\omega) \tag{3.5}$$

and

$$Q(\omega) = \frac{K_4}{2}\left\{\ln\frac{\Lambda^2}{-i\omega/\Gamma + r} - \frac{r}{-i\omega/\Gamma}\left[\ln\left(\frac{r+\Lambda^2}{r}\right) - \ln\left(\frac{-i\omega/\Gamma + r + \Lambda^2}{-i\omega/\Gamma + r}\right)\right]\right\}. \tag{3.6}$$

We finally obtain

$$G^{-1}(q,\omega) = \chi^{-1}(q) - i\omega/\bar{\Gamma}(q,\omega), \tag{3.7}$$

where, to $O(\epsilon)$,

$$\bar{\Gamma}(q,\omega) = \Gamma[1 - \Delta Q(\omega)]. \tag{3.8}$$

In the scaling region where $\omega/\Gamma\Lambda^2 \ll 1$, $r/\Lambda^2 \ll 1$ we can write

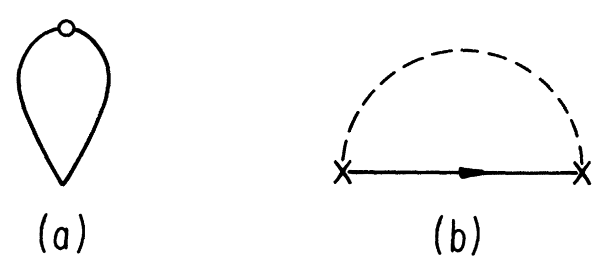

FIG. 2. Graphs contributing to $\Sigma(q,\omega)$ to $O(\epsilon)$.

$$\bar{\Gamma}(q,\omega)=\Gamma\left[1-\frac{\Delta K_4}{2}\ln\left(\frac{\Lambda^2}{r}\right)+\frac{K_4\Delta}{2}\left(1+\frac{\Gamma r}{-i\omega}\right)\ln\left(1-i\frac{\omega}{\Gamma r}\right)\right]. \tag{3.9}$$

As this expression stands it is not well defined in the limit where the two variables ω and r go to zero due to the logarithms. The scaling properties for $\bar{\Gamma}$,

$$\Gamma(q,\omega,\xi^{-1})=b^{-z+2-\eta}\Gamma(qb,\omega b^z,b\xi^{-1}), \tag{3.10}$$

where ξ is the correlation length, follow from the RG result

$$G^{-1}(q,\omega,\xi^{-1})=b^{2-\eta}G^{-1}(qb,\omega b^z,b\xi^{-1}) \tag{3.11}$$

and Eq. (3.7). If we successively set two of the three variables q,ω,ξ^{-1} to zero and make appropriate choices for b we obtain the scaling relations

$$\bar{\Gamma}(q,0,0)=(q/\Lambda)^{z-2+\eta}\bar{\Gamma}(\Lambda,0,0), \tag{3.12a}$$

$$\bar{\Gamma}(0,\omega,0)=(\omega/\Gamma\Lambda^2)^{(z-2+\eta)/z}\Gamma(0,\Gamma\Lambda^2,0), \tag{3.12b}$$

$$\bar{\Gamma}(0,0,\xi^{-1})=(\xi\Lambda)^{-z+2-\eta}\Gamma(0,0,\Lambda). \tag{3.12c}$$

After performing two exponentiations we can write correct to $O(\epsilon)$,

$$\bar{\Gamma}(q,\omega)=\Gamma(\Lambda\xi)^{-y}(1-i\nu_0)^{y/2}\times\left(1+\frac{y}{-2i\nu_0}\ln(1-i\nu_0)\right), \tag{3.13}$$

where we have set $r=\xi^{-2}$ and defined $y=K_4\Delta=z-2$ and $\nu_0=\omega/\Gamma r$. Thus

$$\bar{\Gamma}(0,\omega,0)=\Gamma(-i\omega/\Gamma\Lambda^2)^{y/2}, \tag{3.14a}$$

$$\bar{\Gamma}(0,0,\xi^{-1})=\Gamma(1+\tfrac{1}{2}y)(\Lambda\xi)^{-y}. \tag{3.14b}$$

Since $(z-2-\eta)/z=\frac{1}{2}y+O(\epsilon^2)$ and $-y=-z+2+O(\epsilon^2)$, (3.14a) and (3.14b) are consistent with (3.12b) and (3.12c), respectively. Equation (3.12a), on the other hand, is indeterminate since $\bar{\Gamma}$ is independent of q. It reduces to $0=0$ since $\bar{\Gamma}(\Lambda,0,0)$ is zero according to (3.13). The vanishing of $\bar{\Gamma}$ for small ω and large ξ will be discussed further below.

In order to write the response function in the dynamical scaling form it is convenient to introduce a characteristic frequency. Following the discussion of Freedman and Mazenko[22] we define

$$\omega_c(q,\xi)=\Gamma\chi_0^{-1}(q)\frac{(1+x^2)^{y/2}}{(\Lambda\xi)^y}, \tag{3.15}$$

which can be written

$$\omega_c(q,\xi)=\Gamma\Lambda^{-y}\xi^{-y/2}(1+x^2)^{1+y/2}=\Gamma\Lambda^{-y}\xi^{-z}(1+x^2)^{z/2}. \tag{3.16}$$

In the limit $x\to 0, q\to 0$,

$$\omega_c(\xi)=\Gamma\Lambda^{-y}\xi^{-z}, \tag{3.17a}$$

while in the limit $x\to\infty, T\to T_c$,

$$\omega_c(q)=\Gamma\Lambda^{-y}k^z. \tag{3.17b}$$

We now deal with the scaled frequency $\nu=\omega/\omega_c$. In $\bar{\Gamma}$, to lowest order in ϵ, we can replace ν_0 by $\nu(1+x^2)$ so that

$$G^{-1}(q,\omega)=\chi^{-1}(q,\xi)[-i\nu/F_x(\nu)+1], \tag{3.18}$$

with

$$F_x(\nu)=[(1+x^2)^{-1}-i\nu]^{y/2}\times\left(1+\frac{y}{(-2i\nu)(1+x^2)}\ln[1-i\nu(1+x^2)]\right). \tag{3.19}$$

The correlation function $C(q,\omega)$ can be obtained from the response function via Eq. (1.8). Writing our result in the scaling form (1.10) we can identify the shape function as

$$f_x(\nu)=\frac{2F_x'(\nu)}{[\nu-F_x''(\nu)]^2+[F_x'(\nu)]^2}, \tag{3.20}$$

where F' and F'' are the real and imaginary parts of F, respectively. Let us first consider the case $\nu=0$. Then

$$F_x(0)=(1+x^2)^{-y/2}(1+y/2) \tag{3.21}$$

so that $F_x''=0$ and

$$F_x(0)=\frac{2}{1+\frac{1}{2}y}(1+x^2)^{y/2}. \tag{3.22}$$

In the hydrodynamic regime this is simply $2/(1+\frac{1}{2}y)$ while in the critical regime we find $f_x(0)\to 2x^y/(1+\frac{1}{2}y)$. The shape function evidently blows up at $\nu=0$ as $T\to T_c$; this divergence is a direct result of the presence of the impurities since $y\sim\Delta^*$. Examining the shape function at T_c we see that

$$F_\infty(\nu)=(-i\nu)^{y/2}=(\cos\tfrac{1}{2}\pi y-i\sin\tfrac{1}{2}\pi y)\nu^{y/2}, \tag{3.23}$$

and

$$f_\infty(\nu)=\frac{2\cos(\frac{1}{4}\pi y)\nu^{y/2}}{\nu^2+2\nu^{1+y/2}\sin\frac{1}{2}\pi y+\nu^y}, \tag{3.24}$$

or near $\nu=0$,

$$f_\infty(\nu)=\frac{2\cos\frac{1}{2}\pi y}{\nu^{y/2}}. \tag{3.25}$$

Note that this divergence at small ν does not violate any sum rules since $f_\infty(\nu)$ is still integrable under the assumption that $\frac{1}{2}y<1$.

The physical basis for this small-ν divergence rests with the quenched or "frozen" nature of the impurities. The fluctuations can scatter from these impurities much like Bragg scattering in a solid. Since the impurities do not move the interaction between the impurities and very-low-frequency fluctuations can be arbitrarily long in duration. Thus the probability of two low-frequency fluctuations simultaneously interacting with the same impurity is large and leads to the build up of correlation for $\omega\sim 0$ that is reflected in the shape function. We expect an enhancement of this effect as $T\to T_c$ and "critical slowing down" develops.

Without the quenched impurities, the shape function for the TDGL model is simply $\{1+\nu^2[1+O(\epsilon^2)]\}^{-1}$, which does not diverge at $\nu=0$. The details of the $O(\epsilon^2)$ terms can be found in Ref. 11. The case of annealed impurities with very long relaxation times (i.e., nearly frozen) is discussed in Sec. V.

IV. PROPERTIES BELOW T_c

A. Magnetization

For $T<T_c$ we expect a nonzero average magnetization in zero external field. Assuming the spontaneous field points in the "1" direction we have

$$M=[m_1(x)]_{\text{av}}, \tag{4.1}$$

where $m_i(x)\equiv\langle\sigma_i(x)\rangle$. It is useful to work in terms of the field

$$\psi_i(x,t)=\sigma_i(x,t)-M\delta_{i,1}, \tag{4.2}$$

whose average value $[\langle\psi_i(x,t)\rangle]_{\text{av}}$ vanishes identically and to define the correlation function

$$C_{ij}(x-x',t-t')\equiv[\langle[\sigma_i(x,t)-m_i(x)]\times[\sigma_j(x',t')-m_j(x')]\rangle]_{\text{av}}, \tag{4.3}$$

which is the analog in the random system of the correlation function commonly treated in pure systems. A unique feature of the quenched problem is the existence of a second correlation function which becomes long ranged as T approaches T_c from below. This correlation function, a purely static quantity, is defined by

$$\delta_{i,1}\delta_{j,1}C^{(s)}(x-x')\equiv[[m_i(x)-M\delta_{i,1}][m_i(x')-M\delta_{j,1}]]_{\text{av}}. \tag{4.4}$$

A third correlation function,

$$C'_{ij}(x-x',t-t')\equiv C_{ij}(x-x',t-t')+\delta_{i,1}\delta_{j,1}C^{(s)}(x-x'), \tag{4.5}$$

is simply $[\langle\psi_i(x,t)\psi_j(x',t')\rangle]_{\text{av}}$. The Fourier transform $C'_{ij}(k,\omega)$ is the dynamic structural factor observable by scattering experiment, which contains a $\delta(\omega)C^{(s)}(k)$ piece. We shall also be concerned with the response function

$$G_{ij}(x-x',t-t')=\delta[\langle\psi_i(x,t)\rangle]_{\text{av}}/\delta h_j(x',t)\,|_{h=0}. \tag{4.6}$$

Note that since one established the fluctuation-dissipation theorem for any given fixed configuration, $\varphi(x)$ of the impurities, the theorem relates G to C rather than to C'. Because the isotropy of spin space is preserved in the presence of impurities, G_{ij} is diagonal in i and j and we define

$$G_i=\begin{cases}G_L & \text{for } i=1,\\ G_\perp & \text{for } i\neq 1.\end{cases} \tag{4.7}$$

Consider the situation where there is a small uniform magnetic field pointing in the 1 direction. A static uniform δh perpendicular to h is effectively a rotation of h by an angle $\delta h/h$. Thus the total magnetization must follow the same rotation, i.e., $[\langle\sigma_i(x,t)\rangle]_{\text{av}}/M=\delta h/h$. We then have

$$G_\perp(k=0,\omega=0)=[\langle\sigma_i(x,t)\rangle]_{\text{av}}/\delta h=M/h. \tag{4.8}$$

This result is well known for the pure system, and is preserved here because the impurities do not affect the invariance of the Hamiltonian under spin rotations.

B. Equation of state

The equation of state relating r_0, h, and M can be obtained by calculating $[\langle\psi_1\rangle]_{\text{av}}$ with fixed m and h and then setting the result to zero as required by (4.2). In the case $n\gtrsim 2$, the equation of state can also be obtained by calculating $G_\perp(0,0)$ and then setting the result to M/h as in (4.8). We perform a straightforward perturbation expansion in φ and u using the bare response functions

$$G_L^0(k)=(r+k^2)^{-1},\qquad G_\perp^0(k)=(h/M+k^2)^{-1}, \tag{4.9}$$

where $r\equiv G_L(0)^{-1}$ and $h/M=G_\perp(0)^{-1}$. Note that for purposes of calculating the equation of state we have set all frequency variables to zero.

The expression for $[\langle\psi_1\rangle]_{av}$ to first order in ϵ is represented graphically in Fig. 3, where each wiggly line represents a factor of M. Although both u and Δ are of $O(\epsilon)$, the quantity uM^2 is readily seen to be of $O(1)$,[23] so some care is required in accounting for all diagrams which contribute to $O(\epsilon)$. A direct evaluation of the graphs in Fig. 3 leads to the expression

$$r_0+\frac{uM^2}{2}-\frac{h}{M}=\frac{K_4\Delta}{2}\left[\Lambda^2-r\ln\left(\frac{\Lambda^2}{r}\right)\right]-\frac{K_4u}{4}\left[(n+2)\Lambda^2-3r\ln\left(\frac{\Lambda^2}{r}\right)-(n-1)\frac{h}{M}\ln\left(\frac{\Lambda^2M}{h}\right)\right]$$
$$-\frac{3K_4\Delta}{4}uM^2\left[-1+\ln\left(\frac{\Lambda^2}{r}\right)\right]. \tag{4.10}$$

The value, r_0^c, of r_0 corresponding to T_c is determined by setting M, r, and h to zero:

$$r_0^c=\tfrac{1}{2}K_4\Delta\Lambda^2-\tfrac{1}{4}K_4u(n+2)\Lambda^2. \tag{4.11}$$

Defining $\tau\equiv|r_0-r_0^c|\sim|T-T_c|$ we obtain

$$-\tau+\frac{uM^2}{2}-\frac{h}{M}+\frac{K_4}{2}(\tfrac{3}{2}u-\Delta)r\ln\left(\frac{r}{\Lambda^2}\right)-\tfrac{3}{4}K_4\Delta uM^2\ln\left(\frac{r}{\Lambda^2}\right)-\tfrac{3}{4}K_4\Delta uM^2+\frac{n-1}{4}K_4u\frac{h}{M}\ln\left(\frac{h}{M\Lambda^2}\right)=0. \tag{4.12}$$

By eliminating r from (4.12) in favor of τ, M, and h we obtain the equation of state. Only the $O(1)$ expression

$$r=-\tau+\tfrac{3}{2}uM^2 \tag{4.13}$$

is needed to produce the equation of state to $O(\epsilon)$:

$$(h/M)\{1-(\tfrac{3}{4}K_4u-\tfrac{1}{2}K_4\Delta)\ln(\tfrac{3}{2}uM^2-\tau)-\tfrac{1}{4}K_4u(n-1)\ln(\tfrac{1}{2}uM\;\;-\tau)\}$$
$$=-\tau+\tfrac{1}{2}uM^2[1+(\tfrac{3}{2}K_4u-\tfrac{5}{2}K_4\Delta)\ln(\tfrac{3}{2}uM^2-\tau)]. \tag{4.14}$$

The critical exponents β and δ are readily extracted to $O(\epsilon)$ from (4.14); setting $h=0$ we obtain $\tau\sim M^{1/\beta}$ with

$$\beta=\tfrac{1}{2}(1-\tfrac{3}{2}K_4u+\tfrac{5}{2}K_4\Delta), \tag{4.15a}$$

while with $\tau=0$ one finds $h\sim M^{\delta}$ where

$$\delta=3+\tfrac{1}{2}\frac{K_4u(n+8)}{2}-6K_4\Delta. \tag{4.15b}$$

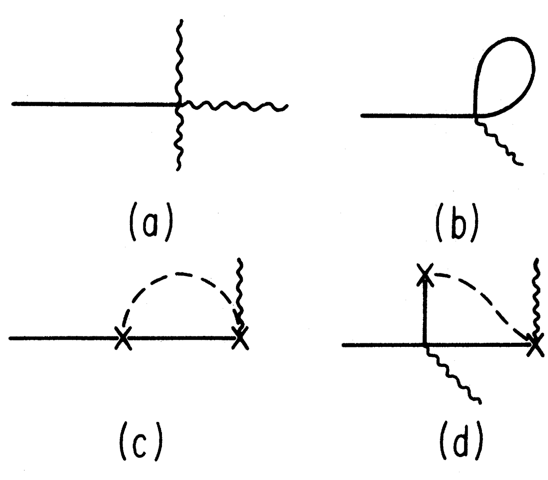

FIG. 3. Graphs contributing to $[\langle\psi_1\rangle]_{av}$ to $O(\epsilon)$. Solid lines represent bare static propagators and each wiggly line represents a factor of M.

Substitution of the critical values for K_4u and $K_4\Delta$ given in (2.14) yields, in agreement[3-5] with scaling laws,

$$\beta=\frac{1}{2}\left(1+\frac{(8-5n)\epsilon}{16(n-1)}+O(\epsilon^2)\right), \tag{4.16a}$$

$$\delta=3+\epsilon+O(\epsilon^2) \tag{4.16b}$$

when $n\neq1$, and

$$\beta=\tfrac{1}{2}[1-\tfrac{1}{2}(\tfrac{6}{53}\epsilon)^{1/2}+O(\epsilon)], \tag{4.16c}$$

$$\delta=3+O(\epsilon) \tag{4.16d}$$

for $n=1$.

Equation (4.14) is easily cast in the scaling form

$$h/M^{\delta}=\tfrac{1}{2}uf[\tau/\tfrac{1}{2}uM^2)^{1/2\beta}], \tag{4.17a}$$

where the scaling function $f(x)$ is

$$f(x)=1-x-\tfrac{3}{2}K_4\Delta+(1-x)[\tfrac{1}{4}K_4u(n+8)-3K_4\Delta]\ln(u/2\Lambda^2)$$
$$+(\tfrac{3}{2}K_4u-\tfrac{5}{2}K_4\Delta)(\ln|3-x|-x\ln\Lambda^2)$$
$$+(1-x)[\tfrac{1}{4}K_4u(n-1)\ln|1-x|$$
$$+\tfrac{1}{2}K_4(\tfrac{3}{2}u-\Delta)\ln|3-x|]+O(\epsilon^2). \tag{4.17b}$$

This function has the behavior depicted schematically in Fig. 4; the zero of f at $x_0\equiv1+O(\epsilon)$ ensures that $M\sim\tau^{\beta}$ in the absence of a field.

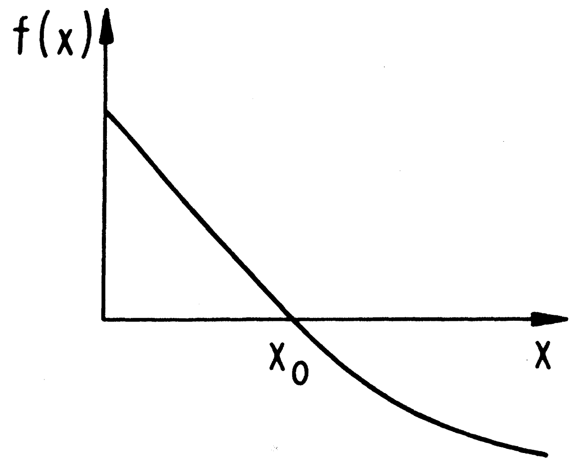

FIG. 4. Schematic representation of the equation-of-state scaling function $f(x)$.

C. Response functions—perturbation theory

In developing our perturbation theory below T_c it is convenient to eliminate σ in favor of the field ψ, defined in (4.2). The equation of motion

$$\frac{1}{\Gamma}\frac{\partial\psi_i(x,t)}{\partial t}=-\delta_{i,1}M[r_0+\tfrac{1}{2}uM^2-\varphi(x)]-h_i(x,t)+\frac{1}{\Gamma}\eta_i(x,t)+\int\chi_i^{-1}(x-\bar{x})\psi_i(\bar{x},t)\,d\bar{x}-\varphi(x)\psi_i(x,t)+3Mu_{i1jk}\psi_j(x,t)\psi_k(x,t)+u_{ijkl}\psi_j(x,t)\psi_k(x,t)\psi_l(x,t), \quad (4.18a)$$

with

$$u_{ijkl}\equiv-\tfrac{1}{6}u(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \quad (4.18b)$$

$$\chi_i^{-1}(x-\bar{x})\equiv[r_0-\nabla^2+uM^2(\tfrac{1}{2}+\delta_{i,1})\delta(x-\bar{x}), \quad (4.18c)$$

and repeated subscripts j,k,l summed from 1 to n is the direct analog for $T<T_c$ of (2.8); iterated in powers of u and φ with zeroth-order propagator $G_i^{0-1}(q,\omega)=-i\omega/\Gamma+\chi_i^{-1}(q)$, it gives rise to a graphical perturbation expansion for the response function G similar to that discussed in Sec. II C, the only additional feature here being the presence of the wiggly lines representing factors of M. All

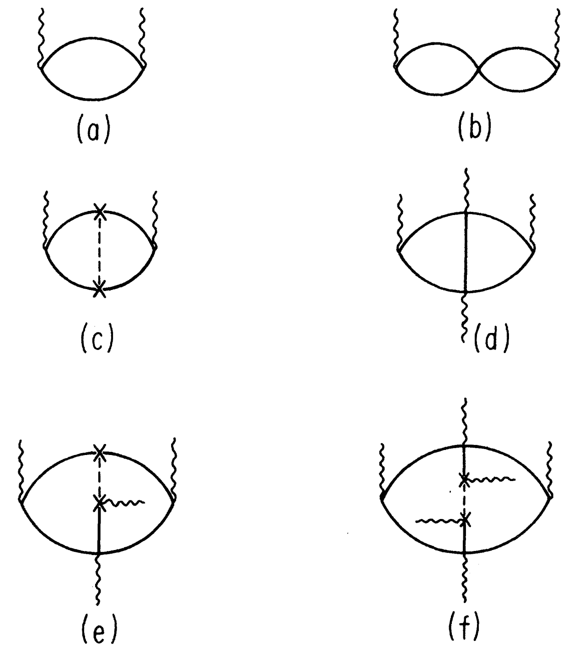

FIG. 5. Graphs contributing to the self-energy $\Sigma_i(q,\omega)$ to $O(\epsilon)$ below T_c. Each wiggly line represents a factor of M.

graphs contributing to $\Sigma_i(q,\omega)$, the self-energy for G, to $O(\epsilon)$ are shown in Fig. 5. The analytic expressions corresponding to these graphs are given in the appendix.

D. Static correlation functions

The static correlation functions can be calculated as the $\omega=0$ limits of the response functions $\chi_i^{-1}(q)=G_i^{-1}(q,0)=\chi_i^{0-1}-\Sigma_i(q,0)$.

1. Longitudinal correlation function

The longitudinal susceptibility can be written

$$\chi_L^{-1}(q)=r_0+\tfrac{3}{2}uM^2+q^2-\Sigma_L(q,0), \quad (4.19)$$

where $\Sigma_L(q,0)$ is given in the Appendix. Evaluating the integrals in the appendix and making use of the zero-field equation of state (4.14) we obtain

$$\chi_L^{-1}(q)=A(q)+\tfrac{1}{4}(n-1)K_4(uM)^2\ln(q^2/uM^2), \quad (4.20)$$

where

$$A(q)=q^2+uM^2\left\{1-W\left[1+\ln\left(\frac{\Lambda^2}{uM^2}\right)\right]+\frac{(x^2+4)^{1/2}}{x}\left(6K_4\Delta-\tfrac{9}{8}K_4u+\frac{9K_4\Delta}{x^2+4}\right)\ln\left[\left(1+\frac{x^2}{4}\right)^{1/2}-\frac{x}{2}\right]\right\}, \quad (4.21)$$

with $x^2\equiv q^2/uM^2$ and $W\equiv\frac{1}{4}K_4[(n+8)u-12\Delta]$.

This expression can be exponentiated to yield

$$A(q)=\Lambda^{-2W}(u^{1/2}M)^{\gamma/\beta}[1+f_R(q^2/(uM^2)^{\gamma/\beta})], \quad (4.22)$$

where to $O(\epsilon)$, $\gamma/\beta=2\gamma/\beta=2W$, in agreement with Refs. 3–5, and

$$f_R(x^2)=x^2-W+Wx^2\ \ln\Lambda^2+\frac{(x^2+4)^{1/2}}{x}\left(6K_4\Delta-\tfrac{9}{8}K_4u+\frac{9K_4\Delta}{x^2+4}\right)\ln\left[\left(1+\frac{x^2}{4}\right)^{1/2}-\frac{x}{2}\right]. \quad (4.23)$$

Equation (4.20) is thus conveniently written

$$\chi_L^{-1}(q)=\Lambda^{-2W}(u^{1/2}M)^{\gamma/\beta}[f_R(x^2)+1+\tfrac{1}{2}(n-1)K_4u\ \ln x]. \quad (4.24)$$

The term proportional to $n-1$ is evidently divergent as $q\to 0$ for all $T<T_c$. This divergence is a manifestation of the coexistence-curve singularities which are a consequence of the Nambu-Goldstone bosons present in a state with continuously broken symmetry. Such singularities have been treated in pure systems by several authors.[17,18]

Dealing with the term proportional to $\ln x$ (or $\ln q$) is always somewhat delicate since there is no unambiguous prescription for exponentiating this term. In order to reduce the ambiguity somewhat we evaluate to $O(\epsilon^2)$ the $(\ln x)^2$ pieces of $\chi_L^{-1}(q)$. These contributions can be obtained from the second-order self-energy graphs of Fig. 6. Straightforward evaluation of these diagrams yields

$$\chi_L^{-1}(q)=\Lambda^{-2W}(u^{1/2}M)^{\gamma/\beta}\times[f_R(x^2)+1+\tfrac{1}{4}\epsilon\ \ln x-\tfrac{1}{16}\epsilon^2\ \ln^2 x], \quad (4.25)$$

where we have used the $O(\epsilon)$ critical value for K_4u given in (2.14a). Following the treatment of Mazenko[18] we try to fit this expression with the form

$$\chi_L^{-1}(q)=\Lambda^{-2W}(u^{1/2}M)^{\gamma/\beta}\left(f_R(x^2)+\frac{a}{C+x^{-\sigma\epsilon}}\right) \quad (4.26)$$

with a, c, and σ constants of $O(1)$. It is trivial to show that the choices $\sigma=1$, $c=3$, and $a=4$ do produce an exact fit of (4.25) to $O(\epsilon^2)$, whereupon we finally obtain

$$\chi_L^{-1}(q)=\Lambda^{-2W}(u^{1/2}M^{\gamma/\beta})\left(f_R(x^2)+\frac{4}{3+x^{-\epsilon}}\right). \quad (4.27)$$

When $n=1$ the coexistence-curve singularities are absent since there are no Nambu-Goldstone modes and $\chi_L^{-1}(q)$ takes on the perfectly orthodox scaling form $\Lambda^{-2W}(u^{1/2}M)^{\gamma/\beta}f_R(x^2)$.

2. *Transverse correlation function*

The transverse correlation function is given by

$$\chi_\perp^{-1}(q)=q^2+r_0+\tfrac{1}{2}uM^2-\Sigma_\perp(q,0), \quad (4.28)$$

where the six $O(\epsilon)$ contributions to $\Sigma_\perp(q,0)$ are given in the Appendix. Straightforward evaluation and use of the equation of state yields

$$\chi_\perp^{-1}(q)=q^2-K_4uM^2\left[\left(\Delta-\frac{u}{2}\right)\left(\frac{\alpha(x)}{x^2}+\alpha(x)+x^2\right)+\frac{\Delta\alpha(x)}{2x^2}\right], \quad (4.29)$$

with $\alpha(x)\equiv\ln(1-x^2)-x^2$. For small q,

$$\chi_\perp^{-1}(q)=q^2[1+\tfrac{1}{8}K_4(u-\Delta)], \quad (4.30)$$

which explicitly exhibits the expected Nambu-Goldstone modes.

3. *Function $C^{(s)}(q)$*

Recall that the function $C^{(s)}(x-x')$, defined in Eq. (4.4), is unique to the random problem. Its Fourier transform, $C^{(s)}(q)$, is expected to be very different from the quantity $\chi_L(q)$, which we have already studied. To lowest order in Δ and u we have

$$C^{(s)}(q)=(r+q^2)^{-2}M^2\Delta \quad (4.31)$$

and in general it is convenient to write

$$C^{(s)}(q)=\chi_L(q)^2D(q), \quad (4.32)$$

where $D(q)$ is represented diagrammatically by the shaded blob in Fig. 7(a). The zeroth- and first-order terms of D are shown in Figs. 7(b) and 7(c), respectively; since D is a purely static object it is easiest to adopt the graphical convention used in Sec. IV B, where in the presence of a magnetic field the bare propagators are given by Eq. (4.9). Evaluation of the diagrams in Fig. 7 then gives rise to the expression

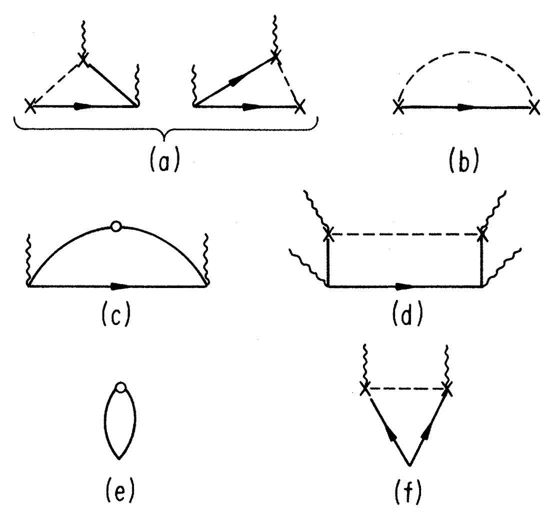

FIG. 6. Graphs contributing $(\ln x)^2$ terms of $O(\epsilon^2)$ to $\chi_L^{-1}(q)$.

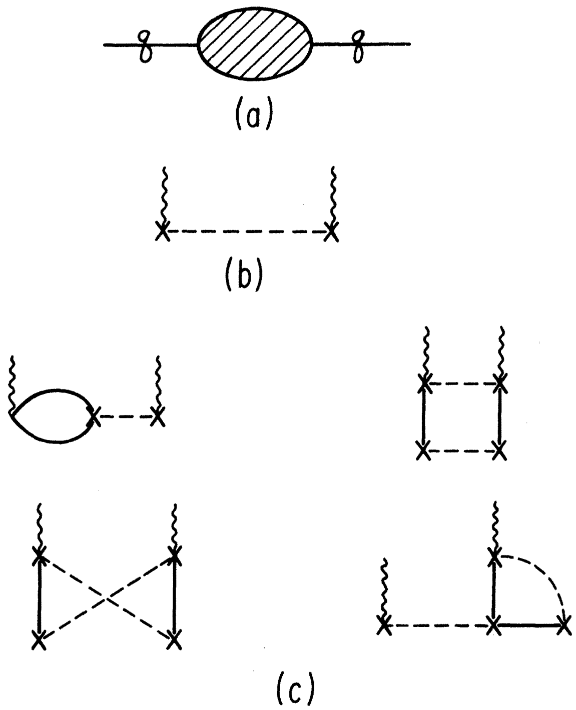

FIG. 7. Graphs contributing to $D(q)$ to $O(\epsilon)$.

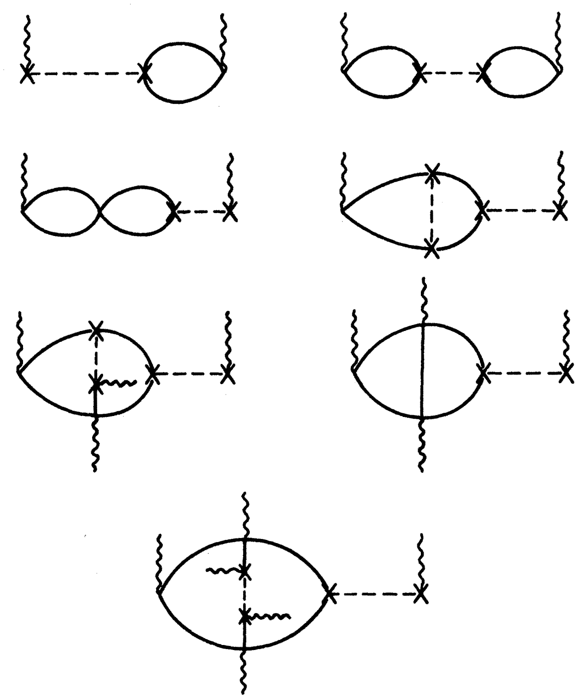

FIG. 8. Graphs contributing $(\ln q)^2$ terms of $O(\epsilon^2)$ to $D(q)$.

$$D(q)=M^2\Delta[1-\tfrac{1}{2}K_4(4\Delta-3u)\ln(r/\Lambda^2)+(3\Delta-3u)Q(r/q^2)-K_4u(n-1)Q(h/Mq^2)+\tfrac{1}{2}K_4u(n-1)\ln(h/M\Lambda^2)], \tag{4.33a}$$

where

$$Q(x^2)=K_4\left[(1+4x^2)^{1/2}\ln\left(\frac{(1+4x^2)^{1/2}-1}{2x}\right)+1\right]. \tag{4.33b}$$

For $n=1$ we obtain, setting $r=uM^2+h/M$,

$$D(q)=M^2\Delta\left[1-\left(\frac{6\epsilon}{53}\right)^{1/2}Q\left(\frac{uM^2}{q^2}(1+y)\right)+O(\epsilon)\right], \tag{4.34a}$$

where $y\equiv h/uM^6$. For $n>1$ we find

$$D(q)=M^{2+\epsilon}\Delta\left[1+\frac{\epsilon}{4}\ln\left(\frac{u^2y(1+y)}{\Lambda^2}\right)-\frac{3n\epsilon}{8(n-1)}Q\left(\frac{uM^2}{q^2}(1+y)\right)-\frac{\epsilon}{2}Q\left(\frac{uM^2}{q^2}y\right)\right]. \tag{4.34b}$$

Like $\chi_L^{-1}(q), D(q)$ contains $\ln q$ terms which signal the presence of coexistence-curve singularities. As before, the logarithms can be exponentiated in many possible ways. Putting $h=y=0$ we proceed as before and compute the $(\ln q)^2$ terms of $O(\epsilon^2)$ contributing to $D(q)$. All such contributions come from the graphs in Fig. 8, keeping only terms singular as $q\to 0$ we obtain

$$D_{\text{sing}}(q)=\Delta M^2\left(1+\frac{\epsilon}{2}\ln q-\frac{\epsilon^2}{16}\ln^2 q\right), \tag{4.35}$$

Once again assuming that this expression should be fitted by the form $a/(C+q-\sigma^{\epsilon})$ we find $\sigma=\frac{5}{4}$, $C=\frac{3}{2}$, and $a=\frac{5}{2}$. Thus

$$D_{\text{sing}}(q)=\Delta M^2\frac{\frac{5}{2}}{\frac{3}{2}+q^{-5\epsilon/4}}, \tag{4.36}$$

whence it follows that for small q

$$C^{(s)}(q)\sim q^{-2\epsilon+5\epsilon/4}\sim q^{-3\epsilon/4}. \tag{4.37}$$

E. Response functions

We turn now to a discussion of the frequency-dependent response functions. For simplicity we will restrict our analysis to the case $q=0$ where we expect the collective effects to be largest in the response functions as a function of frequency.

1. Longitudinal case

Consider first the longitudinal case. Starting with the diagrams given by Fig. 5 we find, after using the equation of state,

$$
\begin{aligned}
G_L^{-1}(0,\omega) \\
&= -\frac{i\omega}{\Gamma} + uM^2 + 3\Delta uM^2K_4\left[\ln\left(\frac{\Lambda^2}{-i\omega/\Gamma + r_L}\right) + \frac{r_L}{-i\omega/\Gamma}\ln\left(\frac{r_L}{-i\omega/\Gamma + r_L}\right)\right] \\
&\quad -\frac{\Delta K_4}{2}\left[r_L\ln\left(\frac{\Lambda^2}{r_L}\right) - \left(\frac{-i\omega}{\Gamma}+r_L\right)\ln\left(\frac{\Lambda^2}{-i\omega/\Gamma+r_L}\right)\right] - \frac{9M^2u^2K_4}{2(i\omega/\Gamma)}\left[r_L\ln\left(\frac{-i\omega/2\Gamma + rL}{r_L}\right) + \frac{i\omega}{2\Gamma}\ln\left(\frac{\Lambda^2}{-i\omega/2\Gamma + r_L}\right)\right] \\
&\quad -\tfrac{1}{4}(n-1)u^2M^2K_4\ln\left(\frac{2\Gamma\Lambda^2}{-i\omega}\right) - \frac{9\Delta u^2M^4K_4}{2(-i\omega/\Gamma)}\left[\ln\left(\frac{\Lambda^2}{r_L}\right) - 1 - \ln\left(\frac{\Lambda^2}{-i\omega/\Gamma + r_L}\right) - \frac{r_L}{-i\omega/\Gamma}\ln\left(\frac{r_L}{-i\omega/\Gamma+r_L}\right)\right]. \qquad (4.38)
\end{aligned}
$$

We can rewrite this equation in the form [correct to $O(\epsilon)$]

$$G_L^{-1}(0,\nu_0) = uM^2(uM^2/\Lambda^2)^{\epsilon/2}\{1 + (-i\nu_0)(1+\tfrac{1}{2}\Delta K_4\ln\Lambda^2/uM^2) + Q_L(\nu_0) + \tfrac{1}{4}(n-1)uK_4\ln(-\tfrac{1}{2}i\nu_0)\}, \qquad (4.39a)$$

$$Q_L(\nu_0) = -\Delta K_4(3-\tfrac{1}{2}i\nu_0)\frac{\Omega}{(-i\nu_0)}\ln\Omega - \frac{9uK_4}{2i\nu_0}\Omega^1\ln\Omega^1 - \frac{9\Delta K_4}{2(-i\nu_0)^2}(\Omega\ln\Omega + i\nu_0), \qquad (4.39b)$$

where

$$\Omega = 1 - i\nu_0, \qquad (4.39c)$$

$$\Omega^1 = 1 - \tfrac{1}{2}i\nu_0, \qquad (4.39d)$$

and

$$\nu_0 = \frac{\omega}{\Gamma uM^2(uM^2/\Lambda^2)^{\epsilon/2}}. \qquad (4.39e)$$

We note that $Q_L(0) = \frac{9}{4}K_4(u - \frac{7}{3}\Delta)$. The $(uM^2/\Lambda^2)^{\epsilon/2}$ factor in $G_L^{-1}(0,\nu)$ comes from the same type graphs as for $\chi_L^{-1}(q)$. A useful choice for a characteristic frequency is

$$\omega_L^c = \Gamma uM^2(uM^2/\Lambda^2)^{\epsilon/2 + K_4\Delta/2}. \qquad (4.40)$$

Since $\omega_L^c \sim M^{2+\epsilon+K_4\Delta}$ and $M \sim \xi^{-\beta/\nu}$ [to $O(\epsilon^2)$] we find, as expected, that $\omega_L^c \sim \xi^{-z}$ since $\nu/\beta = 1 + \frac{1}{2}\epsilon$ and $z\nu/\beta = 2 + \epsilon + K_4\Delta$. We then find that

$$\nu = \omega/\omega_L^c = \nu_0(uM^2/\Lambda)^{-K_4\Delta/2} \qquad (4.41)$$

so $\nu = \nu_0$ to lowest order in ϵ. We can then write, correct to $O(\epsilon)$,

$$G_L^{-1}(0,\nu) = \xi^{-2}[-i\nu + Q_L(\nu) + 1 + \tfrac{1}{4}(n-1)uK_4\ln(-\tfrac{1}{2}\nu)], \qquad (4.42)$$

where we have defined

$$\xi^{-2} = uM^2(uM^2/\Lambda^2)^{\epsilon/2}. \qquad (4.43)$$

We can write Eq. (4.42) in the more conventional form [correct to $O(\epsilon)$]

$$G_L^{-1}(0,\nu) = \xi^{-2}\left(\frac{-i\nu}{\gamma_L(\nu)} + 1 + Q_L(0) + \tfrac{1}{4}(n-1)uK_4\ln(-\tfrac{1}{2}i\nu)\right), \qquad (4.44)$$

where

$$\gamma_L(\nu) = 1 - (1/-i\nu)[Q_L(\nu) - Q_L(0)]. \qquad (4.45)$$

We note that $\gamma_L(0) = 1 + (\frac{1}{16}K_4)(20\Delta - 9u)$. In the case $n=1$ one can easily check that $G_L^{-1}(0,0) = \chi_L^{-1}(0)$ in agreement with our previous results. We see in that case that the physical kinetic coefficient

$$\bar{\Gamma}_L^{-1} \equiv \frac{\partial}{\partial(-i\omega)}G_L^{-1}(0,\omega)\bigg|_{\omega=0} \qquad (4.46)$$

is related to $\gamma_L(0)$ by

$$\bar{\Gamma}_L = \Gamma(uM^2/\Lambda^2)^{\epsilon/2}\gamma_L(0). \qquad (4.47)$$

In the case $n>1$ we can treat the term proportional to $\ln(-i\nu)$ in exactly the same manner as was used in treating the $\ln q^2$ term in $\chi_L^{-1}(q)$. We obtain the result

$$G_L^{-1}(0,\nu) = \xi^{-2}\left(\frac{-i\nu}{\gamma_L(\nu)} + \frac{8[1+Q_L(0)]}{7 + (-\frac{1}{2}i\nu)^{-\epsilon}}\right), \qquad (4.48)$$

which is in agreement with dynamical scaling.

2. Transverse case

Consider now the transverse case. It is convenient to make the rearrangements

$$G_\perp^{-1}(q,\omega) = -i\omega/\Gamma - \Sigma_\perp(q,\omega) + \Sigma_\perp(q,0) + \chi_\perp^{-1}(q), \tag{4.49}$$

$$\Sigma_\perp(q,\omega) - \Sigma_\perp(q,0) = (i\omega/\Gamma)Q_\perp(q,\omega), \tag{4.50}$$

and

$$G_\perp^{-1}(q,\omega) = -(i\omega/\Gamma)[1+Q_\perp(q,\omega)] + \chi_\perp^{-1}(q) \tag{4.51}$$

$$= -i\omega/\Gamma_\perp(q,\omega) + \chi_\perp^{-1}(q), \tag{4.52}$$

where the physical wave-number- and frequency-dependent kinetic coefficient is given by

$$\Gamma_\perp(q,\omega) = \Gamma[1 - Q_\perp(q,\omega)]. \tag{4.53}$$

We easily find in the small-q limit, after some surprising cancellations, that

$$Q_\perp(0,\omega) = \tfrac{1}{2}K_4\Delta[\ln(\Lambda^2/uM^2) - 1] - \tfrac{1}{2}K_4 u \ln\tfrac{1}{2}, \tag{4.54}$$

and we see that $Q_\perp(0,\omega)$ is independent of ω. We can write $\Gamma_\perp(q,\omega)$ in the scaling form

$$\Gamma_\perp(0,\omega) = \frac{\Gamma(uM^2)^{K_4\Delta/2}}{\Lambda^2}\left(1 + \frac{K_4\Delta}{2} + \frac{K_4 u}{2}\ln\frac{1}{2}\right). \tag{4.55}$$

Remembering that for small q

$$\chi_\perp^{-1}(q) = q^2[1 + \tfrac{1}{8}K_4(u-\Delta)], \tag{4.56}$$

we can define the characteristic frequency

$$\omega_\perp^c(q) = \Gamma_\perp(0,0)\chi_\perp^{-1}(q) \tag{4.57a}$$

$$= q^z(x\Lambda)^{-K_4\Delta}\{1 + \tfrac{1}{2}K_4[u(\ln\tfrac{1}{2}+\tfrac{1}{4}) + \tfrac{3}{4}\Delta]\}; \tag{4.57b}$$

and write, for small q and $\nu = \omega/\omega_\perp^c$

$$G_\perp(q,\omega) = [\chi_\perp(q)/\omega_\perp^c(q)](-i\nu+1)^{-1}, \tag{4.58}$$

which gives a simple Lorentzian for the shape function.

V. ANNEALED IMPURITIES WITH LONG RELAXATION TIMES

The random impurities considered in the above discussion have been assumed to be fixed and follow a Gaussian probability distribution. Now suppose that we allow the impurities to move, although extremely slowly. In other words, consider annealed impurities with very long relaxation times. As soon as we allow the impurities to move, the static critical behavior (observed over an infinite length of time) is qualitatively the same as that without impurities. The impurities are just additional degrees of freedom which can be integrated out.

If the impurity relaxation time is very long compared to the time of observation of the dynamic phenomena of interest, then for these phenomena the impurities act just like quenched impurities with a probability distribution determined by thermal equilibrium. Such a distribution is affected by the spin fluctuations in contrast to the fixed Gaussian distribution discussed earlier. How would the earlier results be modified? Let us now treat φ of (1.3) and (1.4) as a new degree of freedom. Define the correlation function and response function in the absence of the $\varphi\sigma^2$ interaction as

$$K_0(k,\omega) = 2\gamma\Delta/(\gamma^2+\omega^2),$$
$$F_0(k,\omega) = \Delta\gamma/(\gamma - i\omega). \tag{5.1}$$

In the limit $\gamma \to 0$, K_0 is, in the space-time representation,

$$K_0(x-x', t-t') \equiv \langle\varphi(x,t)\varphi(x',t')\rangle_0$$
$$= \Delta e^{-\gamma|t-t'|} \approx \Delta. \tag{5.2}$$

Under the RG, Δ now transforms according to

$$\Delta' = b^\epsilon\Delta\{1 + [-(n+2)u + (\tfrac{1}{2}n+4)\Delta]K_4\ln b\}, \tag{5.3}$$

while u still follows (2.4b). Let

$$\bar{u} \equiv u - \Delta. \tag{5.4}$$

Then we obtain from (5.3) and (2.4b) that, for $\gamma \to 0$,

$$\bar{u}' = \bar{u}b^\epsilon\{1 - \bar{u}(\tfrac{1}{2}n+4)K_4\ln b\}, \tag{5.5}$$

$$\Delta' = \Delta b^\epsilon\{1 - [(n+2)\bar{u} + (\tfrac{1}{2}n-2)\Delta]K_4\ln b\}. \tag{5.6}$$

Equation (5.5) is familiar in statics, and involves no Δ. It gives the fixed-point value $\bar{u}^* = (8\pi^2)^{-1}2\epsilon/(n+8)$ for statics. For $n>4$, we have the stable fixed point with $\Delta^*=0$. For $n<4$, there is no stable fixed point. (Note that we require $\Delta \geq 0$.) This conclusion holds also when $\gamma = Dk^2$, which is the model C of HHM, as long as $\gamma \to 0$. In other words, there cannot be a fixed point with $D^*=0$. In conclusion, we see that the quenched model and HHM "C" are different. The dynamics for the quenched model obey dynamical scaling [at least to $O(\epsilon^2)$] while HHMC "C" does not.[24]

APPENDIX: $\Sigma_i(q,\omega)$ BELOW T_c

The self-energy $\Sigma(q,\omega)$ to $O(\epsilon)$ below T_c is given by the graphs in Fig. 5. Denoting the respective contributions of Figs. 5(a)–5(f) by $\Sigma_i^{(\alpha)}$, where α runs from 1 to 6, we have

$$\Sigma_i(q,\omega) = \sum_{\alpha=1}^{6}\Sigma_i^{(\alpha)}(q,\omega),$$

where

$$\Sigma_i^{(1)}(q,\omega) = -2\Delta u M^2(1+\delta_{i,1})\int \frac{d^4\bar{q}}{(2\pi)^4} G_i^0(q-\bar{q},\omega)\chi_i(\bar{q}), \tag{A1}$$

$$\Sigma_i^{(2)}(q,\omega) = \Delta \int \frac{d^4\bar{q}}{(2\pi)^4} G_i(\bar{q},\omega), \tag{A2}$$

$$\Sigma_L^{(3)}(q,\omega) = (uM)^2 \int \frac{d^4\bar{q}}{(2\pi)^4}\left(\frac{9\chi_L(\bar{q})}{-i\omega/\Gamma+\chi_L^{-1}(\bar{q})+\chi_L^{-1}(q+\bar{q})} + \frac{(n-1)\chi_\perp(\bar{q})}{-i\omega/\Gamma+\chi_\perp^{-1}(\bar{q})+\chi_\perp^{-1}(q+\bar{q})}\right), \tag{A3a}$$

$$\Sigma_\perp^{(3)}(q,\omega) = (uM)^2 \int \frac{d^4\bar{q}}{(2\pi)^4}\frac{\chi_L(\bar{q})+\chi_\perp(q+\bar{q})}{-i\omega/\Gamma+\chi_L^{-1}(\bar{q})+\chi_\perp^{-1}(q+\bar{q})}, \tag{A3b}$$

$$\Sigma_i^{(4)}(q,\omega) = \Delta(uM^2)^2(1+8\delta_{i,1})\int \frac{d^4\bar{q}}{(2\pi)^4}\chi_i(q+\bar{q})\chi_L^2(\bar{q}), \tag{A4}$$

$$\Sigma_i^{(5)}(q,\omega) = -\tfrac{1}{2}u(1+2\delta_{i,j})\int \frac{d^4\bar{q}}{(2\pi)^4}\chi_j(\bar{q}), \tag{A5}$$

$$\Sigma_i^{(6)}(q,\omega) = -\tfrac{1}{2}u\Delta M^2(1+2\delta i,1)\pi_{11}(0). \tag{A6}$$

The subscripts L and $\perp$ represent [as in Eq. (4.9)] the longitudinal and transverse components, respectively, while

$$\pi_{ij}(q) = \int \frac{d^4\bar{q}}{(2\pi)^4}\chi_i(\bar{q})\chi_j(q+\bar{q}). \tag{A7}$$

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[14]B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).

[15]We shall use the same formalism and notation as Lubensky (Ref. 3).

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