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Textures and NMR in superfluid ^{3}He-(B)

H. Smith*
H. C. Ørsted Institute, Copenhagen, Denmark

W. F. Brinkman
Bell Laboratories, Murray Hill, New Jersey 07974

S. Engelsberg†
University of Massachusetts, Amherst, Massachusetts 01002
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In this paper we discuss in detail the nature of the spatial variation of the order parameter of the B phase of ^{3}He. It is shown that in this phase there is a single vector component of the order parameter which undergoes spatial variation under normal circumstances and the detailed nature of its variation is calculated for cylindrical and parallel plate geometries. The effect of these textures on nuclear magnetic resonance is discussed and it is shown that the textures should allow one to observe standing spin waves as well as the line-shape effects discussed in an earlier letter.

I. INTRODUCTION

The first experimental evidence for the importance of the concept of textures for understanding nuclear-magnetic-resonance behavior in superfluid ^{3}He-B was given by Osheroff and Brinkman.[1] They interpreted the observed line shape and intensity of the B-phase transverse resonance as arising from a surface-induced spatial variation in the orientation of the magnetic anisotropy axis $\hat{n}$ of the Balian-Werthamer state. This state is commonly associated with the B phase. Subsequently, Brinkman, Smith, Osheroff, and Blount[2] (hereafter referred to as I) proposed detailed forms for the competing surface and volume contributions to the free energy in the presence of an external magnetic field. The energy was quite complicated, but considerations of specific geometries allowed definite predictions to be made regarding the line-shape and intensity variations of the NMR. One of the intermediate consequences of the theory developed in I was that a magnetic field parallel to a planar surface would tend to orient $\hat{n}$ at the angle $\cos^{-1}(1/\sqrt{5})$ relative to the field with a consequent large shift in the transverse resonance. Such behavior was observed and further explored for a parallel plate geometry by Osheroff, Engelsberg, Brinkman, and Corruccini[3] and, more recently for general field directions relative to the surface normal, by Ahonen *et al.*[4,5] The measurements reported in Ref. 3 made it possible to extract the product of a characteristic field H_B and a characteristic length R_c which compared well with Osheroff's earlier bulk measurements of that same quantity[6] on the basis of the line-shape analysis given in I. Thus, a fair amount of evidence in favor of the texture theory presented in I has accumulated. The present work aims at providing a detailed derivation of the various terms in the free energy given in I as well as an estimate of their magnitude (Sec. II). Section III describes the free-energy calculations of textures and singularities of the order parameter. The parallel plate geometry is considered in the limiting case where the plate separation is smaller than $R_c H_B/H$, H being the external magnetic field, in order to exhibit the analysis of Ref. 3. Also we derive some results for a cylindrical geometry, which were briefly reported in I. Two textures are found to be of particular interest when the magnetic field is along the cylinder axis: one in which $\hat{n}$ gradually bends away from being parallel to the magnetic field in the center of the cylinder, and one in which $\hat{n}$ lies in a plane perpendicular to the magnetic field. These two textures appear to explain the experimental data for low-field experiments of both Osheroff[6] and Webb *et al.*[7] Finally, in Sec. IV we discuss the validity of a local resonator versus a spin-wave model for calculating the NMR line shape in the presence of textures, and the various regimes are delineated.

As a final introductory remark we stress that although the derivation of the various bulk and surface terms in the free energy may appear quite complicated their presence and functional dependence on $\hat{n}$ is easily understood in terms of scalar quantities formed from the three vectors occurring in the problem: the direction of rotation $\hat{n}$, the magnetic field $\vec{H}$, and the surface normal $\hat{s}$. (This point was also recently emphasized by Fomin and Vuorio.[8]) The form of the bulk bending energy arising from the spatial variation of $\hat{n}$ has its close analogy in the bend, splay, and twist terms

of liquid-crystal theory except that here there is a nonzero cross term between twist and splay. The advantage of studying texture effects in the B phase as contrasted to the A phase is that we need consider only variations of a single (unit) vector $\vec{\hat{n}}$ and that the characteristic length R_c is much longer (a few tenths of a centimeter) than the corresponding length that describes the variation of the vector $\vec{l}$ in the A phase, the latter being only a few micrometers. As emphasized in I and borne out by subsequent experiments[3-5] the B phase is the more attractive candidate for exploring the fascinating interplay of superfluid and liquid-crystal behavior which makes the study of nuclear-magnetic-resonance behavior in superfluid ^{3}He so uniquely rewarding.

II. FREE ENERGY

In deriving the free energy in the presence of walls we take as a starting point the boundary condition on the order parameter at a specularly reflecting wall as derived by Ambegaokar, de Gennes, and Rainer[9] for a semi-infinite half space. Our notation for the order parameter $d_{\alpha i}$ is the conventional one[10] derived from the anomalous expectation value of the product of two annihilation operators for particles in states $\vec{k}\beta$ and $-\vec{k}\gamma$:

$$\langle a_{\vec{k}\beta} a_{-\vec{k}\gamma}\rangle \propto \sum_{\alpha i} d_{\alpha i}\hat{k}_\alpha (i\sigma^i\sigma^y)_{\beta\gamma}\,. \tag{1}$$

Here $\hat{k}_\alpha$ is the unit momentum vector in the α direction ($\alpha = x, y, z$), σ^i the Pauli spin matrices ($i = x, y, z$), and $d_{\alpha i}$ the 3×3 order-parameter matrix, which we shall take to be the rotation matrix characterizing the Balian-Werthamer state. We therefore write $d_{\alpha i}$ as a real rotation matrix which is proportional to the isotropic gap parameter Δ,

$$\begin{aligned} d_{\alpha i} &= (\Delta/\sqrt{3})[\delta_{\alpha i}\cos\theta + n_\alpha n_i(1-\cos\theta) + \epsilon_{\alpha i j} n_j \sin\theta] \\ &\equiv (\Delta/\sqrt{3})R_{\alpha i}\,, \end{aligned} \tag{2}$$

where θ is the angle of rotation and $\vec{\hat{n}} = (n_x, n_y, n_z)$ is a unit vector in the direction of rotation. Repeated indices are always summed over. Note that our overall normalization is chosen such that $d_{\alpha i} d_{\alpha i} = \Delta^2$. By contrast the energy gap in the dispersion relation of the elementary excitations is $\Delta/\sqrt{3}$.

In order to obtain the various energies influencing the directions of $\vec{\hat{n}}$ we use the terms that occur in the Landau-Ginzburg expansion of the free energy and we keep only the lowest-order terms. For example, we do not consider nonlocal aspects of the susceptibility. This approximation should give correctly the most important effects determining the spatial orientation of $\vec{\hat{n}}$.

In the absence of the dipolar energy the infinite set of states described by arbitrary values of $\vec{\hat{n}}$ and θ in (2) are degenerate in energy. However the dipolar coupling forces the angle θ to assume the value $\cos^{-1}(-\frac{1}{4})$ giving the minimum bulk dipole energy. As pointed out by Leggett[11] a bulk magnetic field will furthermore tend to orient $\vec{\hat{n}}$ in its direction (or opposite to it) due to a combination of the depairing effects of the magnetic field and the dipole energy which gives rise to an energy density proportional to $-(\vec{H}\cdot\vec{\hat{n}})^2$. We now turn to the discussion of this bulk orientational effect.

A. Bulk field orientation energy

The form of the bulk field orientational energy F_H^B is quadratic in magnetic field,

$$F_H^B = -a\int d^3r\,(\vec{H}\cdot\vec{\hat{n}})^2 \tag{3}$$

(the superscripts B and S stand for bulk and surface, respectively). The coefficient a has been calculated by Engelsberg, Brinkman, and Anderson[12] within weak coupling theory by minimizing the usual second- and fourth-order terms in the free energy together with the dipolar energy and the susceptibility anisotropy energy F_S^B responsible for the reduction of the susceptibility in the B phase. We use their notation for the susceptibility anisotropy energy which involves the following invariant combination of the spin angular momentum S_i and the order parameter $d_{\alpha i}$,

$$F_S^B = c_1 S_i\, d_{\alpha i}\, d_{\alpha j}\, S_j\,. \tag{4}$$

The constant c_1 determines the reduction in susceptibility in the bulk of the sample,

$$c_1 = \frac{3}{2}\,\frac{\gamma^2}{\Delta^2}\,\frac{\chi_N - \chi_B}{\chi_B \chi_N}\,, \tag{5}$$

where $\chi_N = N(0)(1+F_0^a)^{-1}(\frac{1}{2}\hbar\gamma)^2$ is the normal phase susceptibility with $N(0) = m^* k_F/\pi^2\hbar^2$ being the density of states. In (5) χ_B is the B-phase susceptibility and γ the gyromagnetic factor.

We shall quote the value of the parameter a obtained from the simultaneous minimization of these three energies. Provided that $\hbar\gamma H \ll \Delta$ and that we are close enough to T_c so that the sixth-order terms in the free energy can be ignored, one finds the value for T close to T_c of

$$a = \frac{35\xi(3)}{192\pi^2}\,\frac{1}{(1+F_0^a)^2}\left(\frac{\hbar\omega_0}{k_B T_c}\right)^2 \chi_N\,, \tag{6}$$

where the frequency ω_0 is related to the longitudinal resonance frequency Ω_L of the B phase through the relation

$$\Omega_L^2 = \omega_0^2 (T_c - T)/T_c\,.$$

Note that (6) contains three Fermi-liquid enhancement factors $(1+F_0^a)^{-1}$, one being implicit in χ_N. Although (3) with (6) represents a tiny orientational energy (about 10^{-12} °K/atom in a magnetic field of 1 kG) this energy is responsible for the absence of a shift in the transverse resonance in the bulk of ^{3}He-(B), since when $\hat{n}$ is parallel to $\vec{H}$ there is no transverse shift away from γH, but only a longitudinal resonance at Ω_L.

The assumptions under which the expression (6) was derived were, however, too restrictive to be very useful in providing a quantitatively reliable expression for a. Since a enters the characteristic field H_B to be introduced below and is an important quantity in the application of the theory to experiments which may be done well below T_c (typically in the range $0.45 < T/T_c < 0.75$ at the melting curve) we shall take a slightly different point of view in the following by treating a as a parameter, which can be obtained from experiment. As Leggett first suggested[11] the form of the energy (3) produces a small, but observable shift in the transverse resonance away from the Larmor frequency γH. The shift $\delta\omega$ is calculated in the Appendix by determining the change in energy of the transverse resonance mode in the presence of this anisotropy energy. It may be thought of in terms of a small change Δg in the g factor of the nucleus:

$$\Delta g/g \equiv \delta\omega/\gamma H = \tfrac{4}{5} a \chi_B^{-1} \tag{7}$$

provided the equilibrium direction of $\hat{n}$ is along the magnetic field H (an incorrect value for this coefficient was quoted in Ref. 3). Note that the susceptibility χ_B is the temperature-dependent B-phase susceptibility which occurs in the resonance equation. Since $\delta\omega/\gamma H$ has been measured by Osheroff[6] and by Osheroff and Brinkman[13] in a geometry in which $\hat{n}$ was parallel to $\vec{H}$ over most of the sample, we can use the experimental results for the relative shift $\delta\omega/\gamma H$ ($\sim 10^{-5}$) and the measured B-phase susceptibility χ_B to obtain the quantity a at temperatures well below the transition temperature. Typical values of a derived in this manner are $(a/\chi_N)\times 10^5 = 0.54$, 0.41, and 0.32 at $T/T_c = 0.7$, 0.6, and 0.5, respectively. Recently the full temperature dependence of a has been calculated in the weak coupling theory[13] and the temperature dependence is qualitatively the same as observed, but the theoretical values are $\sim\frac{2}{3}$ the measured results. This is undoubtedly due to strong coupling corrections.

B. Surface dipole energy

Since the presence of a wall forces the $m_L = 0$ component of the triplet order parameter along the surface normal $\hat{s}$ to be depressed quite independently of the nature of the state, we shall now explore the consequences of this depairing effect for the dipolar energy. The boundary condition of Ref. 9 is expressed in terms of the surface normal $\hat{s}$ by

$$s_\alpha d_{\alpha i} = 0 \tag{8}$$

at the boundary. As a trial function satisfying this condition we shall use the form

$$d_{\alpha i}(\vec{r}) = (\Delta/\sqrt{3})(\delta_{\alpha\beta} - s_\alpha s_\beta f) R_{\beta i}(\hat{r}), \tag{9}$$

where the function f varies from being unity at the wall to being zero well inside the sample, that is, at distances more than a coherence length ξ (~ 200 Å) from the wall. $R_{\beta i}(\hat{r})$ is the rotation matrix specified by $\hat{n}(\hat{r})$ and θ. Since $f = 1$ at the surface, the trial function (9) is seen to satisfy the boundary condition (8). Note that, in general, the other components of this order parameter are also reduced near the surface. They will, however, not be zero and the results discussed below will hold generally.

To see what surface dipole energies arise as a consequence of the boundary condition (8) we insert (9) in the general expression for the dipole energy F_D,[10]

$$F_D = \int \Gamma (d_{ii} d_{jj} + d_{ij} d_{ji}) d^3r, \tag{10}$$

where Γ is related to the longitudinal resonance frequency Ω_L through $\Omega_L^2 = 5\Gamma\Delta^2\gamma^2/\chi_B$. We expect the surface terms to involve invariants like $(\hat{n}\cdot\hat{s})^2$ and $(\hat{n}\cdot\hat{s})^4$. By insertion of (9) into (10) we get

$$F_D = \int \frac{\Gamma\Delta^2}{3} \{(4\cos^2\theta + 2\cos\theta)2 + 2f^2[\cos^2\theta + 2\cos\theta(1-\cos\theta)(\hat{n}\cdot\hat{s})^2 + (1-\cos\theta)^2(\hat{n}\cdot\hat{s})^4] - 2f[4\cos^2\theta + \cos\theta - 1 + (\hat{n}\cdot\hat{s})^2(-4\cos^2\theta + \cos\theta + 3)]\} d^3r. \tag{11}$$

We separate out the bulk contribution F_D^B not involving f by writing

$$F_D = F_D^B + F_D^S \tag{12}$$

and noting that F_D^B is minimized in the usual manner by choosing $\cos\theta = -\frac{1}{4}$. The surface term will not be minimized by this value of θ. However, surface energies of order $F_D\xi/V^{1/3}$, where V is the volume, cannot cause θ to appreciably deviate from its value in the bulk. This is due to the fact

that a deviation at the surface can be corrected only on a scale of $(\Delta F/F_D^B)^{1/2}\xi$ which is much longer than ξ since $\Delta F = F_N - F_S$, the free energy gained when the system becomes superfluid. The energy of the deviation is therefore much larger than any possible energy gain of the surface. The sole influence of the wall becomes therefore one of orienting $\hat{n}$ in a direction such that the surface term becomes as small as possible. We then obtain

$$F_D^S = \frac{\Gamma\Delta^2}{3}\int d^3r\,[2f + \tfrac{1}{8}f^2 - (\hat{n}\cdot\hat{s})^2(5f + \tfrac{5}{4}f^2) + (\hat{n}\cdot\hat{s})^4\tfrac{25}{8}f^2]\,. \quad (13)$$

Since f is only nonzero at distances of the order of or less than a coherence length, (13) is really a surface energy. To obtain explicit values of the coefficients of $(\hat{n}\cdot\hat{s})^2$ and $(\hat{n}\cdot\hat{s})^4$ we must specify f and integrate over the volume. We take $f \simeq e^{-r_\perp/\xi}$, where $r_\perp$ is the perpendicular distance from the surface to be a reasonable guess. The details of the functional dependence of f on $r_\perp$ will of course affect the precise value of the coefficient in this surface energy as well as the surface field energy F_H^S introduced below. However, we are not able to calculate these coefficients accurately anyway, since we do not know the additional depairing effects due to nonspecular reflection from the walls and wall curvature, which would modify the starting boundary condition (8). In the same spirit we write the volume element d^3r in the approximate form $d^2r\,dr_\perp$ for the surface region of interest and obtain upon completion of the integral over $r_\perp$ the surface dipole energy

$$F_D^S = -\xi\frac{15\Gamma\Delta^2}{8}\int_S d^2r[(\hat{s}\cdot\hat{n})^2 - \tfrac{5}{18}(\hat{s}\cdot\hat{n})^4]\,. \quad (14)$$

Here the integration d^2r is extended over the surface. The constant, $\hat{n}$-independent term in (13) has been discarded. The expression (14) allows us to identify the coefficient b introduced in I as the factor in front of the integral, but we stress that the numerical coefficient should not be taken seriously. Note that a possible small change of θ in the boundary region would only have the effect of slightly changing the numerical coefficients, leaving the form (14) unaltered. For the purpose of calculating free energies of textures and NMR resonance frequencies we shall frequently ignore the second term involving $(\hat{s}\cdot\hat{n})^4$ in (14), thus writing

$$F_D^S \simeq -b\int_S d^2r(\hat{s}\cdot\hat{n})^2\,, \quad (15)$$

with

$$b \simeq \xi\Gamma\Delta^2\,. \quad (16)$$

It is seen that the effect of this surface term, whether we use (14) or the simplified form (15) is to orient $\hat{n}$ perpendicular to the wall surface. But this surface energy is not the only one present as we shall see below (Sec. II D). In the presence of a magnetic field the "susceptibility anisotropy energy" F_S^B as given in (4) introduces a surface term giving a much more complicated orientational effect, which depends on the angle between the magnetic field and the surface normal. Before we derive this surface energy, however, we turn to consider the bulk bending energy which arises when the vector $\hat{n}$ is forced to vary in space as a result of the competition between the bulk alignment effect of a magnetic field and the surface effects just mentioned.

C. Bulk bending energy

When the order parameter $d_{\alpha i}$ becomes spatially varying a "kinetic" or bending energy results just as in the Ginzburg-Landau theory of an s-wave superconductor. The form of this bending energy is naturally more complicated due to the large number of degrees of freedom of a 3×3 complex matrix "wave function" as compared to the scalar complex wave function introduced in ordinary Ginzburg-Landau theory. The general, invariant form of this bulk bending energy was first written down by de Gennes for an $l=1$ superfluid.[14] It is

$$F_B^B = \left(\frac{\hbar^2\rho_s}{4m}\,\frac{3}{10\Delta^2}\right)\int d^3r(\partial_\alpha d_{\beta i}\partial_\alpha d^*_{\beta i} + \partial_\alpha d_{\alpha i}\partial_\beta d^*_{\beta i} + \partial_\alpha d_{\beta i}\partial_\beta d^*_{\alpha i})\,, \quad (17)$$

where the derivative ∂_α denotes differentiation with respect to the spatial variable α ($\alpha = x, y, z$).

The constant in front of the integral in (17) is fixed in the Ginzburg-Landau regime by considering a state corresponding to a uniform-mass flow. ρ_s is the superfluid density, which in the Ginzburg-Landau region close to T_c is given in terms of the particle number density ρ as $\rho_s = 2\rho(1 - T/T_c) \times (m/m^*)$.

In deriving the expression for the bending energy in the bulk we follow the same procedure as in Sec. II B, taking θ to be fixed in the bulk by the dipole interaction and assuming it to be essentially unchanged by the boundaries to avoid an increased bending energy. We write as before

$$d_{\alpha i} = \bar{d}_{\alpha\beta}R_{\beta i} \quad (18)$$

with $\bar{d}_{\alpha\beta} = (\Delta/\sqrt{3})(\delta_{\alpha\beta} - s_\alpha s_\beta f)$. The result of inserting the derivatives of $d_{\alpha i}$ in (17) is to produce terms of different categories, depending upon which of the matrices $\bar{d}$ and R is being differentiated. The easiest ones to deal with are of the form

$$(\partial_\alpha \vec{d}_{\beta\gamma}) R_{\gamma i} (\partial_\alpha \vec{d}_{\beta\delta}) R_{\delta i} = (\partial_\alpha \vec{d}_{\beta i}) \partial_\alpha \vec{d}_{\beta i}$$

upon use of the orthogonality properties of rotation matrices. These terms are seen only to involve f and not the bulk state at all, which is sufficient reason for discarding them. Next we consider the bulk terms of the form

$$(\partial_\alpha R_{\gamma i}) \vec{d}_{\beta\gamma} (\partial_\alpha R_{\delta i}) \vec{d}_{\beta\delta} .$$

Here we may replace the $\vec{d}$ matrices with the Kronecker delta, since by so doing we change the bending energy only by a negligible amount corresponding to the volume extending a coherence length away from the surface. The evaluation of $(\partial_\alpha R_{\beta i}) \partial_\alpha R_{\beta i}$ and the similar terms arising from the second and third terms of (17) is straightforward, though tedious. Since θ is assumed not to vary in space the resulting bending energy contains terms involving only spatial derivatives of the $\vec{n}$ vector. Specifically we obtain terms involving $(\nabla \cdot \vec{n})^2$ (in liquid crystal terminology "splay" terms), $(\vec{n} \cdot \operatorname{curl} \vec{n})^2$ ("twist" terms) and $(\vec{n} \times \operatorname{curl} \vec{n})^2$ ("bend" terms) as well as terms involving the cross product $(\nabla \cdot \vec{n})(\vec{n} \cdot \operatorname{curl} \vec{n})$. In addition there are terms which may be written as a divergence of a vector $\vec{A}$, where $\vec{A} = (\vec{n} \cdot \nabla)\vec{n} - (\nabla \cdot \vec{n})\vec{n}$. Such divergence terms, which are generally nonzero, are contributed by the first and third term in the bending energy (17), but not by the second, as long as we restrict ourselves to considering these bulk terms.

Apart from such a divergence term whose explicit form is $(\hbar^2 \rho_s / 20m)(1 - \cos\theta) \operatorname{div} \vec{A}$, the second and third terms in (17) give otherwise identical contributions. Now, since we can always add to the free energy (17) a divergence term of the form $\partial_\alpha (d_{\alpha i} \partial_\beta d_{\beta i} - d_{\beta i} \partial_\beta d_{\alpha i})$ which is identical to the difference between the second and third terms of (17), we must clearly make use of our boundary condition (8) to make the free energy unique in the sense that the second and third terms give identical contributions. In other words, we must consider the "mixed" derivatives of the form $(\partial_\alpha \vec{d}_{\beta\gamma}) \vec{d}_{\beta\delta} R_{\gamma i} \partial_\alpha R_{\delta i}$ and analogous expressions coming from the second and third terms of (17) which arise when (18) is inserted in the expression for the bending energy (17). A careful examination of all these "mixed" terms combined with extensive use of the properties of rotation matrices (which cause the tensor $R_{\gamma i} \partial_\alpha R_{\delta i}$ to be antisymmetric with respect to the interchange of γ and δ) leads to the desired uniqueness of the free energy, since the "mixed" terms coming from the second term in (17) conspire to produce exactly the term $(\hbar^2 \rho_s / 20m)(1 - \cos\theta) \operatorname{div} \vec{A}$, which previously arose from the *bulk* contribution to the *third* term in (17). Thus the free energy is explicitly unique, as we would expect, since our trial function (18) can be seen to satisfy the condition

$$s_\alpha (d_{\alpha i} \partial_\beta d_{\beta i} - d_{\beta i} \partial_\beta d_{\alpha i}) = 0 \tag{19}$$

apart from a constant state-independent contribution proportional to $\operatorname{div} \vec{s}$, which vanishes in a parallel plate geometry and in general only adds a trivial constant to the total energy. It should be noted that to obtain the desired uniqueness we have as before discarded terms which were smaller than those we kept by a factor of the coherence length divided by a linear dimension of the sample. The resulting bending energy then becomes

$$F_B^B = \frac{\hbar^2 \rho_s}{40m} \int d^3r \, 2(1 - \cos\theta) \{ (3 - \cos\theta)(\operatorname{div} \vec{n})^2 + (3 + \cos\theta)(\vec{n} \cdot \operatorname{curl} \vec{n})^2 + 4(\vec{n} \times \operatorname{curl} \vec{n})^2 - 2 \sin\theta (\operatorname{div} \vec{n})(\vec{n} \cdot \operatorname{curl} \vec{n}) + 4 \operatorname{div}[(\vec{n} \cdot \nabla)\vec{n} - (\operatorname{div} \vec{n})\vec{n}] \} , \tag{20}$$

where the integration extends over the sample volume. The bending energy becomes identical to that of I, once $\cos\theta = -\frac{1}{4}$ and $\sin\theta = \sqrt{15}/4$ is inserted in this expression and use is made of the identity $(\operatorname{curl} \vec{n})^2 = (\vec{n} \cdot \operatorname{curl} \vec{n})^2 + (\vec{n} \times \operatorname{curl} \vec{n})^2$. We write it here as

$$F_B^B = \frac{c}{13} \int_V d^3r \, \{ 16[\vec{n} \times (\nabla \times \vec{n})]^2 + 13(\nabla \cdot \vec{n})^2 + 11(\vec{n} \cdot \nabla \times \vec{n})^2 - 2\sqrt{15}(\nabla \cdot \vec{n})(\vec{n} \cdot \nabla \times \vec{n}) + 16 \nabla \cdot [(\vec{n} \cdot \nabla)\vec{n} - \vec{n}(\nabla \cdot \vec{n})] \} , \tag{21}$$

where we have introduced in (12) the Ginzburg-Landau value of ρ_s, which makes the constant c in (21) equal to

$$c = (\hbar^2 / 4m^*) \rho (1 - T/T_c)^{\frac{13}{8}} . \tag{22}$$

In conclusion, we investigate the spin currents associated with our trial solution. Since our trial function (18) is real, there is obviously no mass current associated with it, but there may be spin currents flowing. The spin current j_α^i is in general given by an expression of the form

$$j_\alpha^i \propto d_{\beta j} \epsilon_{ijk} \partial_\alpha d_{\beta k} + d_{\alpha j} \epsilon_{ijk} \partial_\beta d_{\beta k} + d_{\beta j} \epsilon_{ijk} \partial_\beta d_{\alpha k} \tag{23}$$

which satisfies a continuity equation involving the magnetization in the usual manner. This form of

the spin-current reduces to the special case studied by Brinkman and Smith[15] in the appropriate limit. It was recently discussed by Fomin and Vuorio.[8] Since $s_\alpha d_{\alpha i} = 0$, the second term in (23) is seen to give zero contribution to $s_\alpha j^i_\alpha$ at the surface. The other two terms are in general nonzero. This fact does not mean that real spin currents will flow through the surface. As discussed by Fomin and Vuorio[8] the surface boundary condition is obtained from the extremum principle for the free energy and it is not equivalent to the spin current being zero. This is presumably because in the region within a coherence length from the surface the trail solution f may be modified in such a way as to divert the spin currents from flowing through the surface.

D. Surface field energy

The susceptibility anisotropy energy (4) leads in the bulk of the sample to the (isotropic) reduction in the B-phase susceptibility χ_B relative to its normal state value χ_N. When considered in connection with a surface, however, it leads to an important and unusual surface orientational effect. To derive a surface energy from (4) we proceed in analogy with the treatment of the surface dipole energy in Sec. II B and insert the trial function (18) into (4) after replacing the (spin) magnetization $\gamma\vec{S}$ with its equilibrium value proportional to the magnetic field, $\gamma S_i = \chi_N H_i$. We then obtain

$$F_H = c_1 \frac{\Delta^2}{3}\left(\frac{\chi_N}{\gamma}\right)^2 \int d^3r[R_{ij}H_jH_kR_{ik} - (2f - f^2)(s_iR_{ij}H_j)^2]\,. \tag{24}$$

Here c_1 is given in Eq. (5). To extract the surface term involving f we again assume f to be of the form $f = e^{-r_\perp/\xi}$, and integrate over $r_\perp$ to get the surface field energy

$$F_H^S = -c_1 \frac{\Delta^2}{3}\left(\frac{\chi_N}{\gamma}\right)^2 \frac{3}{2}\,\xi \int_S d^2r(s_iR_{ij}H_j)^2 \tag{25}$$

so that the coefficient d in the surface field energy

$$F_H^S \equiv -d\int d^2r(\vec{s}\cdot\vec{R}\cdot\vec{H})^2 \tag{26}$$

is found to be

$$d \simeq \xi(\chi_N - \chi_B)\,. \tag{27}$$

That is d is proportional to the coherence length times the difference in susceptibility between the normal state and the Balian-Werthamer state. As before we drop any numerical factor of order unity due to the uncertainty in our knowledge of the precise variation of $d_{\alpha i}$. An alternative form of (26) is obtained upon insertion of the rotation matrix R_{ij} with the angle $\theta = \cos^{-1}(-\frac{1}{4})$:

$$F_H^S = -\tfrac{25}{16}d\int d^2r[(\vec{s}\cdot\vec{n})(\vec{n}\cdot\vec{H}) + \sqrt{\tfrac{3}{5}}\,\vec{n}\cdot(\vec{s}\times\vec{H}) - \tfrac{1}{5}\vec{s}\cdot\vec{H}]^2\,. \tag{28}$$

Note that the anisotropy energy depends on the angle between the surface normal $\vec{s}$ and the magnetic field $\vec{H}$. In the particular case when $\vec{s}\cdot\vec{H} = \pm|\vec{H}|$ the energy F_H^S is minimized when $\vec{n}$ is parallel (or antiparallel) to $\vec{s}$. In this case the surface field energy has the same orientational effect as the surface dipole energy. When the magnetic field is perpendicular to the surface normal the energy F_H^S is minimized when $(\vec{n}\cdot\hat{H})^2 = \frac{1}{5}$. This angle between the rotation vector $\vec{n}$ and magnetic field $\vec{H}$ was clearly seen in the recent NMR experiments reported in Refs. 3–5. The demonstration in Refs. 4 and 5 of the existence of two relative free-energy minima, when the angle between $\vec{H}$ and $\vec{s}$ was between 90° and 76° provided further confirmation of the present theory.

Put in simple terms the peculiar orientation of $\vec{n}$ for $\vec{H}$ perpendicular to $\vec{s}$ arises as the direction about which a rotation of $\cos^{-1}(-\frac{1}{4})$ makes the surface normal $\vec{s}$ go into the direction of $\pm\vec{H}$. Such a rotation of the spin variables relative to the orbital variables gives the minimum energy, because the depairing of the $m_l = 0$ component along the surface normal then corresponds to a depairing of the $m_s = 0$ component along the magnetic field with a consequent minimum susceptibility anisotropy energy.

E. Characteristic lengths and fields

Sections II A–II D complete the discussion of the theory of textures proposed in I as far as the identification of the important terms in the free energy is concerned. Before we turn to the detailed applications of the theory in the following sections, we shall introduce the characteristic length R_c and the two characteristic fields H_B and H_S, which arise naturally upon combination of the coefficients in front of the four terms in the free energy. The four coefficients a, b, c, and d as given by (6) or (7), (16), (22), and (27) define an important characteristic length

$$R_c = c/b \tag{29}$$

as well as two characteristic fields

$$H_S = (b/d)^{1/2}\,, \tag{30}$$

$$H_B = (b^2/ac)^{1/2}\,. \tag{31}$$

Both H_S and H_B are of order 20–50 Oe and within Landau-Ginzburg theory temperature-independent

quantities; i.e., they have the form of a constant plus corrections of order $(1 - T/T_c)$. As demonstrated in I and experimentally verified in Ref. 6 the line shape of the transverse resonance is determined by the parameter $R_c H_B$, where

$$R_c H_B = (c/a)^{1/2} \propto (1 - T/T_c)^{1/2} . \tag{32}$$

The temperature dependence $(1 - T/T_c)^{1/2}$ is obtained by using the theoretical temperature-independent value of a, as given in (5). If one uses the "experimental" value of a obtained from (7) in terms of the measured shift $\delta\omega$ and the B-phase susceptibility χ_B the product $R_c H_B$ becomes nearly linearly dependent on $(1 - T/T_c)$ due to the rapid drop of χ_B with decreasing temperature and the additional temperature dependence of $\Delta g/g$.[13] The product $R_c H_B$ may in this way be estimated to be 14 cm Oe at $T/T_c = 0.7$ and 23 cm Oe at $T/T_c = 0.5$, somewhat larger than the experimentally observed values in Refs. 3 and 6. Given sufficiently accurate measurements of $R_c H_B$ and $\Delta g/g$ one may use these to determine the temperature dependence of (spin) superfluid density ρ_s in (20), since ρ_s in (22) replaces the combination $2\rho(m/m^*)(1 - T/T_c)$ away from the Landau-Ginzburg regime. We remark in passing that such an experimental determination of ρ_s would in principle allow one to extract the value of the Fermi liquid parameter F_1^a, which affects ρ_s outside the Ginzburg-Landau region.[16]

The estimates of the magnitudes of R_c, H_B, and H_S are obtained using measured values of the longitudinal resonance frequency Ω_L, χ_B, and $\Delta g/g$. The characteristic lengths and fields may be expressed in these quantities as follows, after the numerical constants left out in (16) and (27) have been restored:

$$H_S = [\chi_B/2(\chi_N - \chi_B)]^{1/2}(\Omega_L/\gamma) , \tag{33}$$

$$H_B = 0.23[(1 + F_0^a)(\chi_N/\chi_B)(1 - T/T_c)(\Delta g/g)]^{-1/2} \times (\hbar\Omega_L/E_F)(k_F\xi)(\Omega_L/\gamma) , \tag{34}$$

$$R_c = 5.8(1 + F_0^a)(k_F\xi)^{-1}(E_F/\hbar\Omega_L)^2 \times (1 - T/T_c)(\chi_N/\chi_B)k_F^{-1} , \tag{35}$$

and

$$H_B R_c = 1.32[(1 + F_0^a)(\chi_N/\chi_B)]^{1/2}(\Delta g/g)^{-1/2} \times (1 - T/T_c)^{1/2}k_F^{-1}(E_F/\hbar\gamma) . \tag{36}$$

In the formulas (33)–(36), k_F is the Fermi wave vector and $E_F = \hbar^2 k_F^2/2m^*$ the Fermi energy.

At $T = 0.7T_c$ and with a coherence length $\xi \sim 200$ Å these estimates become $H_S \sim 50$ Oe, $R_c \sim 0.5$ cm, and $H_B R_c \sim 13$ Oe cm, when the experimental values $\chi_B/\chi_N = 0.47$ (from Corruccini and Osheroff[17]), $\Delta g/g = 0.93 \times 10^{-5}$ (from Ref. 13), and $\Omega_L = 2\pi \times 2.3 \times 10^5$ sec^{-1} (from Ref. 3) are used in (33)–(36). Note that the estimates of H_S, H_B, and R_c could easily be off by (50–100)% since the surface boundary condition is not known in detail. The characteristic field H_S is seen not to depend on the coherence length ξ, it being a ratio of two surface quantities. The product $R_c H_B$ depends only on bulk quantities since $R_c H_B = (c/a)^{1/2}$. Its theoretical value is therefore by far the most reliable.

III. TEXTURES IN SIMPLE GEOMETRIES

The vector $\hat{n}$ is very similar to the director of a nematic liquid crystal except that $-\hat{n}$ is not equivalent to $\hat{n}$ (i.e., represents a rotation of $-\theta$ instead of θ). The point singularities allowed are in this respect somewhat more restricted than in liquid crystals. There are many point singularities, but the most simple are (a) the point singularity with $\hat{n}$ pointing radially inward or outward and (b) those with the tangential lines flowing toward the point along one axis and out in the plane perpendicular to that axis. The experimental evidence for these singularities is limited and we now discuss the effect of boundaries where experimental evidence is quite convincing.

A. Parallel plates—small-bending limit

Since the terms in the free energy given by Eqs. (3), (14), (21), and (18) cause considerable complexity even in the simple one-dimensional geometry of parallel plates, we shall use a variational method to obtain an expression for the change in direction of $\hat{n}$ as one moves from one side of the plate to the other. The basic point to note is that the surface field energy F_H^S dominates the problem for the small plate separations and the relatively large magnetic fields employed in Ref. 3. The bulk field energy and bending energy as well as the surface dipole energy may therefore be treated as perturbations on the surface field energy as we shall see in the following.

To specify the geometry we take the x axis to be along the magnetic field $\vec{H}$, the y axis along $\hat{s} \times \vec{H}$ and the z axis along $\hat{s}$. The vector $\hat{n}$ therefore depends on the single variable z. Also we consider always the free energy per unit surface area. The plate separation is denoted by l, and the sample volume of interest is thus the region $-\frac{1}{2}l < z < \frac{1}{2}l$.

We introduce θ and ϕ as the polar and azimuth-

al angles of $\vec{n}$ with respect to axes along $\vec{s}$ and $\vec{H}$ (see Fig. 1). In these coordinates the unit vector $\vec{n}$ is

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{37}$$

where $\theta = \theta(z)$ and $\phi = \phi(z)$ describe the spatial variation of $\vec{n}$. Upon insertion of (37) in the energies (28), (14), (3), and (21) we obtain for each of these energies per unit surface area

$$F_H^S = -\tfrac{25}{16} dH^2(\cos\theta_S \sin\theta_S \cos\phi_S + \sqrt{\tfrac{3}{5}} \sin\theta_S \sin\phi_S)^2, \tag{38}$$

$$F_D^S = -b[(\cos\theta_S)^2 - \tfrac{5}{18}(\cos\theta_S)^4], \tag{39}$$

$$F_H^B = -aH^2 \int_{-l/2}^{l/2} dz \sin^2\theta\cos^2\phi, \tag{40}$$

$$F_B^B = \frac{c}{13} \int_{-l/2}^{l/2} dz[\theta'^2(16 - 3\sin^2\theta) + \phi'^2(16\sin^2\theta - 5\sin^4\theta) - 2\phi'\theta'\sqrt{15}\sin^3\theta]. \tag{41}$$

The values of θ and ϕ at the surfaces are denoted by θ_S and ϕ_S and θ', ϕ' denote derivatives with respect to z. Due to the symmetry in the problem we consider only even trial functions. Thus $\phi_S = \phi(\pm\frac{1}{2}l)$, $\theta_S = (\pm\frac{1}{2}l)$.

Consider the surface field energy F_H^S by itself. It is minimized when

$$\tan\phi_S = \sqrt{\tfrac{3}{5}}/\cos\theta_S \tag{42}$$

which upon insertion in (38) yields a quadratic form in the variable $\cos^2\theta_S$. The minimum occurs at

$$\cos^2\theta_S = \tfrac{1}{5} \tag{43}$$

as discussed in Sec. II D. This value corresponds in turn to the azimuthal angle

$$\tan\phi_S = \pm\sqrt{3} \tag{44}$$

by Eq. (42).

Note that the angle α between $\vec{n}$ and $\vec{H}$, which determines the observed resonance behavior, is obtained from

$$\cos^2\alpha = \sin^2\theta\cos^2\phi$$

and hence

$$\cos^2\alpha_S = \tfrac{1}{5} \tag{45}$$

when $\theta = \theta_S$, $\phi = \phi_S$.

We next investigate the influence of the bulk magnetic field energy F_H^B and the bending energy F_B^B (leaving the surface dipole energy F_D^S for later consideration, since it is by far the weakest perturbation under the experimental conditions of Ref. 3).

To do this we choose a variational solution of

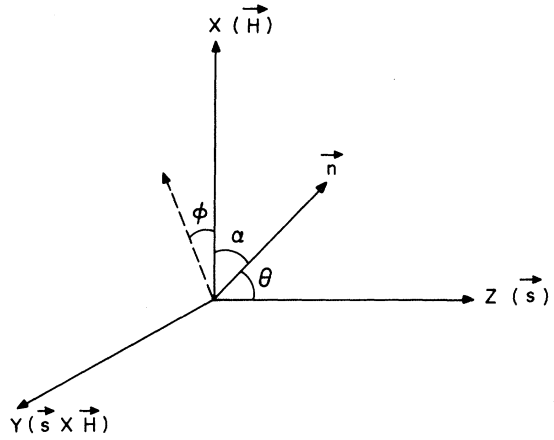

FIG. 1. Coordinate system used in the parallel plate geometry with $\vec{s}$ the surface normal and $\vec{H}$ the magnetic field. In addition to the polar angles θ and φ that specify the orientation of $\vec{n}$ the physically important angle α between $\vec{n}$ and the magnetic field is indicated.

the form

$$\theta(z) = \theta_0 + \delta\theta(z), \qquad \phi(z) = \phi_0 + \delta\phi(z), \tag{46}$$

where $\delta\theta$ and $\delta\phi$ are much less than one in small bending limit. The constant angles θ_0 and ϕ_0 are those angles which minimize the energy in the infinite stiffness limit ($c \to \infty$). With a finite stiffness present the $\vec{n}$ vector tries to take more advantage of the bulk magnetic field at the expense of bending. The infinite stiffness solution is obtained by minimizing $F_H^B + 2F_H^S$ and leads to

$$\cos^2\alpha_0 = \sin^2\theta_0\cos^2\phi_0 = \tfrac{1}{5}(1 + 4al/5d). \tag{47}$$

The correction to (45), $4al/5d$, differs by a factor of 2 from the result given in Ref. 3, Eq. (3). The factor of 2 accounts for the presence of two surfaces in the parallel plate geometry. In accord with the smallness of the correction term $4al/5d$ for the experiments of Ref. 3, we will neglect it when solving for $\delta\theta$ and $\delta\phi$ in the finite stiffness problem.

The energy to be minimized may be written in the form

$$F = \int_{-l/2}^{+l/2} dz[A(\theta)\theta'^2 + B(\theta)\phi'^2 + D(\theta,\phi) + E(\theta)\theta'\phi'] + 2F_H^S(\theta,\phi), \tag{48}$$

where

$$A(\theta) = \tfrac{1}{3}c(16 - 3\sin^2\theta),$$

$$B(\theta) = \tfrac{1}{3}c(16\sin^2\theta - 5\sin^4\theta),$$

$$D(\theta,\phi) = -aH^2\sin^2\theta\cos^2\phi,$$

$$E(\theta) = -c\,\tfrac{2}{13}\sqrt{15}\sin^3\theta.$$

The infinite stiffness solutions θ_0 and ϕ_0 satisfy the equations

$$l\frac{\partial D}{\partial\theta}\bigg|_{\theta_0,\phi_0}+\frac{\partial 2F_H^S}{\partial\theta}\bigg|_{\theta_0,\phi_0}=0$$

and

$$l\frac{\partial D}{\partial\phi}\bigg|_{\theta_0,\phi_0}+\frac{\partial 2F_H^S}{\partial\phi}\bigg|_{\theta_0,\phi_0}=0. \tag{49}$$

Minimizing F with respect to $\delta\theta$ and $\delta\phi$ we obtain

$$\begin{aligned}\delta\theta''&=\left(E\frac{\partial D}{\partial\phi}-2B\frac{\partial D}{\partial\theta}\right)\Big/(E^2-4AB),\\ \delta\phi''&=\left(E\frac{\partial D}{\partial\theta}-2A\frac{\partial D}{\partial\phi}\right)\Big/(E^2-4AB).\end{aligned} \tag{50}$$

The terms on the right-hand side are evaluated at θ_0 and φ_0. In deriving these equations we have neglected terms that are of higher order in the small parameter $H^2l^2/H_B^2R_C^2$, since we shall only consider corrections to the observed NMR frequency shift (see Sec. IV) that are linear in this. The differential Eqs. (50) are completely consistent with the vanishing of the surface terms obtained by minimizing (48) and using conditions (49). Thus no additional boundary conditions need be imposed on the surfaces. Using $\theta_0=\theta_S$, $\varphi_0=\varphi_S$ (we take the positive solutions) as given in (43) and (44) together with (48) and (50) we obtain

$$\begin{aligned}\delta\theta''&=0,\\ \delta\varphi''&=\tfrac{13}{48}\sqrt{3}\;H^2/(H_BR_C)^2.\end{aligned} \tag{51}$$

Thus corresponding to the even solutions for $\theta(z)$ and $\varphi(z)$, the spatial variation of $\vec{\mathrm{n}}$ is such as to keep its angle with $\vec{\mathrm{s}}$ fixed at θ_0, however, the azimuthal angle changes slightly so that at the center $\vec{\mathrm{n}}$ is a bit closer to being parallel to $\vec{\mathrm{H}}$, φ then swings back symmetrically at the walls. The solutions for θ and φ as functions of z are

$$\begin{aligned}\theta&=\theta_0,\\ \varphi&=\varphi_0+\tfrac{1}{2}\delta\varphi''[z^2-(\tfrac{1}{2}l)^2].\end{aligned} \tag{52}$$

The corresponding solution for $\cos^2\alpha=(\vec{\mathrm{n}}\cdot\vec{\mathrm{H}})^2/H^2$ is

$$\cos^2\alpha=\frac{1}{5}\left\{1+\frac{4}{5}\frac{al}{d}+\frac{13}{64}\left(\frac{Hl}{H_BR_C}\right)^2\left[1-\left(\frac{2z}{l}\right)^2\right]\right\}. \tag{53}$$

In Fig. 2 we show this variation of the angle α across the plate for typical values of the parameters in the expression.

In conclusion we remark that the dipole surface energy F_D^S gives a negligible contribution to the total free energy for fields greater than 300 Oe as in Ref. 3, since it introduces corrections of order $b/dH^2\equiv(H_S/H)^2\simeq 10^{-2}$ according to the estimate below Eq. (36). At lower fields it may be included in the calculation as a perturbation. The result is [to lowest order in $(H_S/H)^2$ when only the dominant first term in (39) is considered] to add the term $\frac{4}{25}(H_S/H)^2$ to the right-hand side of the expression (53) for $\cos^2\alpha$. Note that the surface dipole energy tends to align $\vec{\mathrm{n}}$ with the magnetic field contrary to what might intuitively be expected. At lower fields however the effect of the surface dipole energy is to pull $\vec{\mathrm{n}}$ away from $\vec{\mathrm{H}}$.

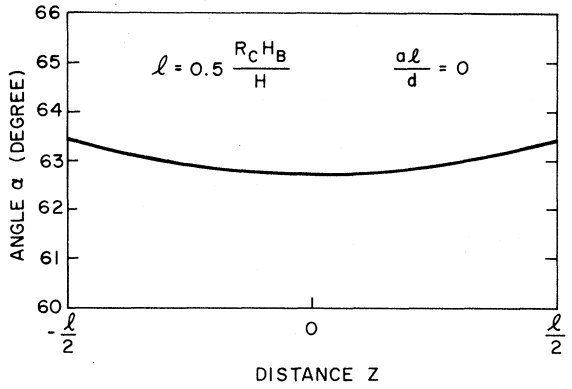

FIG. 2. Variation of the angle α between $\vec{\mathrm{n}}$ and the magnetic field as a function of distance according to formula (53). The plate separation l was chosen to equal $0.5\,R_cH_B/H$, whereas al/d was set equal to zero, resulting in $\alpha=\cos^{-1}(1/\sqrt{5})=63.4°$ at the surfaces $z=\pm\frac{1}{2}l$.

B. Textures in a cylinder

The minimization of the total free energy and the consequent determination of the texture is a nontrivial problem even in a one-dimensional geometry like the parallel plates considered above. Clearly, the extension to the dimensions of a cylinder with a magnetic field along its axis makes the minimization far more complex. The tendency for the surface field energy F_H^S to orient $\vec{\mathrm{n}}$ in a direction which is neither parallel nor perpendicular to the surface is a particularly complicating feature. At low values of the magnetic field, however, the surface field energy may be treated as a perturbation on the appropriate zero field texture. We shall therefore restrict the explicit calculation of texture free energies to the low-field limit and discuss initially the case of zero magnetic field.

In the absence of a magnetic field the texture is determined by a competition of the bulk bending energy and the surface dipolar energy, the latter seeking to orient $\vec{\mathrm{n}}$ parallel to the cylinder surface normal $\vec{\mathrm{s}}$. For values of the cylinder radius R

smaller than or comparable to the characteristic length $R_c = c/b$ one expects an "in-plane" structure like the one shown in Fig. 3(a) to be energetically favored. (This texture is more stable than one with a singular line down the center because of the lack of rigidity of the boundary conditions.) As R grows the bending energy increases like $\ln(R/R_c)$ and eventually one finds a lower free energy for a "flare-out" configuration like the one of Fig. 3(b) in which $\vec{n}$ flares out of the plane. Below we detail the calculation of energies for these two textures and determine the critical cylinder radius for which their energies become equal. The calculation we perform employs trial functions for the variation of the order parameter vector $\vec{n}$. It gives therefore only upper bounds to the free energies and an approximate determination of the crossover between the two textures.

In both textures we shall choose trial functions that are sufficiently simple that $\vec{n}\cdot\mathrm{curl}\vec{n}$ is zero. This means that the third and the fourth term in (21) vanish. Though it is clear from (21) that a lowering of the bending energy could be obtained by keeping $(\vec{\nabla}\cdot\vec{n})$ and $\vec{n}\cdot\mathrm{curl}\vec{n}$ finite, thus taking advantage of the cross term $(\vec{\nabla}\cdot\vec{n})\vec{n}\cdot\mathrm{curl}\vec{n}$, our estimates indicate this to have rather a small effect on the texture. For the in-plane structure its effect would be to tilt $\vec{n}$ out of the plane, since $\vec{n}\cdot\mathrm{curl}\vec{n}$ is zero for $\vec{n}$ lying in the plane perpendicular to the cylinder axis. We shall return to the question regarding the influence of this cross term after the calculation of texture energies below.

Taking the z axis along the axis of the cylinder

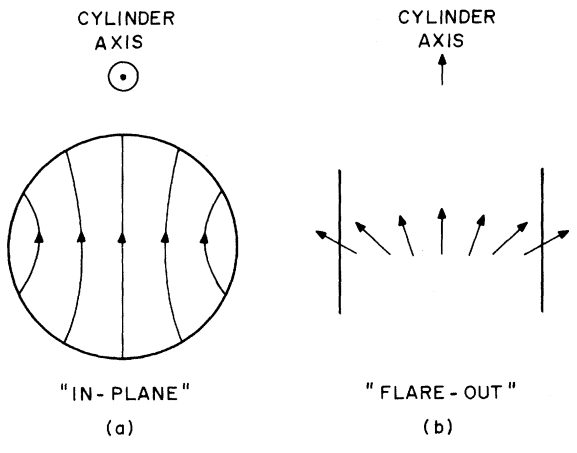

FIG. 3. Schematic drawings of the orientation of the $\vec{n}$-vector in the (a) "in-plane" texture and (b) the "flare-out" texture in the cylindrical geometry. The direction of the cylinder axis is indicated in each case.

the "in-plane" texture in Fig. 3(a) is defined by the following components of the $\vec{n}$ vector,

$$n_x = \cos\chi = \frac{R_0^2 + x^2 - y^2}{[(R_0^2 + x^2 - y^2)^2 + 4y^2x^2]^{1/2}}\ , \tag{54}$$

$$n_y = \sin\chi = \frac{2xy}{[(R_0^2 + x^2 - y^2)^2 + 4y^2x^2]^{1/2}}\ , \tag{54'}$$

$$n_z = 0\ , \tag{54''}$$

where R_0 is a length ($R_0 > R$) that acts as a variational parameter. The longer R_0 is compared to R the less curvature (and hence bending energy) in the texture. On the other hand, when R_0 is close to R one gains considerable surface dipole energy, since $\vec{n}$ is then nearly everywhere perpendicular to the cylinder surface. The value of R/R_0 which minimizes the energy depends on the ratio ρ of the cylinder radius R to the characteristic length R_c, $\rho = R/R_c$. When $\rho \gg 1$, it is energetically favorable to take nearly full advantage of the surface dipole energy, hence $R_0 \sim R$. In the opposite limit $\rho \ll 1$ the balancing of surface dipole and bending energy results in $R_0 \gg R$.

It may be readily shown that the last term in the bending energy (21) is zero, when the trial function (54) is inserted. [This holds irrespective of the functional dependence of n_x and n_y on x and y. The term also vanishes for a texture $\vec{n}(z)$ that depends on a single Cartesian coordinate z-like in the parallel plate situation considered above. However, the term is nonzero if $\vec{n}$ is tilted away from the $x-y$ plane as in the "flare-out" texture, as we shall see below.]

When (54) is inserted in the bending energy (21) and the surface dipole energy (15) one gets the following results for the free energy,

$$F_D^S/2\pi c = -\tfrac{1}{2}(1+\gamma^2)\rho \tag{55}$$

and

$$\frac{F_B^B}{2\pi c} = -\tfrac{29}{26}\ln(1-\gamma^4) - \tfrac{3}{13}\gamma^2 + \tfrac{3}{26}\ln\left(\frac{1+\gamma^2}{1-\gamma^2}\right), \tag{56}$$

where we have introduced the dimensionless quantities

$$\rho = R/R_c \tag{57}$$

and

$$\gamma = R/R_0\ \ (<1)\ . \tag{58}$$

The minimization of the total free energy $F = F_D^S + F_B^B$ is readily performed for general values of ρ. Here we shall only exhibit the result for $\rho \ll 1$ and $\rho \gg 1$. For small cylinder radii $R \ll R_c$ one gets

$$\gamma^2_{\min} = \tfrac{13}{58}\rho, \quad \rho \ll 1 \tag{59}$$

and a total minimum free energy given by

$$F/2\pi c = -\tfrac{1}{2}\rho - \tfrac{13}{232}\rho^2 . \tag{60}$$

In the opposite limit $\rho \gg 1$ one has

$$\gamma^2_{\min} = 1 - \tfrac{16}{13}(1/\rho), \quad \rho \gg 1 \tag{61}$$

and a corresponding minimum free energy given by

$$F/2\pi c = -\rho + \tfrac{16}{13}\ln\rho - 0.8018 . \tag{62}$$

Note that the bending energy introduces a logarithmic increase with ρ in addition to the term linear in ρ originating in the surface dipole energy.

Turning now to the calculation of the free energy of the flare-out texture we shall take as a trial function

$$n_x = \cos\theta \sin(\beta r/R) , \tag{63}$$

$$n_y = \sin\theta \sin(\beta r/R) , \tag{63'}$$

$$n_z = \cos(\beta r/R) , \tag{63''}$$

where $\beta(\leq \frac{1}{2}\pi)$ is a variational parameter, and we have introduced cylindrical coordinates (r, θ). When $\beta = 0$, $\vec{n}$ points everywhere along the cylinder axis. When $\beta = \frac{1}{2}\pi$ the in-plane radial component of $\vec{n}$ varies from being zero at the cylinder axis to unity at the surface. Clearly we must have $\beta \simeq \pi$ when $R \gg R_c$ while one expects $\beta = 0$ in the limit $R \ll R_c$ as confirmed by the explicit calculation below. When (63) is inserted in (15) the surface dipole energy becomes

$$F^S_D/2\pi c = -\rho \sin^2\beta \tag{64}$$

and the bending energy

$$\frac{F^B_B}{2\pi c} = \int_0^\beta d\phi \left[\frac{1}{\phi}(\sin\phi + \phi\cos\phi)^2 + \tfrac{16}{13}\phi\sin^2\phi \right] - \tfrac{16}{13}\sin^2\beta . \tag{65}$$

The last term in this expression originates in the last term of (21) which has been discussed above.

In the limit of small cylinder radii, $R \ll R_c$, the value of β which minimizes the total free energy is zero until the radius becomes bigger than $R = \frac{10}{13}R_c$, as seen by performing a small β expansion.

The corresponding total free energy is therefore likewise zero when $\beta = 0$. It turns negative when $R > \frac{10}{13}R_c$ and it becomes energetically favorable to bend the $\vec{n}$ vector slightly away from being parallel to the cylinder axis.

In the opposite limit $R \gg R_c$ one has

$$\beta_{\min} = \tfrac{1}{2}\pi \tag{66}$$

and

$$F/2\pi c = -\rho + 2.038 . \tag{67}$$

It follows from the comparison of (62) and (67) that the in-plane texture is energetically favored as long as

$$\ln\rho < 2.307 \tag{68}$$

or

$$R < 10.0 R_c . \tag{69}$$

To investigate the influence of a magnetic field on the crossover between these two textures we shall work in the limit $R \gg R_c$ and treat the magnetic field terms F^B_H and F^S_H as perturbations. Using as zeroth-order solutions the variational functions obtained from (63) and (66) one obtains in the flare-out texture a bulk field energy

$$F^B_H/2\pi c = -0.1487(H/H_B)^2\rho^2 , \tag{70}$$

but no contribution from F^S_H because $\vec{n}$ is pointing radially outward from the surface. On the other hand for the in-plane texture the bulk field energy F^B_H does not contribute to the energy because $\vec{n}$ is always perpendicular to $\vec{H}$. The surface field energy F^S_H does, however, because of the singularities near the surface and is

$$F^S_H/2\pi c = -\tfrac{15}{13}(H/H_S)^2 . \tag{71}$$

With the assumption $H_B \sim H_S$ [see (33), (34), and below], we may obtain an explicit value for the critical radius in a field. Let

$$\rho = \rho_0(1 + \delta), \tag{72}$$

where ρ_0 is the previously determined zero-field value, $\rho_0 = 10.0$. Then it follows that the texture energies are equal when

$$\delta = -13.7(H/H_B)^2 . \tag{73}$$

Thus we have an estimate of the field dependence of the critical radius for small values of the magnetic field,

$$R = 10.0R_c - 137(H/H_B)^2 R_c . \tag{74}$$

We briefly comment on the effect of the cross term $(\mathrm{div}\,\vec{n})(\vec{n}\cdot\mathrm{curl}\,\vec{n})$ in the bending energy (21). For the in-plane texture we have considered small deviations δn_z away from the previously considered orientation of $\vec{n}$ in the plane perpendicular to the z axis. One finds that δn_z satisfies an inhomogeneous differential equation for which the solutions in the limit of large $R(R \gg R_c)$ must vanish at the cylinder surface as well as on the lines $x = 0$ and $y = 0$. This implies that δn_z may contribute at most a ρ-independent constant to the free energy of the texture and we expect the value of this constant to be rather small due to the constraints on the variation of n_z. The effect of a

nonvanishing n_z is of course to lower the free energy, but the consequent change in the value of the critical radius depends on how much the cross term may lower the energy of the flare-out texture as well. In this case it is clear that the cross term can at most contribute a constant for large R/R_c since there are no logarithmic terms in the bending energy.

To extend the previous calculations to larger field is complicated, but the qualitative answers are clear. As the field increases the field dependent surface term becomes larger than the dipolar term and $\vec{n}$ orients itself at the peculiar direction discussed earlier. The characteristic length at which a textural transition qualitatively similar to the one calculated above occurs then becomes $R_c H_S^2/H^2 = c/dH^2$. Note however that $\vec{n}$ never lies in the plane perpendicular to the axis because of the effect of the field dependent surface term. The characteristic length is very small, at fields of ~1 kG it is $R_c H_S^2/H^2 \sim 1$ μm. Undoubtedly a transition between two textures with topological differences similar to those at low fields still occurs, but the details are quite different since there must be a sizable nonzero z component of $\vec{n}$ in both textures.

A continuous change always occurs at larger radii in the flare-out texture from a texture whose characteristic bending is determined by the walls as in the zero-field calculation to one where the effects of walls are "healed" in a coherence length $R_H = R_c H_B/H$. This changeover is pictured qualitatively in Fig. 4.

IV. SPIN RESONANCE IN THE PRESENCE OF TEXTURES

In this concluding section we discuss the calculation of the nuclear-magnetic-resonance spectrum in the presence of textures. The procedure for performing such a calculation can be formulated using a Ginzburg-Landau approach similar to that discussed by Brinkman and Smith[15] to obtain spin waves in a uniform medium. However, actual calculations can only be performed in certain limiting cases and the results used for the closely placed parallel plates[3] and the large radius cylinder will be shown to be two opposite extremes of a relatively simple equation. In the general formulation of the calculation of spin-wave spectra one first calculates the change in free energy when the spin coordinates of the order parameter $d_{\alpha i}$ are rotated by small rotation $\vec{\theta}(\vec{r})$ $=(\theta_x(\vec{r}), \theta_y(\vec{r}), \theta_z(\vec{r}))$. This can be done by inserting into all the various expressions used in Sec. II the form

$$\bar{d}_{\alpha i}(\vec{r}) = d_{\alpha\beta}(\vec{r}) R_{\beta i}(\vec{\theta}) , \tag{75}$$

where $d_{\alpha\beta}(\vec{r})$ is the matrix given in Eq. (9) and $R_{\beta i}(\vec{\theta})$ is the rotation matrix of the small rotations, $\vec{\theta}$,

$$R_{\alpha\beta}(\vec{\theta}) = \delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma}\theta_\gamma - \tfrac{1}{2}(\delta_{\alpha\beta}\vec{\theta}^2 - \theta_\alpha\theta_\beta). \tag{76}$$

The result of this calculation is a total free energy density $F(\vec{\theta})$ which in equilibrium satisfies the equation

$$\delta F/\delta\theta_\alpha(\vec{r}) = 0 \tag{77}$$

which has been attained by a previous variational calculation with respect to $\vec{n}$. The simplest example of the terms that occur in F is the bulk dipole energy, Eq. (10)

$$F_D^B = -\tfrac{1}{6}\Gamma\Delta^2 + \tfrac{5}{2}\Gamma\Delta^2(\vec{n}\cdot\vec{\theta})^2 . \tag{78}$$

(We have assumed $\theta = 104°$ in $d_{\alpha i}$.) Other bulk terms arise from the variations of Eq. (3) and from variations of the bending energy.

In addition there are the terms coming from the variation of the surface energies. The expressions for these terms are all complicated and will not be written down explicitly, rather, we will attempt to concentrate only on the largest energies and argue that the other smaller energies have a negligible effect on the spin-wave spectrum. For example, any contributions to the spin-wave frequencies arising from the variation of the bulk anisotropy energy Eq. (3) would be of order $(\Delta g/g)(\gamma H) \simeq 10^{-5}(\gamma H)$. This shift is to be compared to that which will arise from Eq. (78) which is of order $\Omega_L^2/\gamma H$, which is a hundred to a thousand times larger than the above estimate in a field of 1 kG. The surface anisotropy energies may also be neglected. The surface dipolar energy gives a correction of order $(\xi/l)\Omega_L$ which is negligible. The susceptibility anisotropy is rotationally invariant and consequently does not enter the spin wave equations just as the bulk susceptibility

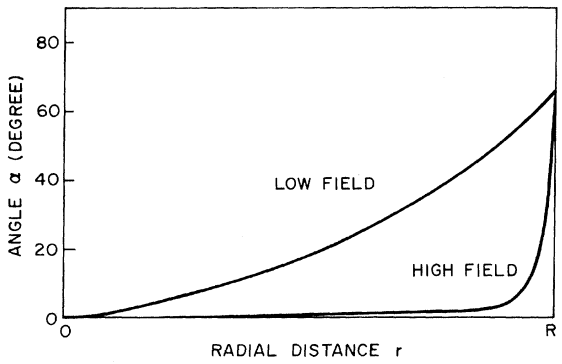

FIG. 4. Qualitative illustration of the behavior of the angle α between $\vec{n}$ and the magnetic field in the high-field ($H \gg H_B R_c/R$) and low-field ($H \ll H_B R_c/R$) regimes for the "flare-out" texture discussed in the text.

anisotropy does not effect the resonance equations.[12]

In the bending energy there are three classes of terms depending on where the differential operators are acting:

(i) $\theta_i\,\theta_j\,\partial_\alpha n_k\,\partial_\beta n_l$,

(ii) $\theta_i\,\partial_\alpha\theta_j\,n_k\,\partial_\beta n_l$,

(iii) $\partial_\alpha\theta_i\,\partial_\beta\theta_j$.

One can show that terms of the first class add to zero using the properties of the rotation matrices. Using Eq. (17) the second class of terms can be shown to be of the form

$$F^\gamma_{\alpha\beta}\theta_\alpha\partial_\gamma\theta_\beta\ ,$$

where $F^\gamma_{\alpha\beta}$ is symmetric in $\alpha\beta$. In the equation of motion for the spin angular momentum, this term will contribute a term of the form

$$(\partial_\gamma F^\gamma_{\alpha\beta})\theta_\beta\ .$$

However $\partial_\gamma F^\gamma_{\alpha\beta}$ must be of order $c(H/R_cH_B)^2\sim aH^2$. Therefore, the class-(ii) terms will contribute terms of the same order as those coming from the g-shift term Eq. (3). Therefore, of the bending terms we must consider only terms of class (iii). These can be reduced to the following contribution to the bending energy density F^B_B:

$$F^B_B=\tfrac{1}{65}\{32\partial_\alpha\theta_\beta\partial_\alpha\theta_\beta-[\nabla\cdot\vec\theta-5\vec n\cdot(\vec n\cdot\nabla)\vec\theta-\sqrt{15}\,\vec n\cdot(\nabla\times\vec\theta)]^2\}. \tag{79}$$

This expression plus Eq. (78) constitute the largest terms in $F(\vec\theta)$ and we propose to use the sum of these two expressions as a lowest-order approximation for $F(\vec\theta)$.

The total free energy determining the spin dynamics is obtained by adding the terms involving $\vec S$ to the free energy $F(\vec\theta)$.

$$F=\gamma^2\vec S^2/2\chi_B-\gamma\vec S\cdot\vec H+F(\vec\theta) \tag{80}$$

and the equations of motion are

$$\frac{d\vec S}{dt}=-\frac{i}{\hbar}\{\vec S,F\}, \tag{81}$$

$$\frac{d\vec\theta}{dt}=-\frac{i}{\hbar}\{\vec\theta,F\}, \tag{81'}$$

where $\{\theta_i,S_j\}=i\hbar\delta_{ij}$, $\{\theta_i,\theta_j\}=0$, $\{S_i,S_j\}=i\hbar\epsilon_{ijk}S_k$.

The resulting equations are

$$\partial_t\vec S=\gamma\vec S\times\vec H-(1/\gamma^2)\chi_B\Omega_L^2\vec n(\vec n\cdot\vec\theta)+\tfrac{1}{65}\{64\nabla^2\vec\theta-2[\vec\nabla-5\vec n(\vec n\cdot\vec\nabla)-\sqrt{15}\,\vec n\times\vec\nabla]\times[\nabla\cdot\vec\theta-5\vec n(\vec n\cdot\nabla)\vec\theta-\sqrt{15}\,\vec n\cdot(\vec\nabla\times\vec\theta)]\}, \tag{82}$$

$$\partial_t\vec\theta=\gamma(\gamma\vec S/\chi_B-\vec H). \tag{82'}$$

In (82) the gradient ∇ operates only on $\vec\theta$, not on $\vec n$. Note that when $\vec n$ is spatially uniform the equations become identical to those of Brinkman and Smith.[15] Equations (82) are complicated and we only discuss them in the high-field limit which applies to many of the experiments performed to date.

In this limit, the transverse resonance is an oscillation of $S_+=S_y+iS_z$, where the field is along the x axis. If we in addition assume that the texture is varying only in the z direction then we need only consider uniform waves in the x and y directions so that we arrive at the equation

$$[\omega^2-\omega\gamma H-\tfrac{1}{2}\Omega_L^2n_\perp^2(z)]\,S_+=\frac{-c\gamma^2}{65\chi_B}(63+10n_z^2-25n_\perp^2n_z^2+10\sqrt{15}\,n_xn_yn_z-15n_x^2)\partial_z^2S_+. \tag{83}$$

We are left with a relatively simple equation to solve for the possible spin waves. It is not difficult to show that in this approximation the boundary condition is that the spin current into the wall is zero and this condition reduces to $\partial_z S_+|_{\text{boundary}}=0$. For any given eigenfunction S_+^m of this equation the contribution to the absorption of a uniform field oscillating at its eigenfrequency ω is given by

$$I_m=\frac{1}{l}\left(\int_{-l/2}^{l/2}S_+^m\,dz\right)^2\Big/\int_{-l/2}^{l/2}[S_+^m(z)]^2dz\ , \tag{84}$$

where we have normalized the contributions so that the total $\sum_m I_m=1$. In practice the normalization would be given by the usual sum rule.[11] The possible regimes in the parallel plate problem can be characterized by two lengths, $R_s=(2c\gamma^2/\chi_B\Omega_L^2)^{1/2}$ which is the characteristic length in the above equation and $R_H=R_cH_B/H$ the characteristic length for the texture itself. The experimental situation is always such that $R_s<R_H$, since R_s is typically 10 μm whereas R_H is typically 100 μm or larger. Therefore, we only need to worry about the plate separation l relative to R_H. Simple regimes that can be analyzed are as follows:

(i) $l>>R_H$ and $R_H>>R_s$. Under these circumstances the "local oscillator" approximation which is simply a classical approximation whereby the energy distribution is given by the potential energy should be correct. This is the regime in which the original experiments by Osheroff and Brinkman in the cylindrical geometry were performed. The spectral distribution is given by

$$P(\omega)=\int\delta\left(\omega-\frac{\Omega_L^2}{2\gamma H}n_\perp^2(z)-\gamma H\right)dz\ . \tag{85}$$

This result was used by Brinkman, Smith, Osher-

off, and Blount to calculate the spectrum of a cylinder whose radius is large compared to R_H. It should be noted that this approximation is equivalent to assuming that the eigenvalues are so close together that they can be averaged over. If $\vec{n}$ is parallel to $\vec{H}$ in the center of the sample the linewidth of each mode may be such that the eigenvalues are still separated and this approximation is not valid.

(ii) $l \ll R_H$ and $l/2\pi \ll (R_s R_H)^{1/2}$. This is the small bending limit discussed in Ref. 3 and Sec. III A. Under these conditions the spins precess in a uniform mode at a single shifted frequency which may be obtained by taking the spatial average $\langle n_\perp^2 \rangle = 1 - \langle \cos^2\alpha \rangle$ in the notation of Sec. III A. The shift in the parallel plate geometry is thus given by

$$\langle \cos^2\alpha \rangle = \tfrac{1}{5}\ (1 + 4al/5d + \tfrac{13}{96}\ l^2/R_H^2)\ . \tag{86}$$

That l should be much less than $(R_s R_H)^{1/2}$ can be obtained by fact that the variation of the potential energy in Eq. (83) is of order $(\frac{1}{2}\Omega_L^2)(l/R_H)^2$ whereas the separation of the spin-wave modes is $(c/\chi_B)(2\pi\gamma/l)^2$. Therefore, for perturbation theory in the potential to be valid we must have

$$\frac{\Omega_L^2}{2}\left(\frac{l}{R_H}\right)^2 \ll \frac{c\gamma^2}{\chi_B}\left(\frac{2\pi}{l}\right)^2 \quad \text{or} \quad l \ll (2\pi R_s R_H)^{1/2}\ .$$

The criterion for the perturbation expansion to be valid may be explored by calculating the second-order correction arising from the mixing-in of higher spin-wave modes. To do this one writes (83) in dimensionless form as

$$[E - n_\perp^2(z)]\, S_+ = -R_s^2 f(\vec{n}) \frac{\partial^2 S_+}{\partial z^2}\ , \tag{87}$$

where

$$E = (\omega - \gamma H) 2\gamma H/\Omega_L^2 \tag{88}$$

and f depends on the direction of $\vec{n}$. For the direction of $\vec{n}$ which minimizes the surface field energy $f = \frac{64}{65}$ [we may safely neglect any spatial variation of $f(\vec{n})$]. The even eigenfunctions u_m obeying the boundary condition $u_m(\pm\frac{1}{2}l) = 0$ for the differential equation (87) with $n_\perp^2 = \text{const}$ are

$$u_m(z) = a_m \cos(2m\pi z/l),\ m = 0, 2\ldots, \tag{89}$$

where

$$a_m = \begin{cases} \sqrt{1/l} & \text{for } m = 0 \\ \sqrt{2/l} & \text{for } m = 1, 2\ldots\ . \end{cases} \tag{90}$$

The solution for a spatially varying $n_\perp^2$ may be characterized by the two dimensionless quantities

$$\alpha = l/R_s \text{ and } \epsilon = l/R_H\ .$$

The calculation of the first- and second-order corrections E^1 and E^2 to the unperturbed energies

$$E_m^0 = \tfrac{64}{65}\ (2\pi)^2 m^2 (1/\alpha^2) + n_\perp^2(\tfrac{1}{2}l) \tag{91}$$

is standard and we shall simply state the results obtained for the spatial variation of $\vec{n}$ given by Eq. (53). With $V = n_\perp^2(z) - n_\perp^2(\frac{1}{2}l)$ acting as the perturbation we get

$$E_0^1 = -\tfrac{13}{480}\ \epsilon^2 \tag{92}$$

in accordance with (86) above and

$$E_0^2 = -\frac{2}{\pi^6}\left(\frac{13}{320}\right)^2 \sum_{m\neq 0} \frac{1}{m^6}\ \alpha^2\epsilon^4 \tag{93}$$

so the change in the dimensionless energy (88) is to second order

$$E_0 - E_0^0 \simeq E_0^1(1 + \alpha^2\epsilon^2 \times 3 \times 10^{-6}). \tag{94}$$

It is obvious that the expansion parameter is not $\alpha^2\epsilon^2$ but more nearly $\alpha^2\epsilon^2/(2\pi)^4$ in accordance with the criterion given above.

Similarly we may find the intensities I_m to lowest order in $\alpha\epsilon$. From the admixture of the $m = 0$ unperturbed wave function in the mth excited state we obtain

$$I_m = |\langle 0|V|m\rangle|^2/(E_m^0 - E_0^0)^2$$

$$= (1/m^8)\alpha^4\epsilon^4 \times 8.97 \times 10^{-8}\ . \tag{95}$$

The m^{-8} dependence is due to the energy differences going as m^2 and the matrix element $\langle 0|V|m\rangle$ going as m^{-2}. Therefore the intensity falls off very rapidly in the modes above $m = 1$. Note also that I_m is proportional to ϵ^4 and hence to the fourth power of magnetic field for fixed energy separation.

(iii) $l \lesssim R_H$ and $l/2\pi \gtrsim (R_s R_H)^{1/2}$. Here explicit calculations must be carried out for the individual modes. This is currently being investigated and will be reported later. It is in this intermediate regime that the full effects of textures and spin-wave velocities are expected. Experimentally, this regime shows a diverse behavior which is not at present understood.

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APPENDIX

In order to obtain the relation between a in Eq. (3) and the g shift we calculate the time-averaged change in free energy under resonance conditions. Since only terms that are first order in a and

zeroth order in the dipole interaction are of interest we need only study the motion of $\vec{S}$ and $\hat{n}$ under the influence of the external field. The equations of motion are in the form given by Brinkman,[18]

$$\frac{d\vec{S}}{dt} = \gamma\vec{S}\times\vec{H}$$

and

$$\frac{d\hat{n}}{dt} = (-\tfrac{1}{2})\gamma\left\{\hat{n}\times\left(\frac{\gamma\vec{S}}{\chi_B} - \vec{H}\right) + f\,\hat{n}\times\left[\hat{n}\times\left(\frac{\gamma\vec{S}}{\chi_B} - \vec{H}\right)\right]\right\},$$

where

$$f = \sin\theta_0/(1-\cos\theta_0).$$

The rotation angle θ_0 can be considered a constant for transverse resonance when $\hat{n}_0$ is parallel to $\vec{H}$.

The motion of $\vec{S}$ is given by the usual expression

$$\vec{S} = (\delta S\cos\omega t,\ \delta S\sin\omega t,\ S[1-(\delta S/S)^2]^{1/2})$$

and solving for $\hat{n}$ to second order in δS we find

$$\hat{n}(t) = \tfrac{1}{2}(\delta S/S)(\cos\omega t + f\sin\omega t,\ \sin\omega t - f\cos\omega t,\ 0) + (0,\ 0,\ [1-(\delta S/2S)^2(1+f^2)]^{1/2}).$$

The change in energy can be obtained by substituting these results into F_H^B written as

$$F_H^B = -(a\gamma^2/\chi_B^2)(\vec{S}\cdot\hat{n})^2.$$

The free energy must be calculated as a function of $\vec{S}$.

$$F_H^B = -(a\gamma^2/\chi_B^2)\{S - [(\delta S)^2/8S](1+f^2)\}^2$$

$$= -(a\gamma^2/\chi_B^2)S^2 + (a\gamma^2/4\chi_B^2)(\delta S)^2(1+f^2).$$

Substituting in $f=\sqrt{\frac{3}{5}}$ we get

$$\delta F_H^B = (2a\gamma^2/5\chi_B^2)(\delta S)^2.$$

This is to be compared with

$$\delta(S^2\gamma^2/2\chi_B - \gamma\vec{S}\cdot\vec{H}) = +[(\delta S)^2/2S]\gamma H.$$

The total energy change δF_T is therefore

$$\delta F_T = [(\delta S)^2/2S]\gamma H(1 + \tfrac{4}{5}\,a/\chi_B)$$

and the relative shift in g value consequently

$$\delta g/g = \tfrac{4}{5}\,a/\chi_B.$$

This result for the shift in the transverse resonance frequency may also be derived directly from the full equations of motion for $\vec{S}$ and $d_{\alpha i}$ after taking into account the combined effects of the dipolar interaction and the susceptibility anisotropy energy. We use the above approach because of its simplicity.

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