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Time evolution of a Bose system passing through the critical point

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Kinetics of Bose condensation of an ideal Bose gas and of the classical Bose gas of hard spheres is considered. Deviations from the equilibrium state are not assumed to be small. An analytic solution for the problem of an ideal Bose gas is found and the role of collisions between the Bose particles themselves is discussed qualitatively. It is shown that Bose condensation is a slowing down process which can be completed at $t \to \infty$ while the formation of the high-energy tail of the distribution function is relatively fast. The transition time is finite if the nuclei of new phase are present in the beginning of the cooling process.

I. INTRODUCTION

Bose condensation of an ideal Bose gas is one of the classic phenomena studied in quantum-statistical mechanics.[1] It is well known that the distribution function of Bose particles is given by the Bose-Einstein law:

$$n(E) = (e^{(E-\mu)/T} - 1)^{-1} \tag{1}$$

with the chemical potential $\mu \leq 0$.

The total number of particles per cm^3, N_0, is thus given by the expression [the factor $(\sqrt{2}\pi^2)^{-1}$ is omitted hereafter]

$$N_0 = \frac{m^{3/2}}{\hbar^3} \int n(E)\sqrt{E}\, dE \tag{2}$$

and from this μ can be evaluated. One can see that μ increases with decrease of temperature T and at a certain $T = T_c$ one finds $\mu(T_c) = 0$. T_c is called the temperature of Bose condensations (critical temperature). Below T_c the equilibrium distribution function is

$$N(E) = \frac{\hbar^3 N_0}{m^{3/2}} \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] \frac{\delta(E)}{\sqrt{E}} + (e^{E/T} - 1)^{-1} \tag{1'}$$

and it is obvious that $[\delta(E)/\sqrt{E}]dE \propto \delta(p)d^3p$, where p is the momentum of the Bose particle. A finite number of particles is collected at the energy level $E = 0$ forming the Bose condensate.

Although the thermodynamical description is not perfect, because it can be applied only to systems in thermal equilibrium, the thermodynamics of Bose condensation is understood quite well.

The time evolution of the system from one equilibrium state to another can be studied only within the framework of kinetics theory. It is interesting to study the steps of the Bose condensation process. Does it take place instantaneously after μ reaches zero or not? Does the system pass through the critical point by a set of quasiequilibrium states or is the evolution an entirely nonequilibrium one? What are the measurable parameters during the relaxation through the critical point? Which kinds of interactions are important at $T = T_c$?

In this work we present an attempt to develop the kinetic theory of Bose condensation of an ideal Bose gas and of a gas of hard spheres. The hard-spheres gas will be considered without taking into account the transformation of the energy spectrum of the system below the critical point[2] and thus our gas model can be called the classical gas of hard spheres. We realize that this model can hardly reflect all the properties of real quantum liquids below their critical points but we hope that this model can serve to some extent to increase understanding of the kinetics of the λ transition in ^{4}He. The time evolution of ^{4}He through the critical point has not been studied thoroughly but some very interesting features of this process can be extracted from the classical works.[3-5] In this paper we will use some of the ideas which have been developed recently for the kinetics Bose condensation of the gas of noninteracting photons.[6-9]

II. KINETIC EQUATION

Let us consider the Bose gas interacting with the heat bath which is chosen to be a Fermi gas of infinite heat capacity. One can write the Hamiltonian of such a system

$$\mathcal{H}=\sum_{p} E_{\vec{p}}a^{\dagger}_{\vec{p}}a_{\vec{p}}+\sum_{\vec{p}_1+\vec{p}_2=\vec{p}_1'+\vec{p}_2'}\langle\vec{p}_1\vec{p}_2\,|\,U\,|\,\vec{p}_1'\vec{p}_2'\rangle\, a^{\dagger}_{\vec{p}_1}a^{\dagger}_{\vec{p}_2}a_{\vec{p}_2'}a_{\vec{p}_1'}$$

$$+\sum_{\vec{p}_1+\vec{p}_2=\vec{p}_1'+\vec{p}_2'}\langle\vec{p}_1\vec{p}_2\,|\,V\,|\,\vec{p}_1'\vec{p}_2'\rangle\, a^{\dagger}_{\vec{p}_1}b^{\dagger}_{\vec{p}_2}a_{\vec{p}_1'}b_{\vec{p}_2'}, \tag{3}$$

where $a^{\dagger}_{\vec{p}}, a_{\vec{p}}$ and $b^{\dagger}_{\vec{p}}b_{\vec{p}}$ are the creation-annihilation operators of the Bose gas and heat bath, respectively. The matrix elements $\langle\vec{p}_1\vec{p}_2\,|\,V\,|\,\vec{p}_1'\vec{p}_2'\rangle$ and $\langle\vec{p}_1\vec{p}_2\,|\,U\,|\,\vec{p}_1'\vec{p}_2'\rangle$ correspond to Bose-Bose and Bose-bath collisions, respectively. Assuming, as is usually done[2,10] that the momentum change per collision is small, one can replace the above matrix elements by the constants U_0 and V_0, respectively. We should mention that the conservation of momentum is assumed to be fulfilled in all the sums in Eq. (3), i.e., $\vec{p}_1+\vec{p}_2=\vec{p}_1'+\vec{p}_2'$.

We consider the Bose condensation of the ideal Bose gas or of the classical Bose gas of hard spheres; thus we will take the energy spectrum of the Bose system to be

$$E_p=p^2/2m$$

or

$$E_p=p^2/2m+(4\pi\hbar^2 a/m). \tag{4}$$

The second spectrum takes into account in a model way the collisions between the particles of the Bose gas.[1] Using the well-known procedure[11] one can derive the kinetic equation for the occupation numbers of Bose particles $n(E,t)$:

$$\frac{\partial n}{\partial t}=I_{BB}+I_{BT}, \tag{5}$$

with

$$I_{BT}=-\frac{U_0^2}{\hbar}\sum\delta(\Delta p)\,\delta(\Delta E)\times[n_{p_1}n_{p_1'}(N_{p_2}-N_{p_2'})+n_{p_1}N_{p_2}(1-N_{p_2'})+n_{p_1'}N_{p_2'}(1-N_{p_2})], \tag{6}$$

$$I_{BB}=-\frac{V_0^2}{\hbar}\sum\delta(\Delta p)\,\delta(\Delta E)[n_{p_1}n_{p_2}(1+n_{p_1'})(1+n_{p_2'})-n_{p_1'}n_{p_2'}(1+n_{p_1})(1+n_{p_2})], \tag{7}$$

where I_{BT} and I_{BB} are the collision integrals corresponding to Bose-bath and Bose-Bose interactions, respectively; $N_p=\{\exp[E-\mu_F)/T]+1\}^{-1}$ are the occupation numbers of the heat-bath particles. $\delta(\Delta p)$ and $\delta(\Delta E)$ take into account the momentum and energy conservation per collision. It should be emphasized that the kinetic equations [Eqs. (5)–(7)] were derived in the random-phase approximation and thus have the limits of validity:

$$\tau>\hbar/\epsilon, \tag{8}$$

where τ is the characteristic time of the distribution function due to the kinetic process and ϵ is the energy per particle. Moreover it is possible to show that the very concept of occupation numbers $n=\langle a^{\dagger}a\rangle$ is not correct[11] if Eq. (8) is not fulfilled. We will show further that such a complication appears only at the final stages of Bose condensation when the average energy per particle in the condensing phase is $\epsilon\simeq 10^{-27}$ erg, and therefore in this work we shall consider the Bose condensation process within the framework of Eqs. (5)–(7), leaving open the question of the kinetics when $E\to 0$.

Let us note the important properties of Eqs. (5)–(7):

(a) $I_{BB}=0$ and $I_{BT}=0$ with $n(E)$ from Eq. (1). It is obvious that $I_{BT}=0$ only if the temperatures of the Bose system and of the heat bath are equal.

(b) $I_{BB}=0$ and $I_{BT}=0$ with the distribution function Eq. (1′).

$$\text{(c)}\quad \sum_{\vec{p}_1}I_{BB}(E_{p_1})=\sum_{\vec{p}_1}I_{BT}(E_{p_1})=0, \tag{9}$$

while

$$\sum_{\vec{p}_1}I_{BB}(E_{p_1})E_{p_1}=0\quad\text{and}\quad\sum_{\vec{p}_2}I_{BT}(E_{p_1})E_{p_1}\neq 0. \tag{9a}$$

Based on Eqs. (9) and (9a) one can write the equation for the time evolution of the total energy of the Bose gas:

$$\frac{dE(t)}{dt}=\sum_{\vec{p}_1}I_{BT}(E_{p_1})E_{p_1}\,. \tag{10}$$

Evidently the Bose-Bose collisional term $I_{BB}(E_p)$ cannot affect the total energy of the system. This fact, which seems obvious at first sight, leads to important conclusions about the structure of the solution of Eq. (5).

In order to elucidate the role of I_{BB} in Eq. (5) let us take the initial state of the Bose system to be

$$n(E,0)=a\delta(E-E_0)/\sqrt{E} \tag{11}$$

with E_0 far from 0.

It is easy to understand that the collisions between the Bose gas atoms lead to the following (the interaction with the heat bath we, for the time being, assume to be very weak): some of the particles move to the direction of energy higher than E_0 while the others are scattered toward the low energies. The resulting distribution should be the Bose one with temperature corresponding to the initial energy of the system. This energy is assumed to be conserved during such a process.

Let us assume further that in the initial state the distribution function is very narrow and situated near $E=0$:

$$n(E,0)=a\delta(E-\epsilon)/\sqrt{E}\,, \tag{11a}$$

where ϵ is very small.

Because of condition (9) almost no particles can be scattered to the high-energy direction if ϵ is small enough. Actually, such a scattering leads to an increase of the total energy of the system and to the violation of Eq. (9). Moreover, one can check that

$$n=a\delta(E)/\sqrt{E}$$

is the solution of Eq. (5). Based on the above general considerations, we conclude that collisions between the Bose particles cannot distort significantly the distribution function if it is narrow enough and located near $E=0$.

We know that Bose condensation is the process in which a finite number of particles is concentrated at $E=0$. Thus, let us assume that at the final stages of the process [when $n(E)$ is narrow enough], I_{BB} is not able to distort the form of the distribution function and let us try to solve the simpler equation:

$$\frac{\partial n}{\partial t}=I_{BT}. \tag{12}$$

After that one will be able to substitute the solution of Eq. (12) into Eq. (5) and to check the limits of validity of the simplification Eq. (12). We are interested in the process of the appearance of the Bose condensate and let us take the Bose system to be initially at the critical point:

$$n(E,0)=(e^{E/T_c}-1)^{-1}. \tag{1a}$$

III. SOLUTION OF THE SIMPLIFIED KINETIC EQUATION (KINETICS OF IDEAL GAS)

Let us consider the temperature of heat bath T to be smaller then T_c but nonzero. We assume further small-energy exchanges per collision $\Delta E \ll T$ and thus:

$$N_{p_2}-N_{p_2'}=\frac{\partial N_{p_2}}{\partial E_2}(E_2-E_2')=\frac{\partial N(E_2)}{\partial E_2}(E_1'-E_1),$$

$$N_{p_2}(1-N_{p_2'})\simeq N_{p_2'}(1-N_{p_2})\simeq N_{p_2}(1-N_{p_2})=-T\frac{\partial N(E_2)}{\partial E_2}. \tag{13}$$

Introducing for the convenience of notations $\vec{p}_1'\equiv\vec{k}$ and $\vec{p}_2\equiv\vec{p}$, we can rewrite the energy conservation law if the momentum exchange per collision is small:

$$E_1-E_1'-\vec{p}\Delta\vec{k}/m=0, \tag{14}$$

where $\Delta\vec{k}\equiv\vec{p}_1'-\vec{p}_1$. This equation is correct because we assumed that the chemical potential of the Fermi system $\mu_F\gg T_c$ and thus: $|\vec{p}|\gg|\vec{k}|$, because $|\vec{k}|\lesssim(2mT_c)^{1/2}$.

Now we can substitute Eq. (14) into the argument of the δ function and let us rewrite the kinetic equation in the following form:

$$\frac{\partial n(E_1)}{\partial t}=I_{BT}=\frac{\partial n(E_1)}{\partial t}\bigg|_i+\frac{\partial n(E_1)}{\partial t}\bigg|_s\,,$$

$$\frac{\partial n(E_1)}{\partial t}\bigg|_s=S_1+S_2, \tag{14a}$$

and

$$\frac{\partial n(E_1)}{\partial t}\bigg|_i=i_1+i_2\,,$$

where

$$s_1=-\frac{U_0^2}{\hbar^7}Tn(E_1)\int\frac{\partial N(E_2)}{\partial E}\delta\left(E_1-E_1'-\frac{\vec{p}_\Delta\vec{k}}{m}\right)d^3k\,d^3p, \tag{15}$$

$$s_2=\frac{U_0^2}{\hbar^7}T\int n(E_1')\frac{\partial N(E_2)}{\partial E}\delta\left(E_1-E_1'-\frac{\vec{p}_\Delta\vec{k}}{m}\right)d^3k\,d^3p, \tag{16}$$

$$i_1=-\frac{U_0^2}{\hbar^7}E_1n(E_1)\int n(E_1')\frac{\partial N(E_2)}{\partial E}\delta\left(E_1-E_1'-\frac{\vec{p}_\Delta\vec{k}}{m}\right)\times d^3k\,d^3p, \tag{17}$$

$$i_2=\frac{U_0^2}{\hbar^7}n(E_1)\int n(E_1')E_1'\frac{\partial N(E_2)}{\partial E}\delta\left(E_1-E_1'-\frac{\vec{p}_\Delta\vec{k}}{m}\right)\times d^3k\,d^3p. \tag{18}$$

It is easy to see that i_1, i_2 and S_1, S_2 correspond to induced and spontaneous scattering, respectively. Far from the temperature of the phase transition the boson occupation numbers are small $n(E)\ll1$ and thus i_1 and i_2 are small because they are of the higher order in n. At the same time below T_c the new phase (Bose condensate) appears where $n(E)\gg1$ and i_1 and i_2 play the dominant role. Moreover, it will be clear that these nonlinear terms are responsible for Bose condensate formation. We would like to stress that the induced and spontaneous scattering are two independent processes. Each of the particles of Bose gas takes part in both of them and the total number of particles should be conserved in these processes separately, i.e., if

$$\frac{d\bar{n}}{dt}=i_1+i_2,\quad \frac{m^{3/2}}{\hbar^3}\int\bar{n}(E)\sqrt{E}\,dE=N_0,$$

while if

$$\frac{d\tilde{n}}{dt}=S_1+S_2\quad\text{and}\quad\frac{m^{3/2}}{\hbar^3}\int\tilde{n}(E)\sqrt{E}\,dE=N_0.$$

At the same time condition (2) should be fulfilled for the solution of the total Eqs. (5)–(7). Let us substitute $\partial N(E_2)/\partial E$ into Eqs. (15)–(18) and carry out the integration over the independent variables p_x and p_y. To do this we choose the axis z to be parallel to $\Delta\vec{k}$ and

$$\begin{aligned}\vec{p}\Delta\vec{k} &= p_z|\Delta\vec{k}| \\ &= p_z m^{1/2}[E_1+E_1'-2(E_1E_2)^{1/2}\cos\phi]^{1/2}\sqrt{2}\end{aligned} \tag{19}$$

and

$$E_1+E_1'-2(E_1E_2)^{1/2}\cos\phi \equiv \epsilon .$$

After a simple procedure one obtains

$$S_1 = -\frac{\sqrt{2}\,\pi U_0^2 M m^{1/2}}{\hbar^7} n(E_1)T \times\int \frac{f(E_1,E_1')d^3k}{[E_1+E_1'-2(E_1E_1')^{1/2}\cos\phi]^{1/2}} , \tag{15$'$}$$

$$S_2 = \frac{\sqrt{2}\,\pi T U_0^2 M m^{1/2}}{\hbar^7} \times\int \frac{n(E_1')f(E_1,E_1')d^3k}{[E_1+E_1'-2(E_1E_1')^{1/2}\cos\phi]^{1/2}} , \tag{16$'$}$$

$$i_1 = -\frac{\sqrt{2}\,\pi U_0^2 M m^{1/2}}{\hbar^7} n(E_1)E_1 \times\int \frac{n(E_1')f(E_2,E_1')d^3k}{[E_1+E_1'-2(E_1E_1')^{1/2}\cos\phi]^{1/2}} , \tag{17$'$}$$

$$i_2 = \frac{\sqrt{2}\,\pi U_0^2 M m^{1/2}}{\hbar^7} n(E_1) \times\int \frac{n(E_1')E_1'f(E_1,E_1')d^3k}{[E_1+E_1'-2(E_1E_1')^{1/2}\cos\phi]^{1/2}} , \tag{18$'$}$$

where

$$f(E_1,E_1') = 1+\exp[(E_1-E_1')^2/\epsilon T - \mu_F/T] . \tag{20}$$

One can note the properties of $f(E_1,E_1')$ when $\mu_F/T \gg 1$ [Eqs. (21)–(23)]:

$$f(E_1,E_1') = 1 \quad \text{if} \quad E_1 \text{ and } E_1' \ll \mu_F ; \tag{21}$$

$$f(E_1,E_1') = f(E_1) \quad \text{or} \quad f(E_1') \quad \text{if } E_1 \ll \mu_F \text{ or } E_1' \ll \mu_F ; \tag{22}$$

$$\int_0^\infty f(E_1)\,dE_1 = \int_0^\infty f(E_1')\,dE_1' = \int_0^\infty \frac{dE}{1+e^{(E-\mu_F)/T}} \simeq \mu_F . \tag{23}$$

One will be able to check further that $i_1 \to 0$ if $E_1 \to 0$. That is why only the large E_1 are important in Eq. (17$'$) and thus let us put approximately $E_1 \gg E_1'$. (Both E_1 and E_1' are $\ll T$. It will be clear from our development that i_1 and i_2 are responsible for Bose condensate creation and thus the important energy interval in i_1 and i_2 is $E \ll T$.)

$$i_1 \simeq \frac{\sqrt{2}\,2\pi U_0^2 M m^{1/2}}{\hbar^4} n(E_1)\sqrt{E_1}\,N_0 \equiv \alpha\sqrt{E_1}\,n(E_1) . \tag{24}$$

$f(E_1,E_1') = 1$ because of Eq. (21). On the other hand it is easy to see that $i_2(E_1)$ is dominant at small E_1, so let us put $E_1 \ll E_1'$ and

$$i_2 \simeq \frac{2\sqrt{2}\,\pi U_0^2 M m^{1/2}}{\hbar^7} n(E_1)F(t) \equiv \beta F(t) n(E_1) , \tag{25}$$

where

$$F(t) \simeq \int n(E_1't)(\sqrt{E_1'})\,d^3k .$$

Expressions (24) and (25), which correspond to the induced scattering process, conserve the total number of particles

$$\int i_1(E)\sqrt{E}\,dE = \int i_2(E)\sqrt{E}\,dE . \tag{26}$$

Applying a similar procedure to the spontaneous scattering integrals (15$'$) and (16$'$) one ends up with

$$S_1 = -\frac{2\sqrt{2}\,\pi T U_0^2 M m^2 \mu_F}{\hbar^7} n(E_1) \equiv -\gamma n(E_1) \tag{27}$$

and

$$S_2 = \frac{2\sqrt{2}\,\pi U_0^2 M m^{1/2}}{\hbar^4} f(E_1) N_0 \frac{T}{\sqrt{E_1}} \equiv \delta f(E_1)\frac{T}{\sqrt{E_1}} . \tag{28}$$

Expression (27) stems from the integral (15$'$). Actually, when $0 < E_1 \lesssim T$ while $0 < E_1' \lesssim \mu_F$ one obtains Eq. (27) using Eqs. (22) and (23) readily. In the last step of the derivation of Eq. (27) we have violated the condition $\Delta E \ll T$ but the more accurate procedure leads to the introduction of the cutting parameter $\simeq T$ instead of μ_F in Eq. (27) and introduces no new features. Equation (16$'$) can be rewritten

$$S_2 = \frac{\sqrt{2}\,T\pi U_0^2 M m^{1/2}}{\hbar^7} \int \frac{n(E_1')f(E_1E_1')\,d^3k}{[E_1+E_1'-2(EE_1)^{1/2}\cos\phi]^{1/2}} . \tag{28$'$}$$

We shall see that it is S_2 which is responsible for the formation of the Bose high-energy tail of the equilibrium distribution function and thus we are interested in Eq. (28$'$) for $E_1 \simeq T$ while $E_1' \ll T$. These conditions give Eq. (28) without difficulty. Let us note that

$$\int S_1(E)\sqrt{E}\,dE = \int S_2(E)\sqrt{E}\,dE ,$$

if S_1 and S_2 are taken from Eqs. (27) and (28). Hence, the particles conservation condition has not been violated by our approximations. After the above simplifications we have

$$\frac{dn(E,t)}{dt}=[-\alpha\sqrt{E}+\beta F(t)]n(E,t)-\gamma n(E,t)+\delta f(E)\frac{T}{\sqrt{E}}. \tag{29}$$

The solution of Eq. (29) is

$$n(E,t)=e^{-\phi}\left(n(E,0)+\frac{\delta T}{\sqrt{E}}\int_0^t e^{\phi}\,dx\right), \tag{29$'$}$$

where

$$\phi=-\alpha\sqrt{E}t-\gamma t+\beta\int_0^t F(x)dx.$$

Let us introduce

$$\bar{n}=n(E,0)\exp\left(-\alpha\sqrt{E}t+\beta\int_0^t F(x)dx\right). \tag{30}$$

As was stressed above

$$N_0=\frac{m^{3/2}}{\hbar^3}\int\bar{n}(E,t)\sqrt{E}\,dE \tag{31}$$

because $\bar{n}$ is the solution of the equation without the terms corresponding to spontaneous scattering:

$$\frac{d\bar{n}}{dt}=i_1+i_2=[-\alpha\sqrt{E}+\beta F(t)]\bar{n} \tag{32}$$

which conserves the *total* number of particles. Using this condition one obtains

$$\bar{n}=\hbar^3N_0\,n(E,0)e^{-\alpha\sqrt{E}t}\Big/m^{3/2}\left(\int n(E,0)e^{-\alpha\sqrt{E}t}\sqrt{E}\,dE\right). \tag{33}$$

For the big $t^2\gg 1/T_c\alpha^2$

$$\bar{n}=2\frac{\alpha\hbar^3N_0\,n(E,0)}{m^{3/2}T_c}te^{-\alpha\sqrt{E}t}\simeq 2\frac{\alpha\hbar^3N_0}{m^{3/2}}\frac{t}{E}e^{-\alpha\sqrt{E}t}. \tag{34}$$

We would like to emphasize that neglecting the spontaneous scattering processes corresponds to the case of heat bath of zero temperature. Thus all the Bose particles form Bose condensate at $t\to\infty$ because expression (34) is proportional to $\delta(E)/\sqrt{E}$ at $t\to\infty$. To obtain the solution of the problem when $T\neq 0$ one can note that

$$\frac{\delta T}{\sqrt{E}}\int_0^t e^{\psi}\,dx=\frac{\delta T}{\sqrt{E}}\int e^{\alpha\sqrt{E}X}e^{\beta(X)}\,dx$$
$$=\frac{T}{E}(e^{\beta(t)+\alpha\sqrt{E}t}-1)-G(t), \tag{35}$$

where

$$\beta(t)=\beta\int_0^t F(X)dx \quad\text{and}\quad G(t)=\frac{T}{E}\int_0^t e^{\alpha\sqrt{E}X}de^{\beta(X)}.$$

Afterwards it is possible to check that $G(t)=0(1/t)$ for sufficiently big times t. Inserting Eq. (35) into Eq. (29′):

$$n=e^{-\alpha\sqrt{E}t-\beta(t)}\left[\left(n_0-\frac{T}{E}\right)+\frac{T}{E}e^{\phi(t)}\right]$$
$$=e^{-\alpha\sqrt{E}t-\beta(t)}\left(n_0-\frac{T}{E}\right)+\frac{T}{E}. \tag{36}$$

Applying one more condition (2) to (36) and using the initial distribution [Eq. (1′)] we readily derive

$$n=\frac{N_0\hbar^3}{m^{3/2}}\frac{t}{E}e^{-\alpha\sqrt{E}t}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right]+\frac{T}{E}. \tag{37}$$

It can be seen that T/E coincides with the first term of the expansion of the Bose distribution function at $E/T\ll 1$, the approximation used previously. Thus it becomes clear that at $t\to\infty$,

$$n\to\frac{N_0\hbar^3}{m^{3/2}}\frac{\delta(E)}{\sqrt{E}}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right]+\phi, \tag{38}$$

where $\phi=(e^{E/T}-1)^{-1}$ is the Bose-Einstein (Plank) function with the final temperature. Obtaining Eq. (37) we took into account that for big enough t

$$n=e^{-\alpha\sqrt{E}t-\beta(t)}\left(n_0-\frac{T}{E}\right)\simeq e^{-\alpha\sqrt{E}t-\beta(t)}\left(\frac{T_0}{E}-\frac{T}{E}\right)$$

and thus

$$e^{-\beta(t)}=(N_0-N_p)\alpha t/(T_0-T),$$

where

$$N_p=\int\phi(E)\sqrt{E}\,dE.$$

The distribution function (38) coincides with the function (1′) which is well known from thermodynamic considerations.

Considering the classical Bose gas with collisions we should use the above procedure but also use the second half of the spectra [Eq. (4)]. It can be shown that

$$n(E)=At^3e^{-\alpha\sqrt{E}t}+\phi \tag{39}$$

and formula (38) is valid in this case also at $t\to\infty$.

We would like to mention that in accordance with results obtained above Bose condensation is the slowing down process which can be completed only at $t\to\infty$ where the initial temperature of the Bose system is $T\geq T_c$. Let us now investigate the same process for the case when at the initial moment $t=0$ the temperature is $T_0<T_c$ and there is already prepared condensate. Thus we take

$$n(E,0)=A\delta(E)/\sqrt{E}+\phi \tag{40}$$

and the temperature of the heat bath is chosen to be $T=0$ for the simplicity. Substituting Eq. (40) into formula (33) we can obtain

$$n=\frac{\hbar^3 N_0}{m^{3/2}}\left(\frac{A\alpha t\,\delta(E)/\sqrt{E}}{A\alpha t+T_c}+\frac{\alpha t T_c e^{-\alpha\sqrt{E}t}}{E(A\alpha t+T_c)}\right). \qquad (41)$$

Expressions (34) and (41) differ drastically one from another. If the initial temperature is $T_0 \geq T_c$ [solution (34)], the Bose condensate can be formed at $t\rightarrow\infty$ only. That means that there is an infinite time gap between the beginning of the cooling process and the new phase formation.

At the same time if at the initial moment even if a negligibly small part of the particles already formed Bose condensate [solution (41)], this new phase grows up continuously till at $t\rightarrow\infty$ all the particles will be in this phase. Thus there is no time gap in this case. In other words we derived that the Bose condensation of ideal gas can proceed much easier if the nuclei of the condensate are formed at the initial stages.

In order to satisfy the assumptions used in deriving Eqs. (35) and (39) one can substitute Eqs. (35) and (39) into Eqs. (15′)–(18′) and evaluate the integrals. After that one finds that the assumptions which led to Eqs. (24)–(28) are justified.

Let us note finally that induced scattering corresponds to the process which can act in one direction only. It leads to the energy flux from the Bose system to heat bath, i.e.,[7]

$$\frac{d\epsilon}{dt}=\int\frac{d\bar{n}}{dt}E\,d^3p<0.$$

Thus, we can conclude *a priori* that the final distribution function obtained without taking into account the spontaneous scattering processes is proportional to δ function. From our analytic solution one can see that the approach to the final state proceeds through a set of essentially nonequilibrium states and is a nonlinear process. We can add that this process is not a relaxational one.

IV. ROLE OF BOSE-BOSE COLLISIONS

Let us consider now the role of the collisions between the Bose particles themselves.

In the small energy region the occupation numbers $n(E)\gg 1$ and the main term corresponding to Bose-Bose interaction is proportional to n^3 while the Bose–heat bath interaction is proportional to n^2 only.

It is clear that at large times the distribution function consists of two parts. One of them is very sharp and another is close to the Bose distribution with the temperature that of heat bath. Let us denote these two parts by α and ϕ, respectively. It is easy to see that keeping the terms proportional only up to the second degree of α the collision term can be rewritten in the following form:

$$I_{BB}\simeq I_{\alpha\alpha}+I_{\alpha\phi}+I_{\phi\alpha}+I^0 \qquad (42)$$

with

$$\begin{aligned}
I_{\alpha\alpha}&=-A^2\int\delta(\Delta\vec{\mathrm{p}})\,\delta(\Delta E)\{[\alpha_1\alpha_2(\alpha_3+\alpha_4)-\alpha_3\alpha_4(\alpha_1+\alpha_2)]+(\alpha_1\alpha_2-\alpha_3\alpha_4)\}d^3p_2\,d^3p_4 d^3p_3\ ,\\
I_{\alpha\phi}&=-A^2\int\delta(\Delta\vec{\mathrm{p}})\,\delta(\Delta E)[\alpha_1\alpha_3(\phi_2-\phi_4)+\alpha_1\alpha_4(\phi_2-\phi_3)]d^3p_3\,d^3p_2\,d^3p_4,\\
I_{\phi\alpha}&=-A^2\int\delta(\Delta\vec{\mathrm{p}})\,\delta(\Delta E)[\alpha_2\alpha_3(\phi_1-\phi_4)+\alpha_2\alpha_4(\phi_1-\phi_3)]d^3p_3\,d^3p_2\,d^3p_4, \qquad (42a)\\
I^0&=-A^2\int\delta(\Delta\vec{\mathrm{p}})\,\delta(\Delta E)[\alpha_1\alpha_2(\phi_3+\phi_4)-\alpha_3\alpha_4(\phi_1+\phi_2)]d^3p_3\,d^3p_2\,d^3p_4,
\end{aligned}$$

and p_3 stands for p_1' while p_2 and p_4 for p_2 and p_2', respectively.

Let us show that there is a significant physical difference between $I_{\phi\alpha}$ and $I_{\alpha\phi}$ on the one hand and I^0 on the other. Actually, we can consider a mixture of two gases assuming further that one gas is described by the distribution function α and the other one by ϕ. It is possible to write the kinetic equation for such a mixture of interacting gases in a symbolic form

$$I_{\mathrm{mix}}=\frac{d\alpha}{dt}+\frac{d\phi}{dt}+\frac{d(\alpha\phi)}{dt}\ , \qquad (43)$$

where

$$\frac{d\alpha}{dt}=I_{\alpha\alpha}+I_{\alpha\phi}\ , \qquad (44)$$

$$\frac{d\phi}{dt}=I_{\phi\alpha}\ , \qquad (45)$$

$$\frac{d(\alpha\phi)}{dt}=I^0. \qquad (46)$$

Equations (43) and (44) form the set of equations describing the changes in the mixture of two different gases α and ϕ. Equation (45) introduces the deviations from this picture and the smaller I^0 is the more separate are the phases α and ϕ which are described by the set (43)–(45). The above statement can be illustrated by the properties of (43)–(45):

$$\int \frac{d\alpha}{dt} d^3p_1 = \int \frac{d\phi}{dt} d^3p_1 = 0, \tag{47}$$

$$\int E_1 \frac{d\alpha}{dt} d^3p_1 = -\int E_1 \frac{d\phi}{dt} d^3p_1 = 0, \tag{48}$$

$$\int \frac{d(\alpha\phi)}{dt} d^3p_1 = \int E_1 \frac{d(\alpha\phi)}{dt} d^3p_1 = 0. \tag{49}$$

There are two kinds of energy exchange between phases α and ϕ. The first one which is described by $d\alpha/dt$ and $d\phi/dt$ conserves the number of particles within each phase as if they were two gases of different sorts. The second type of energy exchange proceeds through the particle exchange between these two phases. Thus α and ϕ behave like two different phases. If α has a sharp and narrow distribution near $E=0$, only a small part of the particles can be transferred to the second phase ϕ. This follows from the energetic agreements given in Sec. II.

We are interested mostly in the elucidation of the behavior of the phase α during the cooling process. We would like to stress the difference between the cooling of the Bose system above and below the critical point. If the initial state of Bose gas corresponds to $\mu \neq 0$, the collisions between Bose particles are able to keep the system in quasiequilibrium with the time dependent μ and T. The role of collisions is thus to choose the quasiequilibrium approach to the critical point at which $\mu = 0$. After that there are no more equilibrium states except the final one [Eq. (1′)] and the system relaxes through the set of essentially nonequilibrium states. The collisions cannot change this.

Let us begin with investigating the second term of Eq. (42) which is proportional to α^2. This term corresponds to collisions between the phases α and ϕ. If distribution of α is narrow enough, it follows from the solution (34) that phase ϕ is in termal equilibrium with its environment and thus we can consider this phase as an additional heat bath. So this term can lead to variation of the coefficients in the equations that cannot change the physical picture obtained above.

Let us discuss now the term proportional to α^3. It is clear that this term describes the collisions within the phase α. If the distribution function of α is narrow with the average energy per particle ϵ, only a very small number of particles proportional to $(\epsilon/T_c)^{3/2} \ll 1$ can be scattered to the high-energy direction. To prove it we must remember that the α^3 process corresponds to scattering in the system which is being cooled to the final temperature $T=0$. (In this case there is no phase ϕ.) If the heat bath is switched off, the final distribution must correspond to the temperature $T \simeq \epsilon$ (average energy of the system per particle) and thus the above statement becomes obvious. So α^3 scattering processes cannot make Bose condensation considerably longer. Of course we did not prove that this term cannot lead to the strong shortening of condensation time.

V. MEASUREMENTS NEAR THE SINGULARITIES OF THERMODYNAMIC FUNCTIONS

First of all let us estimate the time parameters of the Bose condensation process in ideal Bose gas. The energy of condensing phase per particle is

$$\epsilon \simeq (\hbar^8/U_0^4 M^2 m N_0^2)(1/t^2). \tag{50}$$

Such a slow nonexponential decrease of ϵ with time is a result of essentially nonrelaxational approach to the stationary (final) Bose condensate state. We estimated the characteristic time τ_i of the energy washing out from the system to the heat bath using the model parameters $U_0 = 10^{-39}$ erg cm^3, $N_0 = 10^{23}$ cm^3, $M \simeq m = 10^{-23}$ g:

$$\tau_i \approx 10^{-8}–10^{-7} \text{ sec}$$

if $\epsilon \simeq 10^{-16}$ erg. So this time is small when the energy of condensing phase is big enough. At the same time one can see that τ_i increases strongly with decrease of the total energy of this phase; for example, if $\epsilon \simeq 10^{-23}–10^{-24}$ erg:

$$\tau_i \simeq 10^{-4}–10^{-3} \text{ sec}.$$

In order to understand this slowing down of the cooling process τ_i is to be compared with the characteristic time of the second phase formation:

$$1/\tau_s \simeq T U_0^2 M m^2 \mu_F/\hbar^7 \simeq 10^{+9} \text{ sec}^{-1}. \tag{51}$$

Thus

$$\tau_i/\tau_s \simeq 10^5–10^6 \tag{51′}$$

and one can note that the Bose condensate formation is a much slower process than the ordinary cooling of the Bose gas with characteristic time given by Eq. (51).

If τ_e is the time beyond which the kinetic equation is not applicable[8] then $\tau_i/\tau_e \simeq 10^8–10^9$ and the violation of the approximations we have used appears at very large, macroscopic times.

We do not know to what extent our model describes the situation in real helium. We realize that the ideal-gas approximation is a very poor one for helium below its critical temperature.

Our model of the heat bath can hardly represent the properties of the cooling systems usually used in experiments. The observed τ_s is always much larger than the $\tau_s \simeq 10^{-9}$ sec we have estimated. But our picture concludes that in an ideally performed experiment (we do not consider the technical difficulties such as geometry, stirring, heat resistance and so on) $\tau_i \gg \tau_s$.

This can be very important for measurements near the critical point. Because of the finite (nonzero) energy of condensing phase the temperature measurements can be carried out with an error ΔT:

$$\Delta T \gtrsim \epsilon(t). \tag{52}$$

This error must be smaller than $|T - T_c|$ and the condition for precise measurements near the singularities of thermodynamic functions is

$$\epsilon(t) \ll |T - T_c|, \tag{53}$$

or using Eq. (51):

$$t^0 \gg \text{const}/|T - T_c|^{1/2}. \tag{54}$$

Thus t^0 is the time the experimentalist must wait till the energy of the condensing phase becomes small enough and the measured T represents the temperature of the equilibrium Bose system at $t \to \infty$. Remembering that τ_s in real experiments is not too small and using Eq. (51) one concludes that t^0 can be very large when $|T - T_c| \lesssim 10^{-7}$ K even if the experimental (technical) difficulties are avoided. 10^{-7} K is the accuracy of the temperature measurements usually used in experiments.[3-5]

VI. SUMMARY AND CONCLUSIONS

In this work we considered the kinetics of Bose condensation of an ideal Bose gas. The main results can be summarized.

(a) An ideal Bose gas demonstrates the formation of two phases in the momentum space. The distribution function of the first phase narrows and grows with time forming Bose condensate at $t \to \infty$, while the second phase is described by Bose distribution with $\mu = 0$.

(b) The time of Bose condensation, or in other words, the time of the creation of pure quantum state at $p = 0$, was found to be infinite.

(c) The equilibrium high-energy tail of the distribution function is the Bose one and is formed during a relatively short time.

(d) The collisions between Bose particles cannot make the condensation process to be longer. We do not know at the present stage if they cannot shorten the Bose condensation.

(e) It was shown that in order to perform precise measurements near the points of singularities of thermodynamic functions of Bose gas one should wait for quite a long time in order to measure a temperature which can represent the equilibrium property of the system at $t \to \infty$. This time t^0 increases with approach to the critical point $(T_c - 0)$ in ideal Bose gas:

$$t^0 \gg \text{const}/|T - T_c|^{1/2}.$$

Several very interesting questions remained unanswered.

(i) First of all one must explain the finite time of Bose condensation process observed experimentally (if we accept that the λ-transition in helium is connected with Bose condensation).

(ii) We recognize that the energy spectra we have used hardly describe the properties of nonideal Bose gas below the critical point. Thus it is of interest to consider the time evolution of the energy spectrum of the system passing through the critical point. Solving this problem one can follow the appearance of the superfluidity of the phase α (our terminology).

(iii) Although we have shown that Bose-Bose collisions cannot drastically distort the distribution we have obtained, it is interesting to evaluate the accurate shape of the distribution $\alpha(E, t)$ and to derive a more detailed picture of the time evolution. This problem can be treated numerically.

The questions raised here are very intriguing and they will be the subject of future communications.

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responsible for the finite time of the spectrum deformation is not valid when energy $E \rightarrow 0$ (see also Ref. 7).

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