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Microscopic and hydrodynamic theory of superfluidity in periodic solids

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The microscopic theory of fourth sound and of the superfluid fraction for perfect one-component periodic solids has been derived. It is applicable to finite temperatures and is restricted to the case of well-defined excitations. One finds that the superfluid fraction is a tensor $\rho^s_{\alpha\beta}/\rho_0$ and that the fourth-sound velocity C_4 is a tensor $(C_4^2)_{\alpha\beta} = (\partial\rho_0/\partial\mu_0)^{-1}\rho^s_{\alpha\beta}$, where μ_0 and ρ_0 are the spatially averaged values of the chemical potential (per unit mass) and of the number density. In addition, the exact nonlinearized hydrodynamics is derived, and for fourth sound is found to give agreement with the microscopic theory. Because the superfluid velocity for a periodic solid cannot be generated by a Galilean transformation, we find that elastic waves are loaded by the average mass density of the system. This is in contrast to the result of Andreev and Lifshitz, which involves only the superfluid fraction. Therefore one cannot look to (hydrodynamic) elastic waves for an obvious signature of superfluidity. A study of the effect of a transducer indicates that fourth sound will be generated to a non-negligible extent only when the crtystal is imperfect (i.e., it has vacancies, interstitials, or impurities). On the other hand, a heater might be an effective generator of fourth sound, provided that the mean free path for umklapp processes is sufficiently small. In the limit of zero crystallinity the theory shows that second sound, rather than fourth sound, occurs. Detection of superflow by rotation experiments is also considered. It is pointed out that, because the superfluid velocity is not Galilean, two-fluid counterflow does not occur. Hence, it appears that rapid angular acceleration or deceleration would be the best technique for bringing the superfluid into rotation.

I. INTRODUCTION

In 1969 and 1970, two papers appeared which considered the possibility of superfluidity in solids. The first, by Andreev and Lifshitz, considered the ac properties of a superfluid solid.[1] The second, by Leggett, considered the question of dc superflow.[2] Since their appearance, no experimental evidence has accrued to indicate that either phenomenon occurs. Probably the single most significant reason for the absence of such evidence is the damper put on experimental research by the earliest estimates of the superfluid fraction of solid ^{4}He at temperature $T=0$, which ranged from 10^{-4} to 10^{-6}.[2,3] More recently, however, two calculations[4,5] have raised the superfluid fraction to the order of 10^{-1}, so that the phenomenon now must be taken more seriously. (It should be noted that Fernandez and Puma[4] have raised an objection to their own calculation, an objection which depends upon the nature of the ground-state wave function. This point is discussed in detail in Sec. VI of the present paper.) An additional motivation for considering this phenomenon is the recent realization that there are a number of new candidates for superfluidity, both in the liquid and in the solid state.[6] Here we refer to ^{1}H, ^{2}H, and ^{3}H (for which a 50-KG magnetic field can suppress the formation of molecular[7] H_2). By analogy to the quantum theory of corresponding states[8,9] one can expect solid ^{1}H to be more delocalized than is solid ^{4}He; given the sensitivity of the superfluid fraction to the extent of delocalization,[5] the superfluid fraction for solid ^{1}H might well be close to unity. Such a system would truly be a superfluid solid. Note that, in the (uncertain) belief that the elementary excitations of a Fermi solid will show some remnant of their behavior in a Fermi liquid, we tentatively discount the possibility of superfluidity in solid ^{2}H and solid ^{3}He.

We therefore believe it is appropriate to undertake further theoretical study of the superfluidity of periodic solids. In Sec. II we develop the microscopic theory of the superfluid density $\overleftrightarrow{\rho}^s$ at finite temperature T, thus extending the work of Ref. 5. In Sec. III we develop the microscopic theory of fourth sound at finite temperature, finding after much calculation that its velocity C_4 is given by the tensor $(\overleftrightarrow{C}_4^2) = (\partial\rho_0/\partial\mu_0)^{-1}\overleftrightarrow{\rho}^s$, where μ_0 is the average chemical potential (per unit mass) and ρ_0 is the average number density. Thus both ac and dc superflow have the same transition temperature T_c, a not unexpected result. In Sec. IV we derive the nonlinear hydrodynamics of superfluid solids, incorporating the property that the superfluid velocity $\vec{V}^s$ for a periodic solid cannot be generated by a Galilean transformation. In Sec. V we analyze hydrodynamic elastic waves, fourth sound, and second sound. We find that: (i) elastic waves [Eq. (5.32)] are only slightly affected by the presence of the superfluid state (in contrast to the finding of Andreev and Lifshitz); (ii) an ordinary transducer will not generate much fourth sound in a perfect crystal [Eq. (5.36)], and (iii) a heater might generate fourth sound [Eq. (5.41)] if the hydrodynamic regime can be realized. Section VI provides a summary of the paper with particular emphasis on its implications for experiment.

II. THEORY OF $\overleftrightarrow{\rho}^s$ AT FINITE T

We wish to generalize the theory of Ref. 5 to include $T \neq 0$. Consider a single-component, spin zero system of atoms with mass m and Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m}\int d\vec{r}\,\psi^\dagger(\vec{r})\nabla^2\psi(\vec{r}) + \frac{1}{2}\int d\vec{r}\,d\vec{r}\,'\,V(\vec{r}-\vec{r}\,')\psi^\dagger(\vec{r})\psi^\dagger(\vec{r}\,')\psi(\vec{r}\,')\psi(\vec{r}). \tag{2.1}$$

Here $\psi(\vec{r})$ is the field operator, which may obey either Bose or Fermi statistics (commutation or anticommutation relations). The dependence on time is suppressed, in order to condense our notation. $V(\vec{r}-\vec{r}\,')$ is an instantaneous two-body interaction.

It is assumed that the underlying symmetry of H_0 is not shared by the class of equilibrium states which we will consider. Namely, these states will have crystalline symmetry, with a given set of sites to define the crystal lattice. Note that a nonprimitive translation gives a nonidentical system possessing essentially the same physical properties.

We now consider the effect on the energy and momentum of a nonuniform, time-dependent gauge transformation $\psi(\vec{r}) \to \psi'(\vec{r}) = e^{i\phi(\vec{r})}\psi(\vec{r})$, where $\phi(\vec{r})$ is a c number function of space and time. One then finds from the Heisenberg equation of motion for $\psi'(\vec{r})$ that the equivalent Hamiltonian H for $\psi(\vec{r})$ is given by[10]

$$H = H_0 + m\int d\vec{r}\,[\rho(\vec{r})]_{\text{op}}\{\tfrac{1}{2}[\vec{V}^s(\vec{r})]^2 + \delta\mu(\vec{r})\} + m\int d\vec{r}\,\vec{V}^s(\vec{r})\cdot[\vec{j}_0(\vec{r})]_{\text{op}}, \tag{2.2}$$

where the number density ρ_{op} and number flux density $\vec{j}_{0\,\text{op}}$ (or the momentum density per unit mass) are given by

$$[\rho(\vec{r})]_{\text{op}} \equiv \psi^\dagger(\vec{r})\psi(\vec{r}), \tag{2.3}$$

$$[\vec{j}_0(\vec{r})]_{\text{op}} \equiv (\hbar/2mi)[\psi^\dagger(\vec{r})\,\vec{\nabla}\psi(\vec{r}) - \vec{\nabla}\psi^\dagger(\vec{r})\,\psi(\vec{r})], \tag{2.4}$$

and the superfluid velocity $\vec{V}^s$ and the "shift in chemical potential" $\delta\mu$ are given by

$$\vec{V}^s(\vec{r}) \equiv (\hbar/m)\vec{\nabla}\phi, \tag{2.5}$$

$$\delta\mu(\vec{r}) \equiv -\frac{\hbar}{m}\frac{\partial\phi}{\partial t}\;. \tag{2.6}$$

(Properly, this naming of $\delta\mu$ cannot be done until the hydrodynamics is established, in Sec. IV.)

Note that ρ_{op} is unaffected by the gauge transformation, but that the total number flux density [associated with $\psi'(\vec{r})$] is

$$[\vec{j}(\vec{r})]_{\text{op}} = [\rho(\vec{r})]_{\text{op}}\,\vec{V}^s(\vec{r}) + [\vec{j}_0(\vec{r})]_{\text{op}}\;. \tag{2.7}$$

For a uniform system, one can develop a $\vec{V}^s \neq 0$ by giving the system a Galilean "boost" (which would also affect other velocities in the system). In the present case of a periodic solid, one must distinguish between the nonuniform $\vec{V}^s$ developed by the gauge transformation and the uniform $\vec{V}^s$ developed by a Galilean "boost." For the remainder of this paper, by $\vec{V}^s$ we will refer only to that $\vec{V}^s$ developed by the gauge transformation.

We now take the thermal expectation value of Eq. (2.7), thus replacing $[\rho(\vec{r})]_{\text{op}}$ by $\rho(\vec{r}) = \langle[\rho(\vec{r})]_{\text{op}}\rangle$, $[\vec{j}(\vec{r})]_{\text{op}}$ by $\vec{j}(\vec{r}) = \langle[\vec{j}(\vec{r})]_{\text{op}}\rangle$, etc. Assuming that only the thermal excitations contribute to $\langle[\vec{j}_0(\vec{r})]_{\text{op}}\rangle$, and that they are long lived, we can then write

$$\vec{j}(\vec{r}) = \rho(\vec{r})\,\vec{V}^s(\vec{r}) + \sum_{kn} f(E_{kn})\langle kn|[\vec{j}_0(\vec{r})]_{\text{op}}|kn\rangle\;. \tag{2.8}$$

Here k denotes wave vector, n denotes the band index, $|kn\rangle$ denotes the wave function of the kn excitation, and $f(E_{kn})$ is the thermal occupation number of the kn excitation. The second term on the right-hand side simply represents the momentum of the excitations.

At finite $\vec{V}^s(\vec{r})$, the energy E_{kn} becomes (on application of perturbation theory):

$$E_{kn} + m\int d\vec{r}\,\langle kn|\vec{V}^s(\vec{r})\cdot[\vec{j}_0(\vec{r})]_{\text{op}}|kn\rangle + O((V^s)^2) = E_{kn} + (\vec{V}^s\cdot\langle kn|\vec{p}|kn\rangle)_0 + O((V^s)^2), \tag{2.9}$$

where the subscript denotes the $\vec{G} = \vec{0}$ Fourier component, or spatial average, of the appropriate quantity, and $\vec{p} \equiv mV\vec{j}_{0\,\text{op}}$ with V being the volume of the system. (The argument $\vec{r}$ has been deleted from $\vec{V}^s$ and $\vec{p}$.) In addition, the state $|kn\rangle$ becomes

$$|kn\rangle + \sum_m{}' (E_{kn} - E_{km})^{-1}(\vec{V}^s\cdot\langle km|\vec{p}|kn\rangle)_0|km\rangle, \tag{2.10}$$

where only $k' = k$ states are mixed in by the perturbation if $\vec{V}^s$ is of the form[5]

$$\vec{V}^s(\vec{r}) = \sum_G \vec{V}^s_G e^{i\vec{G}\cdot\vec{r}}\;. \tag{2.11}$$

In Eq. (2.10) the prime indicates that the $m = n$ term is not included in the sum. Since $\vec{V}^s(\vec{r})$ is a real quantity, $\vec{V}^s_G = (\vec{V}^s_{-G})^*$. Similar relations hold for ρ_G, $\delta\mu_G$, and $\vec{j}_G$.

As a consequence of Eqs. (2.9)–(2.11), Eq. (2.8) for $\vec{j}(\vec{r})$ becomes, with the excitation drift velocity $\vec{V}_n$ taken to be zero, and $f'(E_{kn}) = \partial f/\partial E_{kn}$,

$$\begin{aligned}\vec{j}(\vec{r}) &= \rho(\vec{r})\vec{V}^s(\vec{r}) \\ &+\frac{1}{mV}\sum_{kn} f'(E_{kn})(\vec{V}^s \cdot \langle kn|\vec{p}|kn\rangle)_0 \langle kn|\vec{p}(\vec{r})|kn\rangle \\ &+\frac{2}{mV}\sum_{knm}' f(E_{kn})(E_{kn}-E_{km})^{-1} \\ &\times \mathrm{Re}[(\vec{V}^s \cdot \langle kn|\vec{p}|km\rangle)_0 \langle km|\vec{p}(\vec{r})|kn\rangle]. \end{aligned} \tag{2.12}$$

Defining the Gth Fourier component of $\vec{j}(\vec{r})$ by

$$\vec{j}_G = \sum_{G'} \overleftrightarrow{\rho}^s_{G,G'} \cdot \vec{V}^s_{G'}, \tag{2.13}$$

one has, from Eq. (2.12),

$$\begin{aligned}\overleftrightarrow{\rho}^s_{G,G'} &= \rho_{G-G'}\overleftrightarrow{1} + \frac{1}{mV}\sum_{kn} f'(E_{kn})\langle kn|\vec{p}|kn\rangle_G \langle kn|\vec{p}|kn\rangle_{-G'} \\ &+\frac{2}{mV}\sum_{knm}' f(E_{kn})(E_{kn}-E_{km})^{-1} \\ &\times \mathrm{Re}[\langle km|\vec{p}|kn\rangle_G \langle kn|\vec{p}|km\rangle_{-G'}]. \end{aligned} \tag{2.14}$$

Note that $\overleftrightarrow{\rho}^s_{G,G'}$ has the following properties:

$$(\rho^s_{G,G'})_{\alpha\beta} = (\rho^s_{-G,-G'})^*_{\alpha\beta} , \tag{2.15a}$$

$$(\rho^s_{G,G'})_{\alpha\beta} = (\rho_{-G',-G})_{\beta\alpha}, \tag{2.15b}$$

$$(\rho^s_{G,G'})_{\alpha\beta} = (\rho^s_{G',G})^*_{\beta\alpha} . \tag{2.15c}$$

These may be interpreted most easily by introducing

$$\overleftrightarrow{\rho}^s(\vec{r},\vec{r}') \equiv \sum_{GG'} \rho^s_{G,G'} \exp(i\vec{G}\cdot\vec{r} - i\vec{G}'\cdot\vec{r}'), \tag{2.16}$$

and noting that one may write

$$\vec{j}(\vec{r}) = \frac{1}{V}\int d\vec{r}'\, \overleftrightarrow{\rho}^s(\vec{r},\vec{r}')\cdot \vec{V}^s(\vec{r}') = \sum_G \vec{j}_G e^{i\vec{G}\cdot\vec{r}}. \tag{2.17}$$

Then Eq. (2.15a) is a consequence of the fact that $\overleftrightarrow{\rho}^s(\vec{r},\vec{r}')$ is real, since $\vec{j}(\vec{r})$ and $\vec{V}^s(\vec{r})$ are real; Eq. (2.15b) implies (and is implied by) the symmetry of $\overleftrightarrow{\rho}^s(\vec{r},\vec{r}')[= \overleftrightarrow{\rho}^s(\vec{r}',\vec{r})]$; and Eq. (2.15c) is the statement (in reciprocal space) that $\overleftrightarrow{\rho}^s(\vec{r},\vec{r}')$ is Hermitian.

To obtain $\overleftrightarrow{\rho}^s$, the superfluid density (which is a tensor in noncubic crystals), one must introduce thermodynamics. The energy density differential of a superfluid solid is taken to be[1, 11]

$$d\epsilon = TdS + \lambda_{ik}\,dw_{ik} + m\,\mu\,d\rho + m\vec{V}_n \cdot d\vec{g} + m\vec{j}_s \cdot d\vec{V}_s . \tag{2.18}$$

Here S is the entropy density,

$$w_{ik} = \partial_i\, u_k \tag{2.19}$$

is the strain (u_k is the vector describing the displacement of the lattice sites from equilibrium), and λ_{ik} is its thermodynamical conjugate (a stress density); ρ is the average number density and $m\mu$ is its thermodynamical conjugate (making μ a chemical potential per unit mass); $\vec{g}$ is the average total momentum density per unit mass and $m\vec{V}_n$ is its thermodynamical conjugate ($\vec{V}_n$ is the normal fluid velocity), and $\vec{V}_s$ is the average superfluid velocity ($=\vec{V}^s_0$) and $m\vec{j}_s$ is its thermodynamical conjugate ($\vec{j}_s$ will shortly be shown to be the average momentum of superflow, per unit mass). Note that

$$\vec{g} = \rho\vec{V}_n + \vec{j}_s . \tag{2.20}$$

Thus $\vec{j}_s$ is given by

$$\vec{j}_s \equiv m^{-1}\left(\frac{\partial\epsilon}{\partial\vec{V}_s}\right)_{S,\, w_{ik},\, \rho,\, \vec{g}} . \tag{2.21}$$

From Eq. (2.2) we find the excess energy of superflow. (Note that a Galilean “boost” by $\vec{V}_n$ changes ϵ and $\vec{g}$, but not $\vec{j}_s$. This implies no counterflow.) Letting $\vec{V}^s \to \vec{V}^s + \delta\vec{V}^s$ in Eq. (2.2), we find, on comparison with Eq. (2.13), that

$$\vec{j}_s = \vec{j}_{G=0} . \tag{2.22}$$

Thus $\vec{j}_s$ is indeed the average momentum of superflow per unit mass. The superfluid density is then defined as

$$(\rho^s)_{\alpha\beta} \equiv \left(\frac{\partial j_{s\alpha}}{\partial V_{s\beta}}\right)_{S,\, w_{ik},\, \rho,\, \vec{g}} \tag{2.23}$$

Employing Eq. (2.13) one finds that, to lowest order in $\vec{V}^s$ and w_{ik},

$$(\rho^s)_{\alpha\beta} = \sum_G (\rho^s_{0,G})_{\alpha\gamma} \frac{\partial(V^s_{G\gamma})}{\partial(V^s_{0\beta})} . \tag{2.24}$$

Equations (2.21) and (2.23) imply that $\rho^s_{\alpha\beta} = \rho^s_{\beta\alpha}$. Note that Eq. (2.23), our definition of $\overleftrightarrow{\rho}^s$, involves only macroscopic averages, just as one would hope. It was not obvious, however, that Eq. (2.22) would be true.

In order to evaluate Eq. (2.24) one may take $(\rho^s_{0,-G})_{\alpha\gamma}$ from Eq. (2.14). However, $\partial(V^s_{G\gamma})/\partial(V^s_{0\beta})$ must be determined from the steady-state flow pattern, which is determined by

$$\vec{\nabla}\cdot\vec{j}_s = 0, \tag{2.25}$$

or

$$\vec{G}\cdot\vec{j}_G = 0, \tag{2.26}$$

so that

$$\vec{G}\cdot \sum_{G'} \overleftrightarrow{\rho}^s_{G,G'}\cdot\vec{V}^s_{G'} = 0. \tag{2.27}$$

Considering $\vec{V}^s_{G=0}$ as the driving term, one may solve Eq. (2.27) for $\vec{V}^s_{G\neq 0}$, as done in Ref. 5 for a model of (fictitious) fcc ^{4}He at $T=0$. Then write, for $\vec{G}'\neq\vec{0}$,

$$\vec{V}^s_{G'}=(i\hbar/m)\vec{G}'\phi_{G'}, \qquad (2.28)$$

$$\sum_{G'}{}' \vec{G}\cdot\overleftrightarrow{\rho}^s_{G,G'}\cdot\vec{G}'\frac{i\hbar}{m}\phi_{G'}=-\vec{G}\cdot\overleftrightarrow{\rho}^s_{G,0}\cdot\vec{V}^s_0, \qquad (2.29)$$

where $\sum'$ indicates that the $\vec{G}'=\vec{0}$ term is deleted. Setting

$$B^{(0)}_{GG'}\equiv\vec{G}\cdot\overleftrightarrow{\rho}^s_{G,G'}\cdot\vec{G}', \qquad (2.30)$$

an operator which can be inverted only in the $\vec{G}\neq\vec{0}$, $\vec{G}'\neq\vec{0}$ subspace, we have

$$\frac{i\hbar}{m}\phi_G=-\sum_{G'}{}'[B^{(0)}]^{-1}_{GG'}\vec{G}'\cdot\overleftrightarrow{\rho}^s_{G',0}\cdot\vec{V}^s_0, \qquad (2.31)$$

so

$$\vec{V}^s_G=-\vec{G}\sum_{G'}{}'[B^{(0)}]^{-1}_{GG'}\vec{G}'\cdot\overleftrightarrow{\rho}^s_{G',0}\cdot\vec{V}^s_0. \qquad (2.32)$$

This permits us to write Eq. (2.24) as

$$\overleftrightarrow{\rho}^s=\overleftrightarrow{\rho}^s_{0,0}-\sum_{GG'}{}'\overleftrightarrow{\rho}^s_{0,G}\cdot\vec{G}[B^{(0)}]^{-1}_{GG'}\vec{G}'\cdot\overleftrightarrow{\rho}^s_{G',0}. \qquad (2.33)$$

Equation (2.33) will be useful in simplifying certain important results in Sec. III, which treats fourth sound.

III. MICROSCOPIC THEORY OF FOURTH SOUND

In this section we derive the microscopic theory of fourth sound. The fourth-sound velocity turns out to have a very simple form, closely resembling that for an ordinary superfluid. The reader who wishes to know the result only, or the reader who wishes to know the result before undertaking to follow the lengthy calculation, is advised to turn to Eq. (3.41).

There are four basic equations describing this form of superflow. Two of them are familiar equations of motion (mass conservation and the equation for superflow):

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}_s=0, \qquad (3.1)$$

$$\frac{\partial\vec{V}^s}{\partial t}+\vec{\nabla}\delta\mu=0. \qquad (3.2)$$

[In Eq. (3.1) we have set $\vec{g}=\vec{j}_s$, under the assumption that $\vec{V}_n\simeq\vec{0}$ for fourth sound. See Sec. V. Note that Eq. (3.2) follows from Eqs. (2.5) and (2.6).] Two of them are constitutive relations; in reciprocal space they read

$$\vec{j}_G=\sum_{G'}\overleftrightarrow{\rho}^s_{G,G'}\cdot\vec{V}^s_{G'}, \qquad (3.3)$$

$$\delta\mu_G=\sum_{G'}\left(\frac{\partial\mu}{\partial\rho}\right)_{GG'}\delta\rho_{G'}, \qquad (3.4)$$

where $(\partial\mu/\partial\rho)_{GG'}\equiv\partial\mu_G/\partial\rho_{G'}$. These equations introduce two coefficients of linear response, $\overleftrightarrow{\rho}^s_{G,G'}$ and $(\partial\mu/\partial\rho)_{GG'}$. Calculation of the first has been discussed in Sec. II; calculation of the second will be discussed now.

The quantity $(\partial\mu/\partial\rho)_{GG'}$ (at fixed $\rho_{G''}$, for all $\vec{G}''\neq\vec{G}'$) may be obtained by inverting the matrix $(\partial\rho/\partial\mu)_{GG'}$ (at fixed $\mu_{G''}$, for all $\vec{G}''\neq\vec{G}'$). The latter may be obtained by considering the perturbation $V=\int d\vec{r}[\rho(\vec{r})]_{\mathrm{op}}\delta\mu(\vec{r})$ and then finding the induced density change $\delta\rho(\vec{r})$ by linear-response theory. It is not difficult, using, e.g., the methods of Ref. 12, to show that

$$\delta\rho(\vec{r})=\int_0^1 ds\,\langle V(s)\,[\delta\rho(\vec{r})]_{\mathrm{op}}\rangle, \qquad (3.5)$$

where

$$V(s)=e^{\beta H_0 s}Ve^{-\beta H_0 s},\quad \beta=(k_BT)^{-1},$$

$$\delta[\rho(\vec{r})]_{\mathrm{op}}=[\rho(\vec{r})]_{\mathrm{op}}-\langle\rho(\vec{r})\rangle,$$

and the angular brackets denote an equilibrium average:

$$\langle A\rangle=\mathrm{Tr}(e^{-\beta H_0}A)/\mathrm{Tr}(e^{-\beta H_0}). \qquad (3.6)$$

Since $\langle V(s)\rangle=\langle V\rangle$, one finds that

$$\left(\frac{\partial\rho}{\partial\mu}\right)_{GG'}=\int_0^1 ds\{\langle[\rho_{-G'}(s)]_{\mathrm{op}}\rho_{G\,\mathrm{op}}\rangle-\langle\rho_{G\,\mathrm{op}}\rangle\langle\rho_{-G'\,\mathrm{op}}\rangle\}, \qquad (3.7)$$

which at $T=0$ becomes

$$\left(\frac{\partial\rho}{\partial\mu}\right)_{GG'}=\langle 0|\rho_{G\,\mathrm{op}}\rho_{-G'\,\mathrm{op}}|0\rangle-\langle 0|\rho_{G\,\mathrm{op}}|0\rangle\langle 0|\rho_{-G'\,\mathrm{op}}|0\rangle, \qquad (3.8)$$

where $|0\rangle$ is the ground state.

The matrix $\partial\rho/\partial\mu$ has symmetries quite similar to those of $\overleftrightarrow{\rho}^s$, and for the same reasons. First, because $\rho(\vec{r})$ and $\delta\mu(\vec{r})$ are real, the quantity $\partial\rho(\vec{r})/\partial\mu(\vec{r}')$ defined by

$$\delta\rho(\vec{r})=\frac{1}{V}\int d\vec{r}'\frac{\partial\rho(\vec{r})}{\partial\mu(\vec{r}')}\delta\mu(\vec{r}')$$

$$=\sum_{G_1}\left[\sum_{G_2}\left(\frac{\partial\rho}{\partial\mu}\right)_{G_1G_2}\delta\mu_{G_2}\right]e^{i\vec{G}_1\cdot\vec{r}} \qquad (3.9)$$

is real. This implies that

$$\left(\frac{\partial\rho}{\partial\mu}\right)_{G,G'}=\left(\frac{\partial\rho}{\partial\mu}\right)^*_{-G,-G'}, \qquad (3.10a)$$

as can be seen explicitly from Eq. (3.7) when one makes use of the fact that $[\rho(\vec{r})]_{\mathrm{op}}$ is real: $\rho_{G\,\mathrm{op}}=\rho^*_{-G\,\mathrm{op}}$. Second, $\langle[\rho_G(s)]_{\mathrm{op}}\rho_{-G'\,\mathrm{op}}\rangle=\langle[\rho_{-G'}(s)]_{\mathrm{op}}\rho_{G\,\mathrm{op}}\rangle$ implies that

$$\left(\frac{\partial \rho}{\partial \mu}\right)_{G,G'} = \left(\frac{\partial \rho}{\partial \mu}\right)_{-G',-G} , \tag{3.10b}$$

which says that $\partial\rho(\vec{r})/\partial\mu(\vec{r}\,') = \partial\rho(\vec{r}\,')/\partial\mu(\vec{r})$ is symmetric. Combination of Eqs. (3.10a) and (3.10b) gives

$$\left(\frac{\partial \rho}{\partial \mu}\right)_{G,G'} = \left(\frac{\partial \rho}{\partial \mu}\right)^{*}_{G',G} , \tag{3.10c}$$

which is the statement (in reciprocal space) that $\partial\rho(\vec{r})/\partial\mu(\vec{r}\,')$ is Hermitian.

Inversion of $\partial\rho/\partial\mu$ gives $\partial\mu/\partial\rho$, which has the same properties described by Eqs. (3.10). Given that $\partial\mu/\partial\rho$ is calculable (in the long-wavelength low-frequency limit), we turn to the solution of Eqs. (3.1)–(3.4) for slowly-varying disturbances (ω, $|\vec{k}| \to 0$):

$$\delta\rho(\vec{r}) = e^{i\vec{k}\cdot\vec{r}-i\omega t} \sum_{G} \delta\rho_{G} e^{i\vec{G}\cdot\vec{r}} , \tag{3.11}$$

$$\vec{V}^{s}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}-i\omega t} \sum_{G} \vec{V}^{s}_{G} e^{i\vec{G}\cdot\vec{r}} . \tag{3.12}$$

Then Eqs. (3.1) and (3.2) can be written

$$-i\omega\delta\rho_{G} + i(\vec{k}+\vec{G}) \cdot \sum_{G'} \overleftrightarrow{\rho}^{s}_{G,G'} \cdot \vec{V}^{s}_{G'} = 0 , \tag{3.13}$$

$$-i\omega\vec{V}^{s}_{G'} + i(\vec{k}+\vec{G}\,') \sum_{G''} \left(\frac{\partial \mu}{\partial \rho}\right)_{G'G''} \delta\rho_{G''} = 0 . \tag{3.14}$$

Therefore fourth sound must satisfy

$$-\omega^{2}\delta\rho_{G} + (\vec{k}+\vec{G}) \cdot \sum_{G'G''} \overleftrightarrow{\rho}^{s}_{G,G'} \cdot (\vec{k}+\vec{G}\,') \left(\frac{\partial \mu}{\partial \rho}\right)_{G'G''} \delta\rho_{G''} = 0 . \tag{3.15}$$

Note that for a fluid this becomes

$$-\omega^{2}\delta\rho_{0} + k^{2}\rho^{s}\frac{\partial \mu}{\partial \rho}\,\delta\rho_{0} = 0 , \tag{3.16}$$

as should be the case.

It should be observed that Eq. (3.15) has more than one solution. However, since the response functions $\overleftrightarrow{\rho}^{s}_{G',G''}$ and $(\partial\mu/\partial\rho)_{G'G''}$ are calculated in the $\omega \to 0$, $|\vec{k}| \to 0$ limit, only solutions having $\omega \to 0$ as $|\vec{k}| \to 0$ can be expected to be valid. More specifically, one expects that $\omega^{2} \propto |\vec{k}|^{2}$ as $|\vec{k}| \to 0$. We first show that a solution of Eq. (3.15) exists with $\omega = 0$ and $\vec{k} = \vec{0}$. This is done by setting these quantities to zero, so Eq. (3.15) then reads, on using Eq. (2.30),

$$\sum_{G'} B^{(0)}_{GG'}\,\delta\mu_{G'} = 0 . \tag{3.17}$$

When $\omega = 0$, Eq. (3.2) implies that $\delta\mu_{G'} = 0$ except for $\vec{G}\,' = \vec{0}$. Since $B^{(0)}_{GG'} = 0$ if either $\vec{G} = \vec{0}$ or $\vec{G}\,' = \vec{0}$, Eq. (3.17) permits the unique nontrivial solution $\delta\mu_{0} \neq 0$. This corresponds to uniformly changing the chemical potential of the system.

We next establish that Eq. (3.15) has a solution with $\omega^{2} \propto |\vec{k}|^{2}$ as $|\vec{k}| \to 0$. This is not entirely obvious. We can prove this, however, by introducing the quantity

$$\left(\frac{\partial \mu}{\partial \rho}\right)^{1/2}_{GG'} \equiv \sum_{G''} S^{-1}_{G\,G''}(A_{G''\,G''})^{1/2} S_{G''G'} , \tag{3.18}$$

where

$$A_{G''G''} = \sum_{G_1 G_2} S_{G''G} \left(\frac{\partial \mu}{\partial \rho}\right)_{G_1 G_2} S^{-1}_{G_2 G''} \tag{3.19}$$

is the diagonalized form of the Hermitian matrix $\partial\mu/\partial\rho$, and S is the similarity transformation (a unitary matrix) which performs the diagonalization. With

$$B_{GG'} = (\vec{k}+\vec{G}) \cdot \overleftrightarrow{\rho}^{s}_{GG'} \cdot (\vec{k}+\vec{G}\,') , \tag{3.20}$$

we may rewrite Eq. (3.15) as

$$\sum_{G'} M_{GG'} C_{G'} = \omega^{2} C_{G} , \tag{3.21}$$

where

$$C_{G'} = \sum_{G''} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2}_{G'G''} \delta\rho_{G''} \tag{3.22}$$

and

$$M_{GG'} = \sum_{G_1 G_2} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2}_{GG_1} B_{G_1 G_2} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2}_{G_2 G'} . \tag{3.23}$$

We now argue as follows. First, $B_{GG'}$ is Hermitian because by Eq. (2.15c) $\overleftrightarrow{\rho}^{s}_{G,G'}$ is Hermitian. Second, $(\partial\mu/\partial\rho)^{1/2}$ is Hermitian because S is unitary and A is Hermitian (in fact, real and diagonal). Therefore M, being the product of three Hermitian matrices, is itself Hermitian. As a consequence, its eigenvalues ω^{2} are real. Note that ω^{2} may be negative, indicating unstable behavior. This occurs for $T > T_{c}$, thus defining the transition temperature T_{c}.

To show that $\omega^{2} \propto |\vec{k}|^{2}$ as $|\vec{k}| \to 0$, we employ perturbation theory about the dc solution ($\omega = 0$) which occurs when $\vec{k} = \vec{0}$. Because of the form of B, we may write M in the form

$$M = M^{(0)} + \vec{k} \cdot \vec{M}^{(1)} + \vec{k} \cdot \overleftrightarrow{M}^{(2)} \cdot \vec{k} , \tag{3.24}$$

where

$$\begin{aligned}
\overleftrightarrow{M}^{(2)} &= \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} \overleftrightarrow{B}^{(2)} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} , \\
\vec{M}^{(1)} &= \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} \vec{B}^{(1)} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} , \\
M^{(0)} &= \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} B^{(0)} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} , \\
(M^{(0)})^{-1} &= \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} [B^{(0)}]^{-1} \left(\frac{\partial \mu}{\partial \rho}\right)^{1/2} ,
\end{aligned} \tag{3.25}$$

and

$$\vec{\mathrm{B}}^{(2)}_{G_1G_2}=\overleftrightarrow{\rho}^{s}_{G_1G_2}\,,$$

$$(\vec{\mathrm{B}}^{(1)}_{G_1G_2})_\alpha=G_{1\beta}(\rho^s_{G_1G_2})_{\beta\alpha}+(\rho^s_{G_1G_2})_{\alpha\beta}G_{2\beta}\,, \tag{3.26}$$

$$B^{(0)}_{G_1G_2}=\vec{\mathrm{G}}_1\cdot\overleftrightarrow{\rho}^{s}_{G_1G_2}\cdot\vec{\mathrm{G}}_2\,.$$

[In Eqs. (3.24) and (3.25), matrix indices have been suppressed.] Application of first-order perturbation theory then gives, with

$$\omega^2=0+\omega^2_{(1)}+\omega^2_{(2)}+\cdots\,, \tag{3.27}$$

that

$$\omega^2_{(1)}=(0\,|\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}|\,0)\,. \tag{3.28}$$

Here $|0)$ is the eigenfunction of M with eigenvalue $\omega^2=0$. It is taken to be normalized to unity, so that

$$|0)=|0)'/(0'|0')^{1/2}\,, \tag{3.29}$$

where $|0')$ is obtained from Eq. (3.22). Thus

$$(0'|0')=\sum_{G_1G_2G_3}\left[\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}_{G_1G_2}\delta\rho_{G_2}\right]^*\left[\left(\frac{\partial\mu}{\partial\rho}\right)_{G_1G_3}\delta\rho_{G_3}\right]$$

$$=\sum_{G_2}\delta\rho_{-G_2}\delta\mu_{G_2}=\delta\rho_0\delta\mu_0\,. \tag{3.30}$$

In the above we have employed the Hermiticity of $\partial\mu/\partial\rho$, the reality of $\delta\rho$, and the fact that $\delta\mu_G=0$ for $\vec{\mathrm{G}}\neq\vec{0}$. We now show that

$$(0\,|\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}|\,0)\equiv\sum_{G,G'}C^*_G\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}_{GG'}C_{G'}=0\,, \tag{3.31}$$

so that $\omega^2_{(1)}=0$. To do so we will apply

$$\vec{\mathrm{M}}^{(1)}_{G,G'}=-\vec{\mathrm{M}}^{(1)}_{-G',-G} \tag{3.32}$$

and

$$C_G=C^*_{-G}\,. \tag{3.33}$$

This last equation is true because $\delta\rho_G=\delta\rho^*_{-G}$. As a consequence of Eqs. (3.32) and (3.33), Eq. (3.31) becomes

$$\sum_{G,G'}C^*_G\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}_{G,G'}C_{G'}=-\sum_{G,G'}C_{-G}\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}_{-G',-G}C^*_{-G'}$$

$$=-\sum_{G,G'}C^*_G\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}_{G,G'}C_{G'}\,, \tag{3.34}$$

thus establishing Eq. (3.31). Note that it was essential, in the above demonstration, that the $\vec{\mathrm{k}}=\vec{0}$ state was a state of uniform change of $\delta\mu$. The perturbation theory is built about this state, not about the state of dc superflow. Physically, by Eq. (3.2) a nonuniform chemical potential induces the superflow.

It is now straightforward to obtain an explicit expression for $\omega^2_{(2)}$ by the use of second-order perturbation theory. We have

$$\omega^2_{(2)}=(0\,|\vec{\mathrm{k}}\cdot\overleftrightarrow{\mathrm{M}}^{(2)}\cdot\vec{\mathrm{k}}\,|\,0)+\sum_{N\neq0}\frac{(0\,|\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}|N)(N|\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}|\,0)}{-\lambda_N}\,, \tag{3.35}$$

where

$$M^{(0)}|N)=\lambda_N|N) \tag{3.36}$$

defines the eigenvalues λ_N and the eigenstates $|N)$. The above expression may be rewritten without the sum on N by noting that $(0\,|\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}|\,0)=0$ permits us to write (with ω^2 for $\omega^2_{(2)}$)

$$\begin{aligned}\omega^2&=(0\,|\vec{\mathrm{k}}\cdot\overleftrightarrow{\mathrm{M}}^{(2)}\cdot\vec{\mathrm{k}}\,|\,0)\\&\quad-\sum_N(0\,|\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}^{(1)}|N)(N|\,[M^{(0)}]^{-1}\vec{\mathrm{k}}\cdot\vec{\mathrm{M}}\,|\,0)\\&=\vec{\mathrm{k}}\cdot(0\,|\overleftrightarrow{\mathrm{M}}^{(2)}-\vec{\mathrm{M}}^{(1)}[M^{(0)}]^{-1}\vec{\mathrm{M}}^{(1)}|\,0)\cdot\vec{\mathrm{k}}\\&\equiv\vec{\mathrm{k}}\cdot\overleftrightarrow{\mathrm{C}}^2_4\cdot\vec{\mathrm{k}}\,,\end{aligned} \tag{3.37}$$

thus defining the fourth-sound velocity. It should be noted that our result is similar in form to the result of Born for elastic waves, also obtained by using a version of second-order perturbation theory.[13]

We now evaluate ω^2 explicitly. From Eq. (3.37), it has two pieces, the first of which involves

$$\begin{aligned}(0'\,|\overleftrightarrow{\mathrm{M}}^{(2)}|\,0')&=(0'\,|\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}\overleftrightarrow{\rho}^{s}\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}|\,0')\\&=\sum_{i=1}^{6}\sum_{G_i}\left[\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}_{G_1G_2}\delta\rho_{G_2}\right]^*\\&\qquad\times\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}_{G_1G_3}\overleftrightarrow{\rho}^{s}_{G_3G_4}\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}_{G_4G_5}\\&\qquad\times\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}_{G_5G_6}\delta\rho_{G_6}\\&=\sum_{G_3G_4}\delta\mu_{-G_3}\overleftrightarrow{\rho}_{G_3G_4}\delta\mu_{G_4}=(\delta\mu_0)^2\overleftrightarrow{\rho}_{00}\,.\end{aligned} \tag{3.38}$$

The second piece involves

$$\begin{aligned}&(0'\,|\vec{\mathrm{M}}^{(1)}[M^{(0)}]^{-1}\vec{\mathrm{M}}^{(1)}|\,0')\\&\quad=(0'\,|\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}\vec{\mathrm{B}}^{(1)}[B^{(0)}]^{-1}\vec{\mathrm{B}}^{(1)}\left(\frac{\partial\mu}{\partial\rho}\right)^{1/2}|\,0')\\&\quad=(\delta\mu_0)^2(\vec{\mathrm{B}}^{(1)}[B^{(0)}]^{-1}\vec{\mathrm{B}}^{(1)})_{00}\\&\quad=(\delta\mu_0)^2\sum_{G_1G_2}{}'\,\overleftrightarrow{\rho}^{s}_{0G_1}\cdot\vec{\mathrm{G}}_1[B^{(0)}]^{-1}_{G_1G_2}\vec{\mathrm{G}}_2\cdot\overleftrightarrow{\rho}^{s}_{G_20}\,.\end{aligned} \tag{3.39}$$

Combining terms, we then have

$$\vec{C}_4^2=(\delta\mu_0\,\delta\rho_0)^{-1}(\delta\mu_0)^2$$

$$\times\left(\overleftrightarrow{\rho}^s_{0,0}-\sum_{G_1G_2}{}'\,\overleftrightarrow{\rho}^s_{0G_1}\cdot\vec{G}_1[B^{(0)}]^{-1}_{G_1G_2}\vec{G}_2\cdot\overleftrightarrow{\rho}^s_{G_20}\right)$$

$$=\left(\frac{\partial\rho_0}{\partial\mu_0}\right)^{-1}\left(\overleftrightarrow{\rho}^s_{0,0}-\sum_{G_1G_2}{}'\,\overleftrightarrow{\rho}^s_{0G_1}\cdot\vec{G}_1[B^{(0)}]^{-1}_{G_1G_2}\vec{G}_2\cdot\overleftrightarrow{\rho}^s_{G_20}\right). \tag{3.40}$$

We write $(\partial\rho_0/\partial\mu_0)^{-1}$ because it is at fixed $\delta\mu_G=0$ for $\vec{G}\neq\vec{0}$. Note that the primes on the summations indicate that the $\vec{G}_1=\vec{0}$ and $\vec{G}_2=\vec{0}$ terms are to be deleted. This is because $[B^{(0)}]^{-1}_{G_1G_2}$ exists only in the $\vec{G}_1\neq\vec{0}$, $\vec{G}_2\neq\vec{0}$ subspace. It is the presence of the $\vec{B}^{(1)}$ terms to the right and left of $[B^{(0)}]^{-1}$ that permitted us to write $[B^{(0)}]^{-1}$ at all. One may consider the $\vec{B}^{(1)}$ terms sandwiching $[B^{(0)}]^{-1}$ to serve as projection operators into the $\vec{G}_1\neq\vec{0}$, $\vec{G}_2\neq\vec{0}$ subspace for which $[B^{(0)}]^{-1}$ exists.

Comparison with Eq. (2.33) permits us to rewrite Eq. (3.40) as

$$\vec{C}_4^2=\left(\frac{\partial\rho_0}{\partial\mu_0}\right)^{-1}\overleftrightarrow{\rho}^s. \tag{3.41}$$

This simple result was by no means obvious. Equation (3.41) has a number of important consequences. First, both dc and ac superflow in a periodic solid have the same transition temperature T_c. (Probably this is also true for nonperiodic solids, where Fourier sums must be replaced by Fourier integrals.) Second, of all the complicated response functions $(\partial\rho/\partial\mu)_{GG'}$, only the $\vec{G}=\vec{G}'=\vec{0}$ component need be computed. This is an immense simplification of the problem. One can thus immediately employ Eqs. (3.7) or (3.8). If, as is the case for liquid ^{4}He, $(\partial\rho_0/\partial\mu_0)$ is not very temperature dependent, then one can employ Eq. (3.8) to a high degree of accuracy. Further, since $\langle 0|\rho_{0\,\mathrm{op}}|0\rangle=\rho_0$ is trivial, and since

$$\langle 0|\rho_{0\,\mathrm{op}}\rho_{0\,\mathrm{op}}|0\rangle=V^{-2}\int d\vec{r}_1\,d\vec{r}_2\langle 0|\rho(\vec{r}_1)\rho(\vec{r}_2)|0\rangle$$

is the double integral over the pair-correlation function (a quantity which is studied in actual calculations on quantum solids), it should not be too difficult to evaluate Eq. (3.8). Lastly, $\overleftrightarrow{\rho}^s$ can easily be evaluated by Fourier techniques at $T=0$.[5] Therefore fourth-sound velocities at $T=0$ should not be difficult to obtain.

IV. HYDRODYNAMICS

The hydrodynamics of Andreev and Lifshitz[1] was derived under the assumption that the superfluid velocity $\vec{V}_s$ is a Galilean velocity. However, as shown in Sec. II, $\vec{V}_s$ is generated by a gauge transformation, and cannot properly be considered to change under a Galilean "boost." Therefore a rederivation of the hydrodynamics is required. The derivation is given in some detail because it contains some simple but essential steps which permit the full nonlinear equations of hydrodynamics to be derived. By contrast, Ref. 1 derives only the linearized equations.

The derivation begins by writing down the energy-density differential, the energy density, and the associated Gibbs-Duhem relation. To conform with the usual notation in hydrodynamics, in this section we let ρ be the mass density, and $\vec{g}$ and $\vec{j}_s$ be momentum densities. Then

$$d\epsilon=TdS+\lambda_{ik}dw_{ik}+\mu d\rho+\vec{V}_n\cdot d\vec{g}+\vec{j}_s\cdot d\vec{V}_s, \tag{4.1}$$

$$\epsilon=TS-P+\lambda_{ik}w_{ik}+\mu\rho+\vec{V}_n\cdot\vec{g}+\vec{j}_s\cdot\vec{V}_s, \tag{4.2}$$

$$0=SdT-dP+w_{ik}d\lambda_{ik}+\rho d\mu+\vec{g}\cdot d\vec{V}_n+\vec{V}_s\cdot d\vec{j}_s. \tag{4.3}$$

Note that one sometimes see $\vec{h}$ for our $\vec{j}_s$.[11] P is the pressure.

The derivation of the hydrodynamic equations of a superfluid solid is accomplished by: (i) writing down mass and momentum conservation, an equation for superflow, and an equation expressing the law of increase of entropy

$$\dot{\rho}+\partial_i g_i=0, \tag{4.4}$$

$$\dot{g}_i+\partial_k\Pi_{ik}=0, \tag{4.5}$$

$$\mathring{V}_{si}+\partial_i\theta=0, \tag{4.6}$$

$$\mathring{S}+\partial_i(SV_{ni}+q_i/T)=R/T\quad(R>0), \tag{4.7}$$

where Π_{ik}, θ, V_{ni}, q_i, and R are unknown, and (ii) determining these unknowns by requiring that Eqs. (4.1)–(4.3) and Eqs. (4.4)–(4.7) are consistent with conservation of energy

$$\dot{\epsilon}+\partial_i Q_i=0, \tag{4.8}$$

where Q_i is also unknown, and to be determined.

By suitable combination of Eqs. (4.1) and (4.4)–(4.8) and some manipulation using the laws of differentiation, one can write

$$\begin{aligned}R=&-\partial_i(Q_i-TSV_{ni}-q_i+\lambda_{ik}\dot{u}_k-V_{nk}\Pi_{ki}\\&-j_{si}\theta-V_{nk}\lambda_{ik}-\mu\rho V_{ni})\\&-(q_i/T)\partial_iT+(\dot{u}_k-V_{nk})\partial_i\lambda_{ik}+(\mu-\theta)\partial_ij_{si}\\&-(\Pi_{ik}+\lambda_{ki})\partial_kV_{ni}-V_{ni}(S\partial_iT+\rho\partial_i\mu).\end{aligned} \tag{4.9}$$

The last piece of Eq. (4.9) may be rewritten, using Eqs. (4.2) and (4.3), as

$$\begin{aligned}-V_{ni}(S\partial_iT+\rho\partial_i\mu)=&-\partial_i(PV_{ni})+P\partial_iV_{ni}+V_{ni}w_{jk}\partial_i\lambda_{jk}\\&+V_{ni}g_k\partial_iV_{nk}+V_{ni}V_{sk}\partial_ij_{sk}.\end{aligned} \tag{4.10}$$

The third and fifth terms on the right-hand side of Eq. (4.10) also may be rewritten. Making use of the property that $\partial_i w_{jk} \equiv \partial_i(\partial_j u_k) = \partial_j w_{ik}$, and of the laws of differentiation, the third term can be rewritten

$$V_{ni} w_{jk} \partial_i \lambda_{jk} = \partial_i (V_{ni} \lambda_{jk} w_{jk} - \lambda_{ik} V_{nj} w_{jk}) + V_{nj} w_{jk} \partial_i \lambda_{ik} + (\lambda_{kj} w_{ij} - \lambda_{jl} w_{jl} \delta_{ik}) \partial_k V_{ni} \,. \tag{4.11}$$

Similarly, use of the property that $\partial_i V_{sk} = \partial_k V_{si}$, and of the laws of differentiation, permits the fifth term to be rewritten

$$V_{ni} V_{sk} \partial_i j_{sk} = \partial_i (V_{ni} V_{sk} j_{sk} - j_{si} V_{nk} V_{sk}) + V_{sk} V_{nk} \partial_i j_{si} + (V_{si} j_{sk} - j_{sj} V_{sj} \delta_{ik}) \partial_k V_{ni} \,. \tag{4.12}$$

Note that Eqs. (4.11) and (4.12) are essential to the nonlinear hydrodynamics. Combination of Eqs. (4.9)–(4.12) gives

$$\begin{aligned} R = &-\partial_i \{ Q_i - q_i - [(TS - P + \mu\rho + \lambda_{jl} w_{jl} + j_{sl} V_{sl}) \delta_{ik} + \Pi_{ki}] V_{nk} + (\dot{u}_k - V_{nk} + V_{nj} w_{jk}) \lambda_{ik} - (\theta - V_{nk} V_{sk}) j_{si} \} \\ &-(q_i/T) \partial_i T + (\dot{u}_k - V_{nk} + V_{nj} w_{jk}) \partial_i \lambda_{ik} + (\mu - \theta + V_{nk} V_{sk}) \partial_i j_{si} \\ &-[\Pi_{ik} - (P - \lambda_{jl} w_{jl} - j_{sl} V_{sl}) \delta_{ik} + \lambda_{ki} - V_{nk} g_i - V_{si} j_{sk} - \lambda_{kj} w_{ij}] \partial_k V_{ni} \,. \end{aligned} \tag{4.13}$$

Since the five terms on the right-hand side of Eq. (4.13) are independent, and $R > 0$, the first term (which can have either sign) must be zero. The other terms must be such that $R > 0$. Thus we have

$$q_i = -\kappa_{ik} \partial_k T - \alpha_{ik} \partial_l \lambda_{lk} \,, \tag{4.14}$$

$$d_i \equiv V_{ni} - \dot{u}_i - V_{nj} w_{ji} = -(\alpha_{ki}/T) \partial_k T - \beta_{ik} \partial_l \lambda_{kl} \,, \tag{4.15}$$

$$\begin{aligned} \pi_{ik} \equiv \Pi_{ik} - [&(P - \lambda_{jl} w_{jl} - j_{sl} V_{sl}) \delta_{ik} \\ &- \lambda_{ki} + V_{nk} g_i + V_{si} j_{sk} + \lambda_{kj} w_{ij}] \\ = -\eta_{iklm} &\partial_m V_{nl} - \zeta_{ik} \partial_l j_{sl} \,, \end{aligned} \tag{4.16}$$

$$\Psi \equiv \theta - (\mu + V_{sk} V_{nk}) = -\zeta_{ik} \partial_k V_{ni} - \chi \partial_k j_{sk} \,. \tag{4.17}$$

The first term of Eq. (4.13) yields, upon use of Eqs. (4.15)–(4.17):

$$\begin{aligned} Q_i &= q_i + (TS + \mu\rho) V_{ni} + [-\lambda_{ik} + V_{ni} g_k + V_{sk} j_{si} + \lambda_{ij} w_{kj} + \pi_{ki}] V_{nk} + d_k \lambda_{ik} + (\mu + \Psi) j_{si} \\ &= [TS V_{ni} + \mu j_{si}] + [q_i + d_k \lambda_{ik} + \Psi j_{si} + \pi_{ki} V_{nk}] + [V_{ni} g_k + V_{sk} j_{si} - \lambda_{ik} + \lambda_{ij} w_{kj}] V_{nk} \,. \end{aligned} \tag{4.18}$$

The second bracket of Eq. (4.18) contains the dissipative contributions, the third bracket contains the nonlinear flow and the stress terms, and the first bracket contains the usual terms for superflow without dissipation and with the neglect of nonlinear flow and crystalline stress. As discussed in Ref. 1, the κ, α, β, η, ζ, and χ are kinetic coefficients: κ is a thermal conductivity; η, ζ, and χ are viscosities; and α and β describe diffusion and thermodiffusion of defects.[14] Fleming and Cohen have derived the (linearized) hydrodynamics for ordinary solids, where ζ and χ do not appear.[15]

To complete our characterization of the full nonlinear hydrodynamics of a superfluid solid, we give the dissipation function:

$$R = -(q_i/T) \partial_i T - d_k \partial_i \lambda_{ik} - \Psi \partial_i j_{si} - \pi_{ik} \partial_k V_{ni} \,. \tag{4.19}$$

The requirement that $R > 0$ imposes certain inequalities upon the kinetic coefficients; for example, the diagonal elements of κ are positive. We have not investigated these inequalities any further.

The significance of the derivation is twofold. First, the full nonlinear hydrodynamics has been obtained, permitting the treatment of oscillations in crystals subject to a finite static strain. Second, the superfluid velocity $\vec{V}_s$ is driven by gradients in the chemical potential per unit mass μ [thus validating the identification in Eq. (2.6)]. This is the usual result, despite the fact that $\vec{V}_s$ no longer has the property of being a Galilean velocity. Note that one can write $g_i = \rho^s_{ik} \tilde{V}_{sk} + \rho^n_{ik} V_{nk}$, where $\tilde{V}_{sk} \equiv V_{sk} + V_{nk}$ and $\rho^n_{ik} \equiv \rho\delta_{ik} - \rho^s_{ik}$. This is the usual form, but only in the limit of zero crystallinity is $\tilde{V}_{sk}$ driven by $\partial_k \mu$.

In Sec. V we shall study some of the implications of the hydrodynamic equations.

V. HYDRODYNAMIC OSCILLATIONS

We will restrict our considerations to unstrained crystals, and treat only the linearized theory for that case. As a further simplification, dissipation will be neglected. Equations (4.4)–(4.7) then become

$$\dot{\rho} + \partial_i g_i = 0 \,, \tag{5.1}$$

$$\dot{g}_i - \partial_k \sigma_{ik} = 0 \,, \tag{5.2}$$

$$\dot{V}_{sk}+\partial_k\mu=0\,, \tag{5.3}$$

$$\dot{S}+S\partial_i V_{ni}=0\,, \tag{5.4}$$

where

$$g_i=\rho^s_{ik}V_{sk}+\rho V_{ni}\,, \tag{5.5}$$

$$V_{ni}=\dot{u}_i\,, \tag{5.6}$$

$$\sigma_{ik}=-P\delta_{ik}+\lambda_{ki} \tag{5.7}$$

and the following relationships hold, treating ρ, S, and u_i as the independent variables:

$$\partial_i\mu=\frac{\partial\mu}{\partial\rho}\partial_i\rho+\frac{\partial\mu}{\partial S}\partial_i S+\frac{\partial\mu}{\partial w_{jk}}\partial_i w_{jk}\,, \tag{5.8}$$

$$\partial_k\sigma_{ik}=\frac{\partial\sigma_{ik}}{\partial\rho}\partial_k\rho+\frac{\partial\sigma_{ik}}{\partial S}\partial_k S+\frac{\partial\sigma_{ik}}{w_{jl}}\partial_k w_{jl}\,. \tag{5.9}$$

In Eqs. (5.8) and (5.9) the partial derivatives are understood to be taken with the appropriate variables held constant.

Let us now consider all varying quantities to do so with a dependence $e^{i\vec{q}\cdot\vec{r}-i\omega t}$, so that the ac response of the system may be studied. We take ρ, S, V_{si}, and V_{ni} as the independent variables, so that Eqs. (5.1) and (5.4) may immediately be used to eliminate ρ' and S' (where the primes denote the deviation from equilibrium) from Eqs. (5.2) and (5.3). Noting that

$$\rho'=(q_j/\omega)g_j\,, \tag{5.10}$$

$$S'=(q_j/\omega)SV_{nj}\,, \tag{5.11}$$

$$w_{jk}=-(q_j/\omega)V_{nk}\,, \tag{5.12}$$

which follow from Eqs. (5.1), (5.4), and (5.6), Eq. (5.2) can be rewritten

$$-\omega g_i=q_k\left(\frac{\partial\sigma_{ik}}{\partial\rho}\frac{q_j}{\omega}g_j+\frac{\partial\sigma_{ik}}{\partial S}\frac{q_j}{\omega}SV_{nj}-\frac{\partial\sigma_{ik}}{\partial w_{jl}}\frac{q_j}{\omega}V_{nl}\right)\,, \tag{5.13}$$

and Eq. (5.3) can be rewritten as

$$\omega V_{sk}=q_k\left(\frac{\partial\mu}{\partial\rho}\frac{q_j}{\omega}g_j+\frac{\partial\mu}{\partial S}\frac{q_j}{\omega}SV_{nj}-\frac{\partial\mu}{\partial w_{jl}}\frac{q_j}{\omega}V_{nl}\right)\,. \tag{5.14}$$

Equations (5.13) and (5.14) can be handled relatively easily be employing Eq. (5.5) to eliminate g_i, and by considering

$$j_{si}\equiv\rho^s_{ik}V_{sk}\,, \tag{5.15}$$

$$j_{ni}\equiv\rho V_{ni} \tag{5.16}$$

as the independent variables. Multiplying Eq. (5.14) by ρ^s_{ik}, and eliminating g_i from Eqs. (5.13) and (5.14) we find that

$$0=\omega^2(j_{si}+j_{ni})+q_jq_k\frac{\partial\sigma_{ik}}{\partial\rho}j_{sj}+q_jq_k\left[\left(\frac{\partial\sigma_{ik}}{\partial\rho}+\rho^{-1}S\frac{\partial\sigma_{ik}}{\partial S}\right)\delta_{jl}-\rho^{-1}\frac{\partial\sigma_{ik}}{\partial w_{jl}}\right]j_{nl}\,, \tag{5.17}$$

$$0=-\omega^2 j_{si}+q_jq_k\rho^s_{ik}\frac{\partial\mu}{\partial\rho}j_{sl}+q_jq_k\rho^s_{ik}\left[\left(\frac{\partial\mu}{\partial\rho}+\rho^{-1}S\frac{\partial\mu}{\partial S}\right)\delta_{jl}-\rho^{-1}\frac{\partial\mu}{\partial w_{jl}}\right]j_{nl}. \tag{5.18}$$

Introducing the notation $[(5.17)]_{ikjl}$ and $[(5.18)]_{jl}$ for the terms within the large brackets in Eqs. (5.17) and (5.18), these equations can finally be put in the form

$$0=\left(\omega^2\delta_{ij}+q_jq_k\frac{\partial\sigma_{ik}}{\partial\rho}\right)j_{sj}+\{\omega^2\delta_{il}+q_jq_k[(5.17)]_{ijkl}\}j_{nl}, \tag{5.19}$$

$$0=\left(-\omega^2\delta_{ij}+q_jq_k\rho^s_{ik}\frac{\partial\mu}{\partial\rho}\right)j_{sj}+q_jq_k\rho^s_{ik}[(5.18)]_{jl}\,j_{nl}. \tag{5.20}$$

Equations (5.19) and (5.20) are still quite complicated. We will not attempt to solve them in full generality. Rather, we will consider a special case which should illuminate the broad features of their solutions. First, however, note that for $T>T_c$ ($\rho^s_{ik}=0$) Eq. (5.20) implies that $j_{sj}=0$, so that Eq. (5.19) has the elastic waves given by the solutions of

$$0=\{\omega^2\delta_{il}+q_jq_k[(5.17)]_{ijkl}\}j_{nl}\,. \tag{5.21}$$

This is what should occur for an ordinary solid. Second, note that in the limit of zero crystallinity, the longitudinal elastic wave becomes ordinary sound.

Our special case will be that of a cubic crystal with $\vec{q}$, $\vec{j}_s$, and $\vec{j}_n$ along the [100] direction. This is a longitudinal motion, which a superfluid can be expected to engage in. Noting that $\rho^s_{ik}=\rho^s\delta_{ik}$ in a cubic crystal, and letting

$$\alpha\equiv\frac{\partial\sigma_{11}}{\partial\rho}, \tag{5.22}$$

$$\beta\equiv[(5.17)]_{1111}=\frac{\partial\sigma_{11}}{\partial\rho}+\rho^{-1}S\frac{\partial\sigma_{11}}{\partial S}-\rho^{-1}\frac{\partial\sigma_{11}}{\partial w_{11}}\equiv-C^2_{10}, \tag{5.23}$$

$$\gamma=\rho^s\frac{\partial\mu}{\partial\rho}\equiv C^2_{40}, \tag{5.24}$$

$$\delta=\rho^s[(5.18)]_{11}=\rho^s\left(\frac{\partial\mu}{\partial\rho}+\rho^{-1}S\frac{\partial\mu}{\partial S}-\rho^{-1}\frac{\partial\mu}{\partial w_{11}}\right), \tag{5.25}$$

Eqs. (5.19) and (5.20) become

$$0=(\omega^2/q^2+\alpha)j_s+(\omega^2/q^2+\beta)j_n, \tag{5.26}$$

$$0=(-\omega^2/q^2+\gamma)j_s+\delta j_n. \tag{5.27}$$

The wave velocities $c=\omega/q$ are thus given by

$$c^2=\tfrac{1}{2}\{(\gamma-\delta-\beta)\pm[(\gamma-\delta-\beta)^2-4(\alpha\delta-\beta\gamma)]^{1/2}\}. \tag{5.28}$$

Note that β is the largest term in this equation, and that δ may be rewritten, using the chain rule for differentiation, as

$$\delta=\rho^s\left(\frac{\partial\mu}{\partial\rho}\right)_{S,w_{ik}}\left[1-\rho^{-1}S\left(\frac{\partial\rho}{\partial S}\right)_{\mu,w_{ik}}+\rho^{-1}\left(\frac{\partial\rho}{\partial w_{11}}\right)_{S,\mu}\right]$$
$$\equiv C_{40}^2\bar{\delta}. \tag{5.29}$$

This is a small quantity for two reasons: (i) the bracket $\bar{\delta}$ is small because the third term will cancel the first term in a perfect crystal, and because the second term will be rather small at the low temperatures of interest here; (ii) ρ^s may be much less than ρ (in Ref. 5 we estimated that $0.05\lesssim\rho^s/\rho\lesssim0.20$ in solid ^{4}He at $T=0$, for an imaginary fcc phase). The quantity α can be rewritten

$$\alpha=\left(\frac{\partial\sigma_{11}}{\partial\rho}\right)_{S,w}=\left(\frac{\partial\lambda_{11}}{\partial\rho}\right)_{S,w}-\left(\frac{\partial P}{\partial\rho}\right)_{S,w}=\left(\frac{\partial\lambda_{11}}{\partial\mu}\right)_{S,w}\left(\frac{\partial\mu}{\partial\rho}\right)_{S,w}-\rho\left(\frac{\partial\mu}{\partial\rho}\right)_{S,w}-S\left(\frac{\partial T}{\partial\rho}\right)_{S,w}$$
$$=-\left(\frac{\partial\mu}{\partial\rho}\right)_{S,w}\left[\left(\frac{\partial\rho}{\partial w_{11}}\right)_{S,\mu}+\rho+S\left(\frac{\partial T}{\partial\mu}\right)_{S,w}\right]=-\rho\left(\frac{\partial\mu}{\partial\rho}\right)_{S,w}\left[1+\rho^{-1}\left(\frac{\partial\rho}{\partial w_{11}}\right)_{S,\mu}-\rho^{-1}S\left(\frac{\partial\rho}{\partial S}\right)_{\mu,w}\right]=-\frac{\rho}{\rho^s}C_{40}^2\bar{\delta}. \tag{5.30}$$

Here we have used the rules of partial differentiation, Eq. (4.3), and the Maxwell relations $(\partial\lambda_{11}/\partial\mu)_{S,w}=-(\partial\rho/\partial w_{11})_{S,\mu}$, $(\partial T/\partial\mu)_{S,w}=-(\partial\rho/\partial S)_{\mu,w}$. Equation (5.28) can thus be written, for small $\bar{\delta}$, as

$$C_4^2\simeq C_{40}^2\left(1+\frac{\bar{\delta}(1-\bar{\delta}\rho/\rho^s)C_{40}^2}{C_{10}^2-C_{40}^2}\right), \tag{5.31}$$

$$C_1^2\simeq C_{10}^2[1-\bar{\delta}C_{40}^2/(C_{10}^2-C_{40}^2)]. \tag{5.32}$$

Equation (5.31) describes a type of fourth sound, and Eq. (5.32) describes an ordinary longitudinal elastic wave. Note that Eq. (5.32) differs from the result of Ref. 1. This is in part because Ref. 1 takes $j_{ni}\equiv(\rho\delta_{ik}-\rho_{ik}^s)V_{nk}$, which implies that the effective crystal mass is decreased by the factor $(\delta_{ik}-\rho_{ik}^s/\rho)$. We believe that such a j_{ni} is incorrect. Equation (5.31) is essentially in agreement with Ref. 1, which treated only the $T=0$ case.

From Eq. (5.27) we can find the ratios of j_s to j_n. They are

$$(j_s/j_n)_4\simeq\frac{C_{10}^2-C_{40}^2}{C_{40}^2}\left(1-\frac{\bar{\delta}\rho}{\rho^s}\right)^{-1}, \tag{5.33}$$

$$(j_s/j_n)_1\simeq\bar{\delta}C_{40}^2/(C_{10}^2-C_{40}^2). \tag{5.34}$$

These relations are relevant to the generation of fourth sound. We will consider two hypothetical generators.[16]

Case 1. If a transducer can produce $V_n\neq0$ but $V_s=0$, then

$$0=(j_s/j_n)_1j_n^{(1)}+(j_s/j_n)_4j_n^{(4)}. \tag{5.35}$$

Hence, use of Eqs. (5.33) and (5.34) gives

$$\frac{j_n^{(4)}}{j_n^{(1)}}=-\bar{\delta}\left(1-\frac{\bar{\delta}\rho}{\rho^s}\right)\left(\frac{C_{40}^2}{C_{10}^2-C_{40}^2}\right)^2. \tag{5.36}$$

As $T\to0$, $\bar{\delta}\to0$ for a perfect crystal, so $j_n^{(4)}/j_n^{(1)}\to0$ in that case. As $T\to T_c$, $C_{40}^2\to0$ and $(\rho/\rho^s)(C_{40}^2/C_{10}^2)^2\to0$, so $j_n^{(4)}/j_n^{(1)}\to0$ in that case. Thus, generation of fourth sound by a transducer which produces only V_n appears very unlikely. Since V_s is determined by a gauge transformation (and thus V_s is unaffected by the Galilean "boost" that enables a transducer to produce V_n), such a transducer is probably a typical one. Further, since, when $\partial\rho/\partial w_{11}=-\rho$ (as for a perfect crystal, or in the limit of fluidlike behavior), both δ and α vary as $\partial T/\partial\rho$ (essentially the thermal-expansion coefficient), this case is the analog of the ordinary transducer against an ordinary superfluid (where, analogously, very little second sound is generated).

Case 2. We consider here a heater which produces a temperature variation T' but no momentum g. Then, following Khalatnikov,[16]

$$0=[(j_s/j_n)_1+1]j_n^{(1)}+[(j_s/j_n)_4+1]j_n^{(4)}, \tag{5.37}$$

so use of Eqs. (5.33) and (5.34) gives

$$\frac{j_n^{(4)}}{j_n^{(1)}}\simeq-\left[\left(\frac{C_{10}^2}{C_{40}^2}-1\right)\left(1-\frac{\bar{\delta}\rho}{\rho^s}\right)^{-1}+1\right]^{-1}. \tag{5.38}$$

In terms of the ratio of the time-averaged energy density $\bar{E}$, where[16]

$$\bar{E}=\rho^sV_s^2+\rho V_n^2, \tag{5.39}$$

we have

$$\frac{\bar{E}_4}{\bar{E}_1}\simeq\left[\frac{\rho}{\rho_s}\left(\frac{j_s}{j_n}\right)_4^2+1\right]\left(\frac{j_n^{(4)}}{j_n^{(1)}}\right)^2\simeq\frac{\rho}{\rho^s}. \tag{5.40}$$

Thus the ratio of the time-averaged intensities $\bar{I}$ is

$$\frac{\bar{I}_4}{\bar{I}_1}\simeq\frac{C_{40}\bar{E}_4}{C_{10}\bar{E}_1}\simeq\frac{(\rho\,\partial\mu/\partial\rho)^{1/2}}{C_{10}}\left(\frac{\rho}{\rho^s}\right)^{1/2}. \tag{5.41}$$

If $\rho\,\partial\mu/\partial\rho\simeq C_{10}^2$ at all T, then

$$\bar{I}_4/\bar{I}_1 \simeq (\rho/\rho^s)^{1/2} > 1 \ . \tag{5.42}$$

If $\rho^s/\rho \simeq 0.1$ at $T = 0$, then $\bar{I}_4/\bar{I}_1 \simeq 3$ at $T = 0$. This is the least-favorable case for the generation of fourth sound.

We thus conclude that a heater, rather than a transducer, would be the better fourth-sound generator. Similarly, one can expect a thermometer to be a better fourth-sound detector. This assumes that, during a period of the heater's oscillation, collisions occur rapidly enough for the excitations (i.e., phonons) to equilibrate to the lattice. Such an assumption is questionable, since it is well-known that at low temperatures (e.g., $T < 0.5$ K) heaters in He II tend to produce a spray of ballistic phonons rather than a heat pulse.[17] Therefore the generation of fourth sound by this method is quite uncertain.

There is another reason to be uncertain about the relevance of the above calculation. It was assumed, not proved, that the heater puts all of its energy into just two modes. This is by no means obvious. Indeed, there are reasons for believing this is not the case. Specifically, a heater may merely produce a thermal wave which does not propagate. If we set ρ', V_{ni}, u_i, and θ equal to zero in the equations of motion, then one has thermal diffusion from $\dot{S} + \partial_i(q_i/T) \simeq 0$, provided that $(\partial\sigma_{11}/\partial S)_{\rho,\omega} \simeq 0$ and $(\partial\mu/\partial S)_{\rho,\omega} \simeq 0$. These conditions are probably satisfied at very low temperatures. However, at higher temperatures it is not clear that only thermal diffusion occurs, in which case a heater might also produce appreciable amounts of fourth sound. Also note that the presence of impurities or finite strain might cause some mode mixing, thus improving the coupling of a heater to the fourth-sound mode. (The impurity concentration c, like the temperature, is a scalar. Strain squared is also a scalar. One can envision two fourth-sound modes and two diffusion modes, each involving ρ, T, θ, and c.) Further, nonlinear effects in pulse amplitudes might cause a heater or a transducer to generate fourth sound.

Owing to the significance of observing superfluidity in a solid, it is probably most prudent to take the position that experiments can best answer the question of fourth-sound generation.

We will close this section with an inquiry on how one might obtain a closer analog of second sound in an ordinary superfluid. This can be done by considering $\vec{q}$, $\vec{V}_n$, and $\vec{V}_s$ to be along [100] and requiring that $g_i \simeq 0$ for second sound. This implies, by Eqs. (5.1) and (5.2) that both ρ and σ_{11} ($= \lambda_{11} - P$) are constants. In an ordinary superfluid $\lambda_{11} = 0$, so these conditions are equivalent (if the thermal expansion coefficient is negligible). However, in a superfluid solid λ_{11} depends upon $\partial_1\dot{u}_1 (= iq\dot{u}_1)$, so that if $\dot{u} = V_n$, then the motion of V_n implied by Eq. (5.4) will prevent σ_{11} from being a constant. The way to make these restrictions consistent is to relax the condition $\dot{u} = V_n$ by permitting $\dot{u} = 0$ yet $V_n \neq 0$. Although these conditions may seem arbitrary, they in fact describe the situation for second sound in an ordinary solid. There, one observes second sound by operating at a frequency ω such that it is much less than the collision frequency ω_N for normal processes (so that the phonons can equilibrate with respect to one another), yet much greater than the collision frequency ω_U for umklapp processes (so that the phonons do not equilibrate to the lattice): $\omega_U \ll \omega \ll \omega_N$.

Before solving Eqs. (5.3) and (5.4) subject to $g_i = 0$, we note that g_i is no longer given by $g_i = \rho^s_{ik} V_{sk} + \rho V_{ni}$. Returning to Eq. (2.12), but permitting $V_{ni} \neq 0$, one finds that

$$g_i = \rho^s_{ik} V_{sk} + \tilde{\rho}^n_{ik} V_{nk} \ , \tag{5.43}$$

where

$$\tilde{\rho}^n_{ij} = -\frac{1}{mV} \sum_{kn} f'(E_{kn}) \langle kn | p_i | kn \rangle_0 \langle kn | p_j | kn \rangle_0 \ . \tag{5.44}$$

(Only in the limit of zero crystallinity will one have $\rho^s_{ik} + \tilde{\rho}^n_{ik} = \rho\delta_{ik}$.) Then Eqs. (5.3) and (5.4) become

$$-\omega V_s + q\left(\frac{\partial\mu}{\partial S}\right)_\rho S' = 0 \ , \tag{5.45}$$

$$-\omega S' + SqV_n = 0 \ . \tag{5.46}$$

Imposing $g_i = 0$ and using Eq. (5.43), one can solve Eqs. (5.45) and (5.46) to find

$$C_2^2 \equiv \frac{\omega^2}{q^2} = \frac{\rho^s}{\tilde{\rho}^n} S \left(\frac{\partial\mu}{\partial S}\right)_\rho \simeq \frac{\rho^s}{\tilde{\rho}^n} \frac{S^2}{\rho} \left(\frac{\partial T}{\partial S}\right)_\rho \tag{5.47}$$

In the limit of zero crystallinity, this gives the second-sound velocity of an ordinary superfluid.

To summarize, if $\omega_U \ll \omega \ll \omega_N$ one will obtain second sound, whereas if $\omega \ll \omega_U$ one will obtain fourth sound. Note that as $\omega_U \to 0$ (the limit of zero crystallinity) the fourth-sound regime becomes vanishingly small, and only second sound can propagate. Since the limit of zero crystallinity corresponds to an ordinary superfluid, this is the expected result.

VI. DISCUSSION AND SUMMARY

At this point it would do well to provide some justification for the conceptual and calculational labor that has been expended, in this paper and in others, on the speculation that solids might display superfluidity. Aside from the intrinsic interest in finding another superfluid, indeed a wholly new type of superfluid, the discovery of superfluidity in solid ^{4}He would provide completely new ground for testing and extending our understanding of the quantum theory of condensed matter. Most particularly, experimental study of the superfluid density would provide detailed information about atomic localization and about the excitations in the system. Note that, by doping with ^{3}He, by freezing in vacancies, and by straining the crystal, one has a number of ways to provide variations on the theme in question.

One theoretical objection has been made to the possibility of superfluidity in solid ^{4}He. This objection, alluded to in Sec. I, was made by Fernandez and Puma after they estimated the superfluid fraction of hcp ^{4}He to lie between 0.2 and 0.4 at $T=0$.[4] It goes as follows: for a ground state wave function Ψ_0 made up of a properly symmetrized set of single-particle wave functions $\phi_i(x_\alpha)$ (with the αth particle localized about the ith site) multiplied by a Jastrow function Ψ_J[18] one can show, under a number of assumptions, that the system does not possess off-diagonal long-range order (ODLRO)[19] and therefore cannot be a superfluid.[20] Supporting their objection is the work of Matsuda and Tsuneto,[21] which considers the case $\Psi_J=1$ (so that hard-core correlations are neglected), finding no ODLRO for nearest-neighbor overlap only of the $\phi_i(x_\alpha)$. It appears reasonable to assume that the $\Psi_J\neq 1$ wave function of Ref. 4 should possess no more ODLRO than the $\Psi_J=1$ wave function of Ref. 21, since the $\Psi_J\neq 1$ wave function is more restricted. If this assumption is correct, then by Ref. 21 the $\Psi_J\neq 1$ wave function of Ref. 4 should possess no ODLRO. However, such a wave function might yield ODLRO if there were vacancies and interstitials.

On the other hand, the following point cannot be too greatly emphasized: the structure of Ψ_0 is not known. Indeed, recently Lowy and Woo[22] have studied the ground-state properties of solid ^{4}He with a Ψ_0 made up of a product of identical, delocalized but nonuniform, single-particle wave functions multiplied by a Jastrow function, obtaining results comparable to those obtained with the usual Ψ_0. By letting the crystallinity go to zero, one obtains a Ψ_0 appropriate to liquid ^{4}He, which is a superfluid. Since the wave functions go smoothly into one another (unlike the case considered by Ref. 4), one can expect the superfluid densities to go smoothly into one another. This gives one reason to believe that the calculations of Refs. 4 and 5 are meaningful.

Because of the uncertainty about the structure of Ψ_0, I believe that experimental searches for superfluidity (either by dc or ac techniques) should not be impeded by the pessimistic forecast of Ref. 4. However, there are a number of other provisos to keep in mind during the design and interpretation of experiments designed to detect superfluidity in solids. We will now consider these provisos, based primarily on the results of Ref. 5 and the present work.

A number of questions immediately come to mind, the first of which concerns the value of T_c. Since we have performed no numerical calculations of $\rho^s(T)$, this can at best be estimated. Simple arguments, however, are of little help. Consider the following. If we accept that $\rho^s/\rho_0\simeq 0.1$ at $T=0$, one might expect that it should be easier to destroy the smaller superfluid fraction of a solid than to do so in a liquid. On the other hand, fourth-sound cavities in He II have $\rho^s/\rho_0<1$ at $T=0$, yet their T_c is nearly the same as that of open chambers of He II. However, this latter argument can be turned around: if cavity pore dimensions are so small that size effects matter, then the value of T_c is less than in the bulk; thus, if the solid is considered to be a fourth-sound chamber with atom-sized pore dimensions, one can expect a value of T_c much less than that in bulk liquid (which is in coexistence with the solid). Clearly, such qualitative reasoning is most uncertain in value. It is probably best, therefore, to consider the theory of $\rho^s(T)$ as delineated by Eqs. (2.33), (2.14), and (2.30).

The first term in Eq. (2.33) equals at $T=0$ the average density ρ_0. In the limit of zero crystallinity, it is the only term that appears. The other terms decrease the value of ρ^s from ρ_0. At finite T, the first term decreases, behaving much as ρ^s does in an ordinary superfluid [providing that the third term in Eq. (2.14) can be neglected]. It is not clear how the other terms behave, owing to their complexity. If the flow pattern $\vec{V}^s$ of Eq. (2.32) remains nearly independent of T, then only the temperature dependence of $\bar{\rho}_{0,-G}$ [as given by Eq. (2.14)] is needed. Unfortunately, without a knowledge of the properties of the excitations, this temperature dependence is unknown. Assuming that the flow pattern $\vec{V}^s$ has no temperature dependence, this means that $\rho^s(T)-\rho^s(0)$ comes only from the first term in Eq. (2.33), so that only 10% "normal fluid" [as described by the first term in Eq. (2.33)] is needed to destroy superfluidity, if $\rho^s/\rho_0=0.1$ at $T=0$. Since transverse phonons dominate the thermal properties of the system, one might expect them to provide the dominant component of the "normal fluid." However, if one works by analogy to He II, where rotons produce the dominant contribution to ρ_n, then it is zone-edge phonons that must be considered. Since these have a higher energy than do rotons, and since ρ_n

depends exponentially on their energy, it would appear harder to destroy superfluidity in solid ^{4}He than in the liquid, even if only 10% "normal fluid" will destroy superfluidity. Of course, the third term in Eq. (2.14) has not been discussed, and this might have a large effect. Nevertheless, without detailed calculations, I cannot help but conclude that T_c in solid ^{4}He might well be as large as in the liquid (i.e., $T_c \sim 1\,°K$).

One can then ask why there has so far been no experimental manifestation of this superfluidity if T_c is indeed so high. The results of Secs. IV and V help to explain this—the system has no anomalous thermal transport properties [no counterflow by Eq. (2.20)], it is difficult to generate fourth sound by accident, and the elastic waves are hardly at all affected by the presence of the superfluid. Nevertheless, one is still bothered by the absence of any thermal anomalies to indicate a transition from normal to superfluid phase. I can only speculate about why this could be so: the effect is there, but subtle and not easily observed unless very careful measurements are made; the presence of ^{3}He impurities causes attenuation of the critical fluctuations which produce the critical phenomena; the presence of a crystal lattice somehow eliminates the phenomena associated with the λ transition in liquid ^{4}He.

From the considerations of Sec. V we conclude that, irrespective of the value of T_c, ac measurements in a pure, unstrained, perfect crystal are not likely to yield evidence of superfluidity. In the case of elastic waves, the shift in velocity [see Eq. (5.32)] is likely to be very small.[23] (We estimate $\delta C/C \simeq 10^{-4}$ for $C_{40}^2/C_{10}^2 \simeq 10^{-1}$, and $\bar{\delta} \simeq 10^{-3}$ for 0.1% vacancies.) In the case of fourth sound, it is doubtful that available sample dimensions are large enough to enable a spray of ballistic phonons (generated by a heater) to develop into a heat pulse. Further, if a heat pulse does develop, it is not clear that fourth sound will be generated. Also, fourth sound must be distinguished from second sound, a phenomenon which has already been observed in solid ^{4}He.[24] However, it should be noted that impure, strained, and imperfect crystals may behave differently. The impurities, strains, and imperfections might aid lattice equilibration, thus enabling heat pulses to develop. Strain and impurities might aid the generation of fourth sound or might make the shifts in the elastic wave velocities more pronounced. Other, unforeseen, effects might develop.

Another approach is to make dc measurements, as suggested by Leggett.[2] Here, one looks for a nonclassical moment of inertia (NCMI) by measuring the ratio L/ω, where L is the angular momentum and ω is the angular velocity. If the superfluid can be brought into rotation, this appears to be the most direct and unambiguous method to detect superfluidity in solids. However, because there is no counterflow, it is not clear that the superfluid can easily be brought into rotation. For example, stopping a rotating bucket of superfluid liquid leaves the superfluid in rotation. However, stopping a rotating bucket of superfluid solid does not necessarily leave the superfluid in rotation because it probably was not initially in rotation when the system was first prepared. Consider the following. First, observe that by analogy with Eq. (2.20), the total angular momentum should be given by $L = I\omega_n + I_s\omega_s$ (where I is the classical moment of inertia, ω_n is the frequency of lattice rotation, I_s is the superfluid moment of inertia, and ω_s is the frequency of superfluid rotation with respect to the lattice). Now consider the system to be rotating at $T > T_c$. There is no reason to expect a $\omega_s \neq 0$ to develop on passing to $T < T_c$ because L, I_0, and ω_n do not change on passing to $T < T_c$. Hence ω_s remains zero. On the other hand, if the system is rapidly brought to a stop (or to a start), there exists the possibility that nonequilibrium processes can occur which will bring the superfluid into rotation. Therefore rapid angular acceleration or deceleration would appear to provide the best technique for bringing the superfluid into rotation, after which its angular momentum can be detected (e.g., by heating to $T > T_c$).

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[14] Note the following misprints in Ref. 1: In q_i the expression $\partial_l\lambda_{kl}$ (rather than $\partial_l\lambda_{lk}$) appears; in d_i the expression α_{ki} (rather than α_{ki}/T) appears.

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