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Dynamics of sine-Gordon solitons in the presence of perturbations*

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We have examined the dynamical behavior of soliton solutions of the sine-Gordon equation in the presence of weak external perturbations. Three examples, of particular importance in condensed matter, are described in detail: (i) a model impurity is found to bind low-velocity solitons but merely phase shift those with high velocities, (ii) an external static driving "force" together with damping causes the soliton to accelerate to a terminal velocity, and (iii) spatial inhomogeneities in the coefficient of the nonlinear term cause the soliton to adjust its velocity and shape in the regions of imperfection. In all cases we find that solitons maintain their integrity to a high degree. These calculations are based on a linear perturbation theory which emphasizes the use of a translation mode, and we are led to conclude that, in many respects, sine-Gordon solitons behave as classical particles whose dynamics are governed by Newton's law.

I. INTRODUCTION

It is well recognized that the study of nonlinear equations and their solutions is of great importance in many areas of physics. Of particular importance are nonlinear wave equations which admit large-amplitude solitary-wave or soliton solutions[1] that retain their shape during propagation. Such solutions have recently received considerable attention[2] by elementary-particle physicists since they may be regarded as extended particlelike solutions of nonlinear field equations. The study of solitary waves has proved fruitful in many areas of condensed matter physics as well; for example, in theories of Bloch walls[3] which separate domains in magnetic materials, structural phase transitions,[4] liquid ^{3}He,[5] Josephson transmission lines,[6] and most recently in the theory of the low-temperature conductivity of "one-dimensional" Fröhlich charge-density-wave condensates.[7]

The interaction of solitary waves with spatial inhomogeneities is of considerable importance for the above applications in condensed matter physics. Impurities and/or defects are present even in the purest of material samples and their effect on the motion of solitary waves must be considered when the dynamics of such solutions are important in the problem at hand. In addition, the effect of external electric or magnetic fields on the motion of solitary waves is important in theories of the dynamical properties of certain systems[5-7] characterized by nonlinear wave equations.

In this paper, we describe a method for investigating the effect of perturbations on the motion of solitary-wave solutions of nonlinear wave equations.[8] We illustrate the method by examining the motion of soliton solutions of the sine-Gordon equation[1] in three different situations where perturbations are present. The particular choice of the sine-Gordon equation is not crucial; the method can be applied with slight modifications to all cases where the system possesses translational invariance in the absence of perturbations.

The organization of the paper is as follows. In Sec. II we review necessary properties of the sine-Gordon equation and its solutions. We then describe the simple mathematical concepts and methods employed in our investigation. In Sec. III we begin our discussion of perturbations by considering the effect of a weak model impurity potential on the motion of soliton solutions of the sine-Gordon equation. We find that high-velocity solitons pass through the impurity region suffering only a phase shift while low-velocity solitons can become trapped by an attractive impurity potential. In Sec. IV we examine the motion of soliton solutions in the presence of a constant external driving term and a damping or viscous term in the equation of motion. We find that when the damping constant is large the transient perturbations of the soliton decay rapidly and it achieves a terminal velocity which depends on the ratio of the driving constant to the damping constant, in agreement with the numerical results of Ref. 6. In Sec. V we treat the case where the coefficient of the nonlinear term [ω_0^2 in Eq. (2.1)] in the sine-Gordon equation exhibits spatial dependence. It is found that the soliton adjusts its velocity and shape to accommodate local changes in this coefficient. In Sec. VI we summarize our results and conclude that in many respects the soliton behaves as a classical particle which obeys Newton's second law of motion.

II. BASIC CONCEPTS

In this section we first review the properties of the sine-Gordon equation and some of its solutions and then describe the mathematical foundations of the perturbation theory which we employ in the remainder of the paper.

The sine-Gordon equation is a nonlinear wave equation of the form

$$\frac{\partial^2\psi}{\partial t^2}-c_0^2\frac{\partial^2\psi}{\partial x^2}+\omega_0^2\sin\psi=0\,, \tag{2.1}$$

where ψ is a function of x and t, c_0 is a characteristic velocity, and ω_0 is a characteristic frequency. Note that Eq. (2.1) has Lorentz covariant form where c_0 plays the role of the speed of light although in materials applications it is actually some limiting velocity characteristic of the system such as the speed of sound[4] or the Fermi velocity.[7] Small amplitude ($|\psi|\ll 1$) solutions of Eq. (2.1) may be found by replacing $\sin\psi$ with ψ in which case one obtains the ordinary Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2}-c_0^2\frac{\partial^2\psi}{\partial x^2}+\omega_0^2\psi=0\,. \tag{2.2}$$

The solutions of Eq. (2.2) have the harmonic form $\exp[i(\kappa x-\omega_\kappa t)]$, where the frequencies ω_κ obey the dispersion relation

$$\omega_\kappa^2=c_0^2\kappa^2+\omega_0^2\,. \tag{2.3}$$

One class of large-amplitude traveling-wave solutions of Eq. (2.1) have the form[1]

$$\psi_\pm^v(x,t)=4\tan^{-1}\left[\exp\left(\pm\frac{\omega_0}{c_0}\gamma(x-vt)\right)\right], \tag{2.4}$$

where

$$\gamma\equiv(1-v^2/c_0^2)^{-1/2},\quad |v|<c_0\,. \tag{2.5}$$

These solutions are referred to as solitons (+ sign) and antisolitons (− sign), respectively. Note that the velocity v must be less than c_0 in magnitude. In Fig. 1, we have plotted the waveform of the soliton solution in its rest frame. These solutions have the remarkable property that they retain their shape during propagation and hence may be classified as solitary waves.[1] The velocity v is a free parameter subject only to the restriction $|v|<c_0$ and thus the soliton (or antisoliton) may be regarded as a "relativistic" free particle in the absence of perturbations. It is the purpose of this paper to determine the values of v allowed in the presence of certain types of perturbations and also to determine whether the soliton retains its character (shape) in the presence of these perturbations.

The simplest kind of perturbation on the soliton solution is an internal one, namely, small oscillations about the soliton solution which must satisfy Eq. (2.1). Such perturbations may be treated[9]

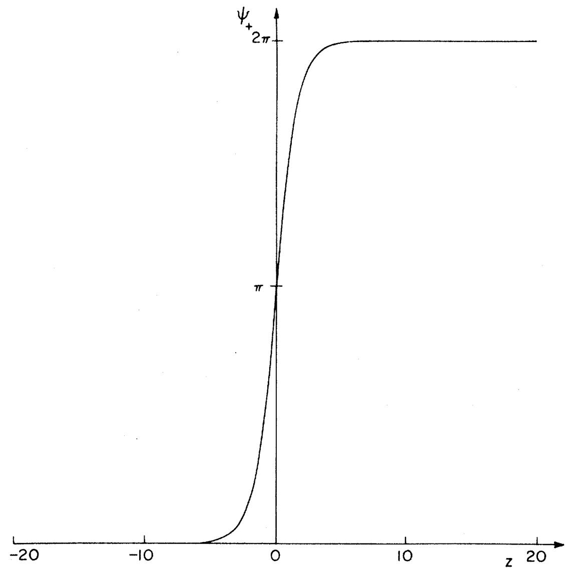

FIG. 1. Single-soliton solution $\psi_+(z)=4\tan^{-1}[\exp(z)]$ plotted in dimensionless units $z=(\omega_0 x/c_0)$.

by assuming a solution of the form

$$\psi_\pm(x,t)=\psi_\pm^v(x,t)+\phi(x,t)\,, \tag{2.6}$$

where $|\phi(x,t)|\ll 1$. By substituting Eq. (2.6) into Eq. (2.1) and linearizing in the small quantity $\phi(x,t)$, one obtains the following equation when $v=0$:

$$\frac{\partial^2\phi}{\partial t^2}-c_0^2\frac{\partial^2\phi}{\partial x^2}+\omega_0^2\left(1-2\,\mathrm{sech}^2\frac{\omega_0}{c_0}x\right)\phi=0\,. \tag{2.7}$$

The more general case with $v\neq 0$ can be reduced to this one by a Lorentz transformation to the rest frame of soliton (or antisoliton). Assuming a solution of Eq. (2.7) with harmonic time dependence

$$\phi(x,t)=f(x)\,e^{-i\omega t}, \tag{2.8}$$

one finds the following equation for $f(x)$:

$$-c_0^2\frac{d^2f(x)}{dx^2}+\omega_0^2\left(1-2\,\mathrm{sech}^2\frac{\omega_0}{c_0}x\right)f(x)=\omega^2 f(x)\,. \tag{2.9}$$

Equation (2.9) has the form of Schrödinger's equation with a "potential" of the form

$$V(x)=\omega_0^2\left(1-2\,\mathrm{sech}^2\frac{\omega_0}{c_0}x\right).$$

The solutions have been studied elsewhere[9] and the principal results are the following. There exists exactly one "bound state" with

$$\omega_b^2=0 \tag{2.10a}$$

and

$$f_b(x)=\frac{2\omega_0}{c_0}\operatorname{sech}\frac{\omega_0}{c_0}x\,. \tag{2.10b}$$

The remaining solutions form a continuum with eigenvalues

$$\omega_\kappa^2=c_0^2\kappa^2+\omega_0^2 \tag{2.11a}$$

and corresponding eigenfunctions

$$f_\kappa(x)=\frac{1}{(2\pi)^{1/2}}\frac{c_0}{\omega_\kappa}e^{i\kappa x}\left(\kappa+i\,\frac{\omega_0}{c_0}\tanh\frac{\omega_0}{c_0}x\right). \tag{2.11b}$$

The solutions (2.10) and (2.11) have simple physical interpretations. The fact that a solution with $\omega^2=0$ [Eq. (2.10a)] exists is a consequence of Goldstone's theorem,[10] i.e., the presence of the soliton breaks the continuous translational symmetry. When $f_b(x)$ is added to ψ^0 [see Eq. (2.6)] one finds that $\psi=\psi^0+\alpha f_b(x)$ corresponds to a soliton (or antisoliton) which is translated by an amount proportional to α. Thus the $\omega^2=0$ solution yields the "translation mode" of the soliton. This result is of paramount importance in the treatment of the dynamics of solitons in the presence of external perturbations. The continuum solutions (2.11) resemble the linearized solutions of Eq. (2.2) except for a localized perturbation [proportional to $\tanh(\omega_0/c_0)x$] in the vicinity of the soliton. Note that the dispersion relation (2.3) is unaffected by the presence of the soliton. By examining the asymptotic form of the solutions (2.11b) for $x\to\pm\infty$ one finds that the linearized solutions of Eq. (2.2) suffer only a phase shift

$$\Delta(\kappa)=\pi\kappa/|\kappa|-2\tan^{-1}(c_0\kappa/\omega_0) \tag{2.12}$$

when passing through the soliton. This is a consequence of the fact that the "potential" in Eq. (2.7) is reflectionless.[11]

Because the solutions $f_i(x)$ $(i=b,\kappa)$ are eigenfunctions of the self-adjoint spatial operator

$$D=-c_0^2\frac{d^2}{dx^2}+\omega_0^2\left(1-2\operatorname{sech}^2\frac{\omega_0}{c_0}x\right), \tag{2.13}$$

they form a complete set which spans the space of functions of x. The orthogonality relations are[9]

$$\int_{-\infty}^{+\infty}f_b(x)f_b(x)\,dx=8\,\frac{\omega_0}{c_0}\,, \tag{2.13a}$$

$$\int_{-\infty}^{+\infty}f_\kappa^*(x)f_{\kappa'}(x)\,dx=\delta(\kappa-\kappa')\,, \tag{2.13b}$$

and

$$\int_{-\infty}^{+\infty}f_\kappa(x)f_b(x)\,dx=0\,, \tag{2.13c}$$

while the completeness relation has the form[9]

$$\int_{-\infty}^{+\infty}d\kappa f_\kappa^*(x)f_\kappa(x')+\frac{1}{8(\omega_0/c_0)}f_b(x)f_b(x')=\delta(x-x')\,. \tag{2.14}$$

In Secs. III–VI we make repeated use of the fact that pertubations of the soliton solution may be expanded in this complete set of functions. Special significance is attached to the translation mode $f_b(x)$ since it governs the motion of the soliton. The continuum contributions correspond to small perturbations of the soliton shape. It should be noted that the linear stability of the soliton and the validity of our perturbation theory follow from the fact that the eigenvalues ω_i^2 $(i=b,\kappa)$ are all nonnegative. There is one subtlety in the use of the functions $f_i(x)$ as a basis for a transform procedure: the inverse integral transform of a function of κ is understood to be replaced by its Cauchy principle value should the function of κ have a singularity on the real κ axis.

III. INTERACTION OF A SOLITON WITH A MODEL IMPURITY POTENTIAL

In this section we examine the effect of a model impurity potential on the motion of a soliton initially moving with velocity v. We consider the Hamiltonian

$$\mathcal{H}=\int dx\,A\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{c_0^2}{2}\left(\frac{\partial\psi}{\partial x}\right)^2+\omega_0^2(1-\cos\psi)-\lambda\,\frac{\partial\psi}{\partial x}g(x)\right]. \tag{3.1}$$

The first three terms in the integrand comprise the usual sine-Gordon Hamiltonian density.[9] The last term represents the interaction of ψ with the impurity potential $g(x)$, which we take to have the simple form

$$g(x)=\Theta(x-x_0)-\Theta(x+x_0)\,. \tag{3.2}$$

In Eq. (3.2), $\Theta(x)$ is the Heaviside step function defined by

$$\Theta(x)\equiv\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}. \tag{3.3}$$

The method described below is not limited to the special form of $g(x)$ in Eq. (3.2). All of the qualitative features of the results obtained with Eq. (3.2), however, are expected to be present with more realistic forms of the impurity potential. The coupling constant λ appearing in Eq. (3.1) is assumed small and may be either positive or negative. The particular form of the coupling term $(\partial\psi/\partial x)g(x)$ has been chosen for definiteness. In the next section we consider perturbations which couple to ψ rather than $\partial\psi/\partial x$. The constant A

sets the energy scale and has units of (mass) ×(length).

From Eq. (3.1) we derive the following equation of motion for ψ:

$$\frac{\partial^2\psi}{\partial t^2} - c_0^2\frac{\partial^2\psi}{\partial x^2} + \omega_0^2\sin\psi = \lambda[\delta(x+x_0) - \delta(x-x_0)]. \tag{3.4}$$

For convenience we introduce the dimensionless quantities

$$\tau \equiv \omega_0 t, \quad z \equiv (\omega_0/c_0)x \tag{3.5a}$$

and

$$k \equiv \frac{c_0}{\omega_0}\kappa, \quad \overline{\omega}_k \equiv \frac{\omega_\kappa}{\omega_0}, \quad \beta \equiv \frac{v}{c_0}, \quad \alpha \equiv \frac{\lambda}{c_0\omega_0}. \tag{3.5b}$$

In terms of these quantities, Eq. (3.4) takes the dimensionless form

$$\frac{\partial^2\psi}{\partial\tau^2} - \frac{\partial^2\psi}{\partial z^2} + \sin\psi = \alpha[\delta(z+z_0) - \delta(z-z_0)]. \tag{3.6}$$

We note that in terms of the quantities (3.5), the basis functions (2.10b) and (2.11b) become, respectively,

$$f_b(z) = 2\,\text{sech}z \tag{3.7a}$$

and

$$f_k(z) = (2\pi)^{-1/2}(\overline{\omega}_k)^{-1}e^{ikz}(k + i\tanh z). \tag{3.7b}$$

When $\alpha = 0$ in Eq. (3.6), an exact solution for ψ is the soliton (or antisoliton) described in Sec. II. We wish to determine the effect of nonzero but small values of α on the soliton solution initially moving with velocity v. For large velocities, we expect the perturbations on the soliton to be small, while for low velocities there may be a significant modification of the soliton solution. We treat these two velocity regimes separately, focusing first on the case of fast solitons. We assume a perturbative solution of Eq. (3.6) having the form

$$\psi(z,\tau) = \psi^v(z-\beta\tau) + \phi(z,\tau), \tag{3.8}$$

where $\psi^v(z-\beta\tau)$ is a soliton moving with velocity $\beta = v/c_0$ and $\phi(z,\tau)$ is assumed small. By making use of Eqs. (2.4) and (3.5), we find upon substituting (3.8) into (3.6) and Lorentz transforming to a frame moving with velocity v that $\phi(z,\tau)$ obeys the following equation:

$$\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial z^2} + (1 - 2\,\text{sech}^2 z)\phi = \frac{\alpha}{\gamma}\left[\delta\left(z+\beta\tau+\frac{z_0}{\gamma}\right) - \delta\left(z+\beta\tau-\frac{z_0}{\gamma}\right)\right], \tag{3.9}$$

where γ is defined by Eq. (2.5), and z and τ are now measured in the coordinate frame moving with velocity v.

To solve Eq. (3.9) we expand $\phi(z,\tau)$ in the complete set of basis functions (3.7) as follows:

$$\phi(z,\tau) = \tfrac{1}{8}\phi_b(\tau)f_b(z) + \int_{-\infty}^{+\infty} dk\,\phi(k,\tau)f_k(z), \tag{3.10}$$

and then we introduce the Fourier time transform of $\phi(k,\tau)$ defined by

$$\phi(k,\overline{\omega}) = \int_{-\infty}^{+\infty} d\tau\, e^{i\overline{\omega}\tau}\phi(k,\tau). \tag{3.11}$$

This procedure yields the following equations for $\phi_b(\tau)$ and $\phi(k,\overline{\omega})$:

$$\frac{d^2\phi_b}{d\tau^2} = \frac{2\alpha}{\gamma}\left[\text{sech}\left(\beta\tau+\frac{z_0}{\gamma}\right) - \text{sech}\left(\beta\tau-\frac{z_0}{\gamma}\right)\right], \tag{3.12a}$$

$$(\overline{\omega}^2 - \overline{\omega}_k^2)\,\phi(k,\overline{\omega}) = \alpha\,\frac{2(2\pi)^{1/2}i}{\gamma\beta\overline{\omega}_k}\sin\left(\frac{z_0\overline{\omega}}{\gamma\beta}\right) \times\left[k\delta\left(k+\frac{\overline{\omega}}{\beta}\right) - \tfrac{1}{2}\,\text{csch}\frac{\pi}{2}\left(k+\frac{\overline{\omega}}{\beta}\right)\right]. \tag{3.12b}$$

It now remains to solve these equations and invert the various transforms in (3.12b). When inverting the Fourier time transform we give $\overline{\omega}$ a small positive imaginary part in order to satisfy casual boundary conditions. From Eq. (3.12a), we find

$$\frac{d\phi_b(\tau)}{d\tau} = \frac{4\alpha}{\beta\gamma}\left[\tan^{-1}\exp\left(\beta\tau+\frac{z_0}{\gamma}\right) - \tan^{-1}\exp\left(\beta\tau-\frac{z_0}{\gamma}\right)\right] \tag{3.13a}$$

and

$$\phi_b(\tau) = \frac{4\alpha}{\beta\gamma}\int_{-\infty}^{\tau} d\tau'\left[\tan^{-1}\exp\left(\beta\tau'+\frac{z_0}{\gamma}\right) - \tan^{-1}\exp\left(\beta\tau'-\frac{z_0}{\gamma}\right)\right], \tag{3.13b}$$

where we have used the initial conditions $\dot{\phi}_b(-\infty) = \phi_b(-\infty) = 0$. We have also obtained the continuum amplitude $\phi(k,\tau)$ by inverting the Fourier-time transform. However, the expression is very complicated and for brevity is not presented here. The essential features of the continuum contribution are discussed below.

We now interpret the contribution (3.13b) of the translation mode to the perturbation $\phi(z,\tau)$ of the soliton. In the coordinate frame moving with velocity v, we write Eq. (3.8) as

$$\psi(z,\tau)=\psi^0(z)+\tfrac{1}{8}\phi_b(\tau)f_b(z)+\phi_c(z,\tau)\,, \qquad (3.14)$$

where ϕ_c is the continuum contribution. The first two terms in (3.14) may be combined to yield the soliton function having $z+\frac{1}{8}\phi_b(\tau)$ as its argument instead of z, i.e.,

$$\psi^0(z)+\tfrac{1}{8}\phi_b(\tau)f_b(z)=\psi(z+\tfrac{1}{8}\phi_b(\tau))+O(\alpha^2)\,. \quad (3.15)$$

The interpretation of $f_b(z)$ as the translation mode is now clear. Note that Eq. (3.12a) may be interpreted as Newton's law for the position of the "center-of-mass" of the soliton. As the impurity potential approaches $z=0$ (soliton center) in the rest frame, the soliton, because of its finite spatial extent, begins to "feel" the presence of the impurity. When the soliton approaches the leading edge of the impurity it acquires a velocity in the negative direction (if α is positive) and hence moves slower in the lab frame until it has passed through the impurity region. If α is negative the soliton speeds up in the impurity region. Exactly the opposite is true for the antisoliton. Asymptotically, the soliton acquires a net phase shift of $\delta=\frac{1}{2}\pi(z_0\alpha/\beta^2\gamma^3)$ in the lab frame.

The continuum contribution to ϕ consists of two pieces, both of which are proportional to α. The first of these persists for all times and is localized in the region of the impurity potential. This contribution was expected and may be regarded as intrinsic to the impurity. The second contribution is nonzero only when the soliton is in the region of the impurity potential and corresponds to a slight modification of the soliton waveform.

We have illustrated this behavior in Fig. 2 where we plot the soliton plus the effects of the impurity (in the initial rest frame) for a large value of α ($\alpha=1.0$) in order to accentuate the impurity effects. In Fig. 3 we focus in detail on only the corrections ($\phi_c(z,\tau)$) to the soliton's shape at seven successive times during the collision process. All plots are in the initial rest frame of the soliton. The cross denotes the position of the soliton center and the vertical arrows denote the boundaries of the impurity region. Figure 3(a) depicts the initial, *unrelaxed* impurity "dressing" that we considered. The following figures [3(b)–3(g)] show the evolution of the dressing as the soliton collides with the impurity. We note that at the end of the process, Fig. 3(g), the perturbation caused by impurity has evolved to the form that it would have in a system without a soliton. In Fig. 4 we plot the effect of the impurity when no soliton is present (calculated by linearizing the equation of motion about $\phi=0$) for comparison.

It is clear from Eqs. (3.12b) and (3.13b) that the perturbation ϕ diverges as $\beta\to 0$, i.e., for slow-moving solitons, thus invalidating the use of the

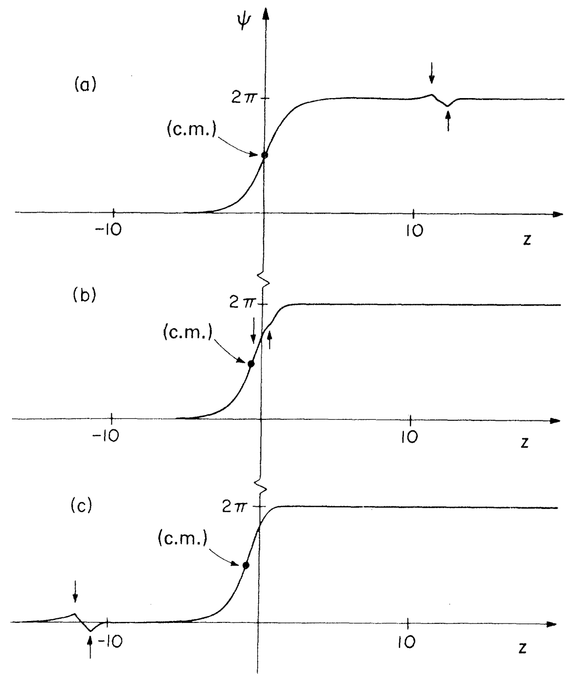

FIG. 2. Representative soliton-impurity collision is shown in the initial rest frame of the soliton (a) before, (b) during, and (c) after the interaction; the parameter values are $\alpha=1.0$, $\beta=0.9$, and $z_0=1.25$. The "center-of-mass" (c.m.) position of the soliton suffers a phase shift (see text). The vertical arrows indicate the bounaries of the "Lorentz-contracted" impurity region.

perturbation theory above in this velocity regime. This divergence suggests that the soliton may become trapped by, or repelled from, the impurity if v is sufficiently small, causing a large perturbation ($\phi\cong 2\pi$) where the soliton would have been had the impurity not been present.

In order to pursue this possibility of trapping at low velocities, we proceed in the following manner. Let us assume for the moment that a static soliton exists with its center a distance $(c_0/\omega_0)\xi$ (ξ is dimensionless) from the center of the impurity potential ($z=0$):

$$\psi^0(z)=4\tan^{-1}[\exp(z-\xi)]\,. \qquad (3.16)$$

By employing Eq. (3.1), we obtain the "potential" energy of this configuration as a function of ξ:

$$V(\xi)=A\omega_0 c_0\left[8+4\alpha\tan^{-1}\left(\frac{\sinh z_0}{\cosh\xi}\right)\right]. \qquad (3.17)$$

The first term in Eq. (3.17) is just the rest energy

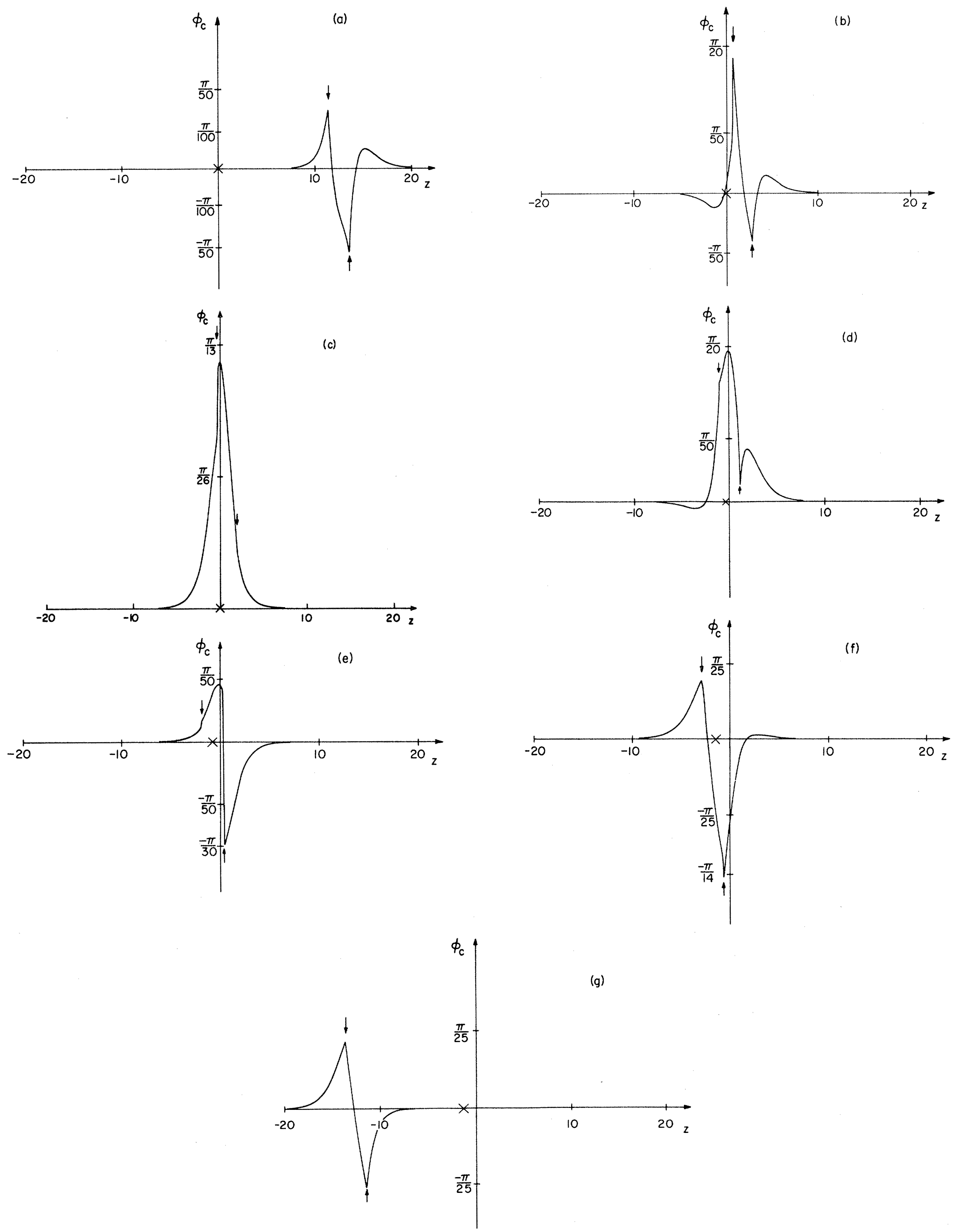

FIG. 3. Perturbation of soliton shape at successive times during collison with impurity. Corrections (ϕ_c) to the soliton shape are plotted in the initial rest frame for parameter values $\alpha=0.2$, $z_0=1.25$, and $\beta=0.5$. The "center-of-mass" position is denoted by the X. Note the change of scale of the various plots especially between (a), (d), and (f).

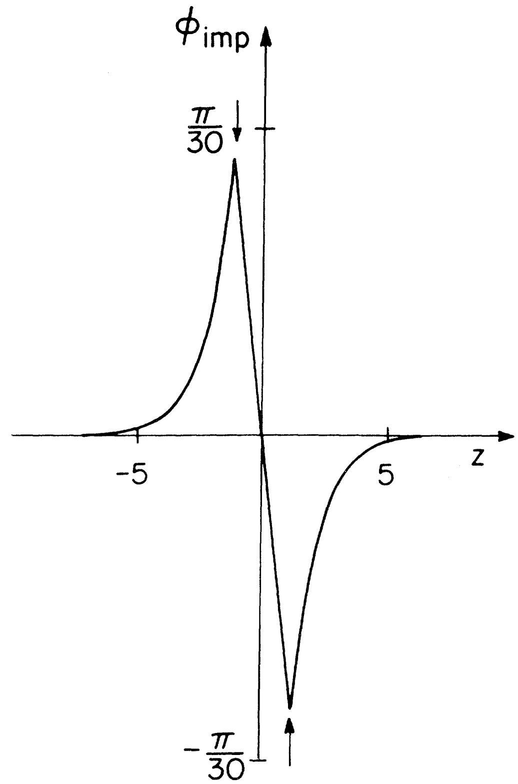

FIG. 4. Intrinsic impurity effect with no soliton present. It is instructive to compare it with the corresponding plots in Figs. 3(a) and 3(g), when the soliton is far from the impurity.

of the soliton and the second term is the change in the static soliton energy due to the impurity. We denote this change by

$$\Delta V(\xi) \equiv 4A\omega_0 c_0 \alpha \tan^{-1}\left(\frac{\sinh z_0}{\cosh \xi}\right). \tag{3.18}$$

We note that $\Delta V(\xi)$ has an extremum at $\xi = 0$ (the center of the impurity potential). This extremum is a minimum if α is negative and a maximum if α is positive. The reverse is true for an antisoliton. Thus we see from energy considerations alone that for α negative the impurity attracts solitons and repels antisolitons and vice versa for α positive.

When the soliton is very far from the impurity and moving with velocity $\beta = v/c_0$, its energy is given by the relativistic form

$$E(\beta) = 8A\omega_0 c_0/(1-\beta^2)^{1/2}. \tag{3.19}$$

The classical condition on the soliton's velocity such that it be trapped ($\alpha < 0$) or reflected ($\alpha > 0$) by the impurity potential can be obtained by comparing its kinetic energy to the depth of the potential $V(0)$, i.e.,

$$E(\beta) - E(\beta = 0) < |\Delta V(\xi = 0)|, \tag{3.20}$$

or, more explicitly,

$$\beta^2 < 1 - (1 + \tfrac{1}{2}|\alpha| \tan^{-1} \sinh z_0)^{-2}. \tag{3.21}$$

Note that if α or z_0 vanishes then the soliton is free. If α becomes large, then only very fast solitons can penetrate the impurity region.

We now investigate further the case when $\alpha < 0$ and β satisfies the inequality (3.21), i.e., when the soliton is trapped or bound to the impurity. In this case one expects the soliton to execute oscillatory motion about $\xi = 0$. This oscillatory motion can be easily studied in detail in two different regimes, namely, (i) when the spatial extent of the soliton is small compared to the width of the impurity potential and (ii) when the soliton is extended over a large region compared to the width of the impurity potential. In case (i) the soliton should behave as a point particle and for small energies execute harmonic motion. In case (ii) the oscillation will be distinctly anharmonic due to the spatial extent of the soliton.

We consider first case (i) where $z_0 \gg 1$ and we expect the soliton to oscillate harmonically if its maximum velocity is small. We expand the potential (3.18) to quadratic order in the position ξ of the soliton

$$\Delta V(\xi) \cong \Delta V(\xi = 0) + 2A\omega_0 c_0 |\alpha| \frac{\sinh z_0}{\cosh^2 z_0} \xi^2. \tag{3.22}$$

By regarding the soliton as essentially a classical particle, we make the ansatz that $\xi(\tau)$ is governed by Newton's law

$$M \frac{d^2\xi}{d\tau^2} = -\frac{1}{c_0^2} \frac{\partial \Delta V(\xi)}{\partial \xi}, \tag{3.23}$$

where the soliton "mass" M is defined by

$$M \equiv 8A\omega_0/c_0. \tag{3.24}$$

Substitution of Eq. (3.22) into (3.23) yields the harmonic oscillator equation with solution

$$\xi(\tau) = \xi_0 \sin(\Omega\tau), \tag{3.25}$$

where the oscillation frequency is given by

$$\Omega \equiv \left(\frac{|\alpha|}{2} \frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}. \tag{3.26}$$

In order to test the validity of the solution $\xi(\tau) = \xi_0 \sin\Omega\tau$, we consider the exact equation of motion governing ψ:

$$\frac{\partial^2\psi}{\partial\tau^2} - \frac{\partial^2\psi}{\partial z^2} + \sin\psi = \alpha[\delta(z + z_0) - \delta(z - z_0)]. \tag{3.27}$$

We assume that

$$\psi(z,\tau)=4\tan^{-1}\{\exp[\gamma(z-\xi_0\sin\Omega\tau)]\}+\phi(z,\tau)\,, \tag{3.28}$$

where the first term represents a soliton oscillating about $z=0$ and $\phi(z,\tau)$ is the difference between the exact solution of (3.27) and the oscillating soliton. Substituting Eq. (3.28) into Eq. (3.27) and assuming ϕ to be small we find the following equation for $\phi(z',\tau)$ $(z'\equiv z-\xi_0\sin\Omega\tau)$:

$$\frac{\partial^2\phi}{\partial\tau^2}-2\xi_0\Omega\cos\Omega\tau\,\frac{\partial^2\phi}{\partial\tau\partial z'}+\xi_0^2\Omega^2\cos^2\Omega\tau\,\frac{\partial^2\phi}{\partial z'^2}-\frac{\partial^2\phi}{\partial z'^2}+(1-2\,\mathrm{sech}^2 z')\phi$$
$$=\alpha[\delta(z'+z_0+\xi_0\sin\Omega\tau)-\delta(z'-z_0+\xi_0\sin\Omega\tau)]+2\xi_0^2\Omega^2\cos^2\Omega\tau\,\sinh z'\,\mathrm{sech}^2 z'$$
$$-2\xi_0\Omega^2\sin\Omega\tau\,\mathrm{sech}\,z'-2\sinh z'\,\mathrm{sech}^2 z'\,, \tag{3.29}$$

where we have taken $\gamma\cong 1$ for low-velocity solitons. We now expand $\phi(z',\tau)$ in the complete set of basis functions (3.7) as

$$\phi(z',\tau)=\tfrac{1}{8}\phi_b(\tau)f_b(z')+\int_{-\infty}^{+\infty}dk\,\phi(k,\tau)f_k(z')\,. \tag{3.30}$$

Equation (3.29) then leads to the following equation for the amplitude of the translation mode

$$\frac{\partial^2\phi_b(\tau)}{\partial\tau^2}+\tfrac{1}{3}\xi_0^2\Omega^2\cos^2\Omega\tau\,\phi_b(\tau)$$
$$=2\alpha[\mathrm{sech}(z_0+\xi_0\sin\Omega\tau)-\mathrm{sech}(z_0-\xi_0\sin\Omega\tau)]$$
$$-8\xi_0\Omega^2\sin\Omega\tau\,. \tag{3.31}$$

Recalling the condition that $z_0\gg 1$, Eq. (3.31) may be approximated by

$$\frac{\partial^2\phi_b(\tau)}{\partial\tau^2}+\tfrac{1}{3}\xi_0^2\Omega^2\cos^2\Omega\tau\,\phi_b(\tau)$$
$$=-8\alpha e^{-z_0}\xi_0\sin\Omega\tau-8\xi_0\Omega^2\sin\Omega\tau\,. \tag{3.32}$$

Using Eq. (3.26) for Ω we then have

$$\frac{\partial^2\phi_b(\tau)}{\partial\tau^2}+\frac{|\alpha|\,\xi_0^2}{3}e^{-z_0}\cos^2\Omega\tau\,\phi_b(\tau)=0\,. \tag{3.33}$$

Since ξ_0 is small at low soliton velocities, we see that

$$\phi_b(\tau)\propto e^{-z_0}\ll 1\,. \tag{3.34}$$

This result verifies our initial ansatz (3.25) since the corrections to the harmonic motion of the soliton arising from the translational mode amplitude (3.34) are exponentially small. The continuum contributions are also small and correspond to slight modifications of the soliton waveform.

We now turn our attention to the case when $z_0\ll 1$, i.e., when the soliton's extent is much larger than the width of the impurity potential. In this case we do not necessarily expect the amplitude of oscillation to be small. Thus we add the next nonvanishing term to the expansion of $\Delta V(\xi)$ given by Eq. (3.22) so that

$$\Delta V(\xi)\cong\Delta V(\xi=0)+2A\omega_0c_0|\alpha|\frac{\sinh z_0}{\cosh^2 z_0}\xi^2$$
$$-\tfrac{1}{6}A\omega_0c_0|\alpha|\frac{\sinh z_0}{\cosh^2 z_0}\left(\frac{6}{\cosh^2 z_0}-1\right)\xi^4\,. \tag{3.35}$$

Once again we use the ansatz (3.23) to assume that

$$\frac{d^2\xi}{d\tau^2}=-\frac{|\alpha|}{2}\left(\frac{\sinh z_0}{\cosh^2 z_0}\right)\xi$$
$$+\frac{|\alpha|}{12}\left(\frac{\sinh z_0}{\cosh^2 z_0}\right)\left(\frac{6}{\cosh^2 z_0}-1\right)\xi^3\,. \tag{3.36}$$

In order to find the solutions to the nonlinear equation (3.36) we first make a change of variables

$$\eta\equiv\frac{\xi}{\cosh z_0}(1-\tfrac{1}{6}\cosh^2 z_0)^{1/2}\,, \tag{3.37a}$$

$$s\equiv\tau\left(\frac{|\alpha|}{2}\frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}\,, \tag{3.37b}$$

so that Eq. (3.36) then takes the convenient form

$$\frac{d^2\eta}{ds^2}+\eta-\eta^3=0\,. \tag{3.38}$$

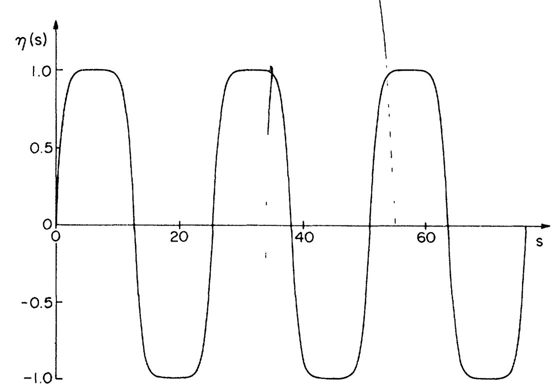

FIG. 5. Plot of the function $\eta(s)=a\,\mathrm{sn}[(b\sqrt{2})s]$ for $(d\eta/ds)_{\eta=0}=1/\sqrt{2}-5\times10^{-6}$.

This equation has oscillatory solutions known as Jacobi elliptic functions.[4,12] The relevant solution for our purposes is given by

$$\eta(s) = a\,\mathrm{sn}[(b/\sqrt{2})\,s]\,, \tag{3.39}$$

where

$$a^2 \equiv 1 - \left[1 - 2\left(\frac{d\eta}{ds}\right)^2_{\eta=0}\right]^{1/2} \tag{3.40a}$$

and

$$b^2 \equiv 1 + \left[1 - 2\left(\frac{d\eta}{ds}\right)^2_{\eta=0}\right]^{1/2}. \tag{3.40b}$$

Each member of this family of solutions is distinguished by specifying a value of $(d\eta/ds)_{\eta=0}$ in the range

$$0 < \left(\frac{d\eta}{ds}\right)_{\eta=0} \leq \frac{1}{\sqrt{2}}. \tag{3.41}$$

In Fig. 5 we have plotted an example of $\eta(s)$ vs s for $(d\eta/ds)_{\eta=0} = 1/\sqrt{2} - 5\times10^{-6}$.

In order to verify this solution for the motion of the soliton we proceed in a manner similar to that following Eq. (3.26). We assume that

$$\psi(z,\tau) = 4\tan^{-1}(\exp\{\gamma[z-h(\tau)]\}) + \phi(z,\tau)\,, \tag{3.42}$$

where

$$h(\tau) \equiv a\cosh z_0(1-\tfrac{1}{6}\cosh^2 z_0)^{-1/2} \times \mathrm{sn}\left[b\left(\frac{|\alpha|}{4}\frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}\tau\right]. \tag{3.43}$$

Substitution of Eq. (3.42) into Eq. (3.27) and the use of the expansion (3.30) with $z' \equiv z - h(\tau)$ leads to the following equation for $\phi_b(\tau)$:

$$\frac{\partial^2\phi_b(\tau)}{\partial\tau^2} + \tfrac{1}{3}[h'(\tau)]^2\phi_b(\tau) = 2\alpha\{\mathrm{sech}[z_0+h(\tau)] - \mathrm{sech}[z_0-h(\tau)]\} + 8h''(\tau)\,, \tag{3.44}$$

where

$$h'(\tau) \equiv \frac{\partial h(\tau)}{\partial\tau} = \frac{ab}{8}\left(\frac{|\alpha|\sinh z_0}{1-\frac{1}{6}\cosh^2 z_0}\right)^{1/2}\mathrm{cn}\left[b\left(\frac{|\alpha|}{4}\frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}\tau\right]\mathrm{dn}\left[b\left(\frac{|\alpha|}{4}\frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}\tau\right] \tag{3.45}$$

and

$$h''(\tau) = -\frac{ab^2|\alpha|}{64}\frac{\tanh z_0}{(1-\frac{1}{6}\cosh^2 z_0)^{1/2}}\,\mathrm{sn}\left[b\left(\frac{|\alpha|}{4}\frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}\tau\right]\left\{1+\frac{a^2}{b^2}-\frac{2a^2}{b}\,\mathrm{sn}^2\left[b\left(\frac{|\alpha|}{4}\frac{\sinh z_0}{\cosh^2 z_0}\right)^{1/2}\tau\right]\right\}. \tag{3.46}$$

In Eqs. (3.45) and (3.46), sn, cn, and dn are Jacobi elliptic functions.[12] Since these functions are bounded in magnitude by $1/a$, we see that for $z_0 \ll 1$,

$$\phi_b(\tau) \propto \alpha z_0 \ll 1\,. \tag{3.47}$$

Once again the corrections to the assumed motion of the soliton arising from the translation mode amplitude are very small, thus verifying our use of Newton's law to determine the motion of the soliton to a good approximation.

For completeness we present the formal solution to Eq. (3.23) for the full form of the potential (3.18):

$$\int_0^{\xi(\tau)} d\xi'\left[\tan^{-1}\left(\frac{\sinh z_0(\cosh\xi_0-\cosh\xi')}{\sinh^2 z_0+\cosh\xi_0\cosh\xi'}\right)\right]^{-1/2} = |\alpha|^{1/2}\tau\,, \tag{3.48}$$

where ξ_0 is the amplitude of oscillation and $\xi(\tau=0)=0$. Equation (3.48) provides an implicit relation determining $\xi(\tau)$ and can be solved numerically for arbitrary amplitude ξ_0.

IV. MOTION OF A SOLITON IN THE PRESENCE OF A FORCING FUNCTION AND DAMPING

In this section we consider the motion of the soliton in the presence of a constant "force" and damping. This situation is somewhat different than the one treated in Sec. III in that the perturbations are independent of the position of the soliton and only affect its velocity and shape.

We consider the following form for the Hamiltonian:

$$\mathcal{H} = \int dx\,A\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{c_0^2}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 + \omega_0^2(1-\cos\psi) + \lambda E\psi\right], \tag{4.1}$$

where E is a constant which plays the role of a force or driving term. From Eq. (4.1) we derive an equation of motion for $\psi(x,t)$:

$$\frac{\partial^2\psi}{\partial t^2} - c_0^2\frac{\partial^2\psi}{\partial x^2} + \omega_0^2\sin\psi = \lambda E\,. \tag{4.2}$$

To this equation we now add a damping term, $\eta\partial\psi/\partial t$, so that

$$\frac{\partial^2\psi}{\partial\tau^2}-\frac{\partial^2\psi}{\partial z^2}+\sin\psi+\Gamma\frac{\partial\psi}{\partial\tau}=\chi\,, \tag{4.3}$$

where we have employed the dimensionless quantities defined by Eq. (3.5a) together with the definitions

$$\Gamma\equiv\eta/\omega_0,\quad \chi\equiv\lambda E/\omega_0^2\,. \tag{4.4}$$

Considering $\chi\ll 1$, we assume a solution to Eq. (4.3) of the form

$$\psi(z,\tau)=\psi^v(z-\beta\tau)+\phi(z,\tau)\,, \tag{4.5}$$

where ψ^v is a soliton solution to the unperturbed equation. We transform to a coordinate frame moving with velocity v and assume that ϕ is small. This leads to the following equation for ϕ:

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial z^2}+(1-2\,\mathrm{sech}^2 z)\phi+\gamma\Gamma\frac{\partial\phi}{\partial\tau}-\beta\gamma\Gamma\frac{\partial\phi}{\partial z}=\chi+2\beta\gamma\Gamma\,\mathrm{sech}\,z\,. \tag{4.6}$$

We now expand $\phi(z,\tau)$ in the complete set of functions (3.7) according to Eq. (3.30) and obtain the following equations for the amplitudes $\phi_b(\tau)$ and $\phi(k,\tau)$:

$$\frac{\partial^2\phi_b(\tau)}{\partial\tau^2}+\gamma\Gamma\frac{\partial\phi_b(\tau)}{\partial\tau}=8\beta\gamma\Gamma+2\pi\chi+i\left(\frac{\pi}{2}\right)^{1/2}\beta\gamma\Gamma\int_{-\infty}^{+\infty}dk\,\frac{k\phi(k,\tau)}{\overline{\omega}_k\sinh\frac{1}{2}\pi k} \tag{4.7a}$$

and

$$\frac{\partial^2\phi(k,\tau)}{\partial\tau^2}+\overline{\omega}_k^2\phi(k,\tau)+\gamma\Gamma\frac{\partial\phi(k,\tau)}{\partial\tau}=\frac{\chi}{(2\pi)^{1/2}\overline{\omega}_k}\left(k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right)+i\beta\gamma\Gamma\left[\frac{1}{8}\left(\frac{\pi}{2}\right)^{1/2}\frac{\overline{\omega}_k}{\cosh\frac{1}{2}\pi k}\phi_b(\tau)+k\phi(k,\tau)+\frac{1}{4\overline{\omega}_k}\int_{-\infty}^{+\infty}\frac{dk'(k'^2-k^2)\,\phi(k',\tau)}{\overline{\omega}_{k'}\sinh\frac{1}{2}\pi(k'-k)}\right]. \tag{4.7b}$$

In general, the coupled equations (4.7a) and (4.7b) are difficult to solve. However, we note that the coupling terms are proportional to $\beta\phi_b(\tau)$ or $\beta\phi(k,\tau)$ and for a small initial velocity of the soliton ($\beta\ll 1$) these terms are of second order in the small quantities β and ϕ. Thus, as a first approximation we neglect the coupling terms altogether and solve the uncoupled equations

$$\frac{\partial^2\phi_b^{(0)}(\tau)}{\partial\tau^2}+\gamma\Gamma\frac{\partial\phi_b^{(0)}(\tau)}{\partial\tau}=8\beta\gamma\Gamma+2\pi\chi \tag{4.8a}$$

and

$$\frac{\partial^2\phi^{(0)}(k,\tau)}{\partial\tau^2}+\overline{\omega}_k^2\phi^{(0)}(k,\tau)+\gamma\Gamma\frac{\partial\phi^{(0)}(k,\tau)}{\partial\tau}=\frac{\chi}{(2\pi)^{1/2}\overline{\omega}_k}\left(k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right). \tag{4.8b}$$

We have denoted these first approximations to $\phi_b(\tau)$ and $\phi(k,\tau)$ by the superscript (0). If we assume that $[\partial\phi_b^{(0)}(\tau)/\partial\tau]_{\tau=0}=0$, i.e., that the soliton is initially at rest in the frame traveling with velocity $\beta=v/c_0$ and that the perturbations are turned on at $\tau=0$, then Eq. (4.8a) is easily solved and we find

$$\phi_b^{(0)}(\tau)=\left(8\beta+\frac{2\pi\chi}{\gamma\Gamma}\right)\left(\tau+\frac{1}{\gamma\Gamma}(e^{-\gamma\Gamma\tau}-1)\right). \tag{4.9}$$

Since $\frac{1}{8}\phi_b^{(0)}(\tau)$ is the amplitude of the translational mode, we see that the soliton acquires a terminal velocity (in the $-z$ direction) equal to

$$\beta^T=\beta(1+2\pi\chi/8\gamma\beta\Gamma)\,. \tag{4.10}$$

If $\chi=0$, then $\beta^T=\beta$ implying that the soliton comes to rest in the lab frame. For large τ, this perturbation theory breaks down since $\phi_b(\tau)$ grows with τ. However, the conclusion that the soliton achieves a terminal velocity may still be valid. In order to verify this hypothesis we first need to obtain the continuum contribution to ϕ. From Eq. (4.8b) we note that for large Γ, the transient part of $\phi(k,\tau)$ decays rapidly leaving

$$\lim_{\tau\to\infty}\phi^{(0)}(k,\tau)\equiv\phi^{(0)}(k)=\chi\left(k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right)\times[(2\pi)^{1/2}\overline{\omega}_k^3]^{-1}\,. \tag{4.11}$$

The total continuum contribution is then

$$\phi_c^{(0)}(z)=P\int_{-\infty}^{+\infty}dk\,f_k(z)\,\phi^{(0)}(k)=-\frac{\chi}{2}P\int_{-\infty}^{+\infty}dk\,\frac{(k+i\tanh z)e^{ikz}}{\overline{\omega}_k^4\sinh\frac{1}{2}\pi k}\,. \tag{4.12}$$

The principle part is necessary as discussed at the end of Sec. II. The integral is easily evaluated using the method of contours and one finds

$$\phi_c^{(0)}(z)=\chi\left[\tanh|z|\left(1-\frac{\pi}{4}e^{-|z|}-\frac{\pi}{4}|z|\,e^{-|z|}+2\sum_{n=1}^{\infty}\frac{(-1)^n e^{-2n|z|}}{(2n+1)^2(2n-1)^2}\right)-\frac{\pi}{4}|z|\,e^{-|z|}+4\sum_{n=1}^{\infty}\frac{(-1)^n n e^{-2n|z|}}{(2n+1)^2(2n-1)^2}\right]. \tag{4.13}$$

We now consider Eq. (4.3) in a coordinate frame moving with velocity $\beta=-\pi\chi/4\Gamma$ (the terminal velocity in the lab frame, assuming $\gamma\cong 1$) and assume a solution of the form

$$\psi=\psi^0(z)+\phi_c^{(0)}(z)\,, \tag{4.14}$$

where $\psi^0(z)$ is a soliton at rest in this frame and $\phi_c^{(0)}(z)$ is given by Eq. (4.13). Then $\phi_c^{(0)}(z)$ must approximately satisfy the consistency equation

$$-\frac{\partial^2\phi_c^{(0)}(z)}{\partial z^2}+\left(1-\frac{2}{\cosh^2 z}\right)\phi_c^{(0)}(z)+\frac{\pi\gamma\chi}{4}\left(\frac{2}{\cosh z}+\frac{\partial\phi_c^{(0)}(z)}{\partial z}\right)=\chi\,. \tag{4.15}$$

For large values of $|z|$, we see from Eq. (4.13) that

$$\lim_{|z|\to\infty}\phi_c^{(0)}(z)=\chi\,. \tag{4.16}$$

Thus Eq. (4.15) is satisfied very well for large $|z|$. Indeed, after some tedious algebra we find that it is satisfied *exactly* (with $\gamma\cong 1$) to linear order in χ [i.e., neglecting the $\partial\phi_c^{(0)}(z)/\partial z$ term] for *all* z. This implies that the approximation (4.14) is very good[13] after the transient part of the continuum contribution has decayed to zero.

In Fig. 6, we have plotted the full solution in the terminal rest frame for $\chi=0.2$. Note the overall shift of the wings of the soliton by an amount χ [see Eq. (4.16)]. In addition there is a change in shape which is more evident in Fig. 7 where we plot $\partial\psi/\partial z$ in the terminal rest frame for three values of χ (0.0, 0.2, 0.4). Note that the shape becomes asymmetric about the soliton's center for nonzero χ. A measure of this asymmetry is given by the integral

$$\int_{-\infty}^{+\infty}dz\,\frac{\partial\phi_c^{(0)}(z)}{\partial z}z\cong 5.675\chi\,. \tag{4.17}$$

The coefficient of χ in Eq. (4.17) may be interpreted as a polarizability in the context of Ref. 7.

Before we conclude this section we note that the approximation (4.14) was obtained by neglecting the coupling terms in Eqs. (4.7a) and (4.7b). It is worth pointing out that Eq. (4.8a) remains unchanged even when $\phi^{(0)}(k)$ [see Eq. (4.11)] is substituted into Eq. (4.7a) to obtain the first correction to $\phi_b(\tau)$ due to coupling. This is a consequence of the fact that $\phi^{(0)}(k)$ is an odd function of k causing the integral in Eq. (4.7a) to vanish identically.

In this section we have shown that in the presence of a constant "force" and damping, the soliton achieves a terminal velocity determined by the ratio of the "force field" to the damping coefficient. This result was derived from Eq. (4.8a) which has the form of Newton's second law for the position of the soliton [position $\propto\phi_b(\tau)$] as a function of time.

After this work was essentially completed, we became aware of numerical studies by Nakajima *et al.*,[6] in which they find that the soliton achieves a terminal velocity in the presence of a driving term and damping. Our results agree with theirs for small values of χ ($\lesssim 0.3$) where our perturbation theory is valid.

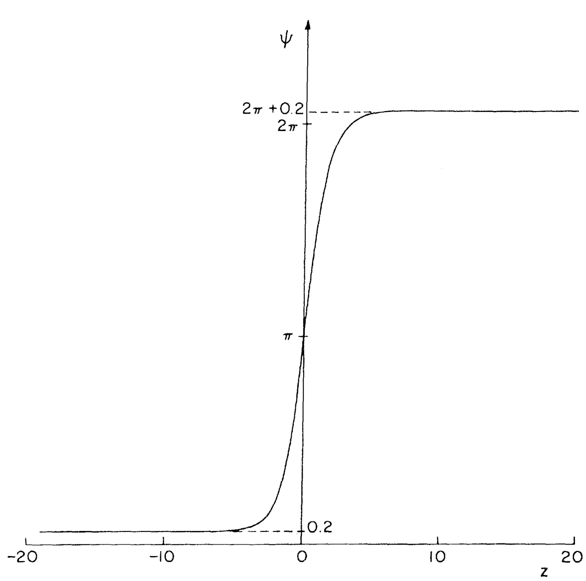

FIG. 6. Waveform of the soliton in its terminal rest frame when $\chi=0.2$. Except for the vertical shift the shape is very similar to that of the unperturbed soliton (see Fig. 1).

V. SPATIAL VARIATIONS OF THE CHARACTERISTIC FREQUENCY

In this section we investigate the effect of spatial variations in the characteristic frequency (ω_0) on the motion of solitons. For simplicity we consider the case where ω_0^2 changes abruptly to $\omega_0^2(1+\alpha)$ ($\alpha\ll 1$) at some point ($x=0$). The results are appropriate to physical situations of soliton transmission from one medium characterized by ω_0^2 to another medium characterized by $\omega_0'^2=\omega_0^2(1+\alpha)$. We find that for $\alpha>0$ the soliton can indeed pass into the second medium if it is moving fast enough in the first medium. The soliton slows down to a new velocity and its width changes to accomodate the new value of ω_0^2.

We formulate the problem in the following way. Consider the sine-Gordon equation with a variable characteristic frequency

$$\frac{\partial^2\psi}{\partial t^2}-c_0^2\frac{\partial^2\psi}{\partial x^2}+\omega_0^2(x)\sin\psi=0\,, \tag{5.1}$$

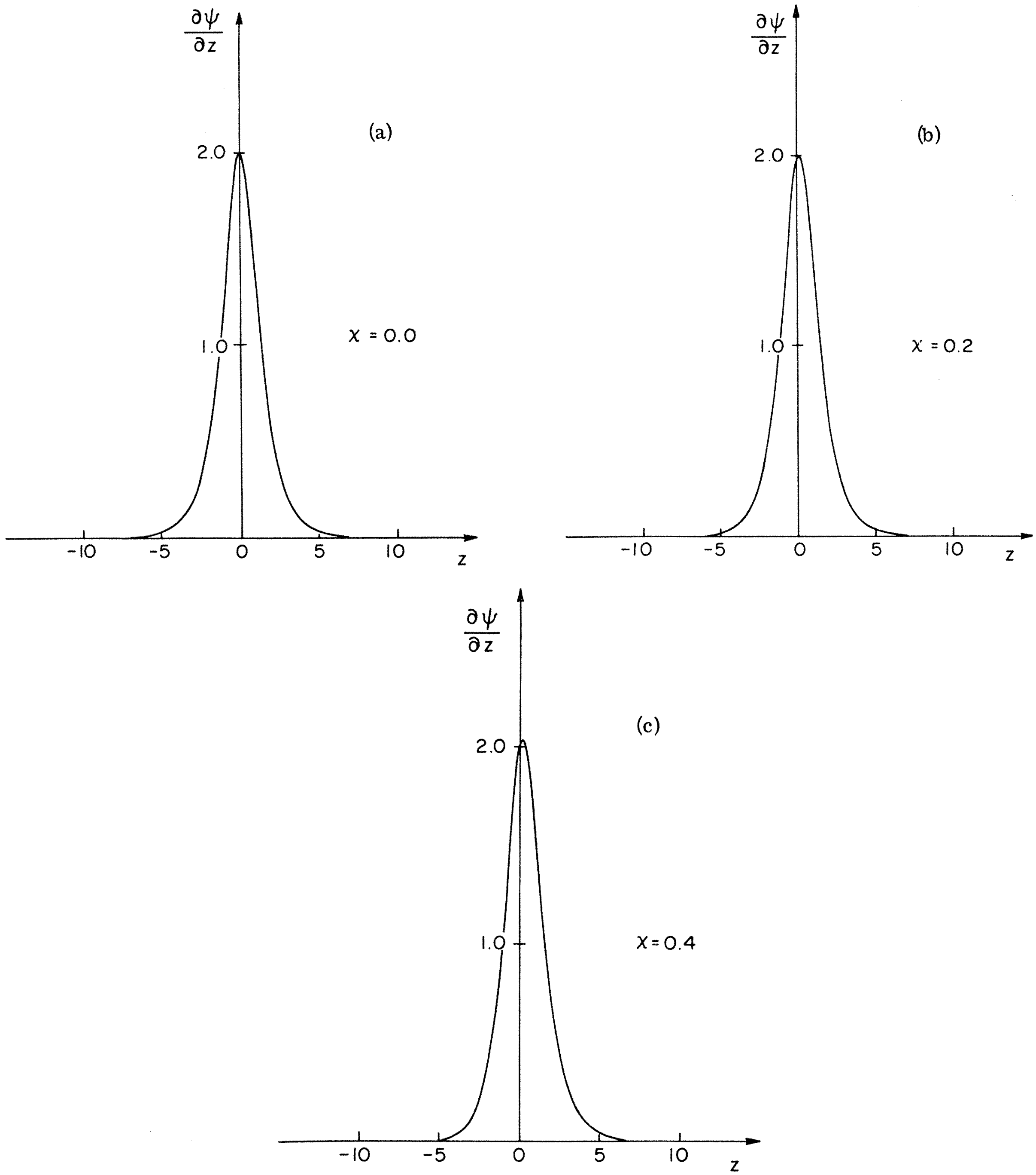

FIG. 7. Derivative of the soliton waveform in its rest frame for (a) $\chi=0.0$, (b) $\chi=0.2$, and (c) $\chi=0.4$. Note the growth of the asymmetry with respect to $z=0$ as χ is increased.

where $\omega_0^2(x)$ has the form

$$\omega_0^2(x)=\omega_0^2+\Delta\omega_0^2(x)\,, \tag{5.2}$$

with $\Delta\omega_0^2(x)\ll\omega_0^2$ for all x. For the moment, the form of the perturbation $\Delta\omega_0^2(x)$ will be left arbitrary. Using the definitions (3.5a), we recast Eq. (5.1) in dimensionless form

$$\frac{\partial^2\psi}{\partial\tau^2}-\frac{\partial^2\psi}{\partial z^2}+[1+g(z)]\sin\psi=0\,, \tag{5.3}$$

where

$$g(z) \equiv \Delta\omega_0^2(z)/\omega_0^2 \ll 1 . \quad (5.4)$$

We wish to determine the effect of a nonzero $g(x)$ on the motion of a soliton with initial velocity v. Thus, we assume a solution to (5.3) of the form

$$\psi(z,\tau) = \psi^v(z,\tau) + \phi(z,\tau) , \quad (5.5)$$

where $\phi(z,\tau)$ is assumed to be small. By substituting (5.5) into (5.3) and linearizing in $\phi(z,\tau)$, we find the following equation for $\phi(z,\tau)$:

$$\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial z^2} + [1 + g(\gamma(z+\beta\tau))]\left(1 - \frac{2}{\cosh^2 z}\right)\phi = 2g(\gamma(z+\beta\tau))\frac{\tanh z}{\cosh z} , \quad (5.6)$$

where as usual we have transformed to a reference frame moving with velocity v [$\beta \equiv v/c_0$; $\gamma \equiv (1-\beta^2)^{-1/2}$]. We note that the terms involving $g\,\phi$ on the left-hand side of Eq. (5.6) are of second order in small quantities. Thus, we neglect them in a first approximation and replace (5.6) by

$$\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial z^2} + \left(1 - \frac{2}{\cosh^2 z}\right)\phi = 2g(\gamma(z+\beta\tau))\frac{\tanh z}{\cosh z} . \quad (5.7)$$

We now expand $\phi(z,\tau)$ in the complete set of states (3.7):

$$\phi(z,\tau) = \tfrac{1}{8}\phi_b(\tau)f_b(z) + \int_{-\infty}^{+\infty} dk\, \phi(k,\tau)f_k(z) . \quad (5.8)$$

Substitution of (5.8) into (5.7) and subsequent projection leads to the following equations for the amplitudes $\phi_b(\tau)$ and $\phi(k,\tau)$:

$$\ddot{\phi}_b(\tau) = 4\int_{-\infty}^{+\infty} dz\, g(\gamma(z+\beta\tau))\frac{\sinh z}{\cosh^3 z} \quad (5.9a)$$

and

$$\ddot{\phi}(k,\tau) + (k^2+1)\,\phi(k,\tau) = 2\int_{-\infty}^{+\infty} dz\, g(\gamma(z+\beta\tau))f_k^*(z)\frac{\sinh z}{\cosh^2 z} , \quad (5.9b)$$

where the dot denotes a derivative with respect to τ. Equations (5.9) are linear inhomogeneous second-order differential equations for the amplitudes $\phi_b(\tau)$ and $\phi(k,\tau)$. These can be solved (at least in principle) and the solutions can be substituted into Eq. (5.8) to yield $\phi(z,\tau)$ for an arbitrary (but small) perturbation g.

As a simple example we treat the case described at the beginning of this section, namely,

$$g(\gamma(z+\beta\tau)) = \alpha\Theta(\gamma(z+\beta\tau)) , \quad (5.10)$$

where the step function is defined by Eq. (3.3). Substitution of (5.10) into (5.9a) yields

$$\ddot{\phi}_b(\tau) = 2\alpha/\cosh^2\beta\tau . \quad (5.11)$$

As in Secs. III and IV, the equation governing $\phi_b(\tau)$ has the form of Newton's law. Equation (5.11) is readily integrated to yield

$$\dot{\phi}_b(\tau) = (2\alpha/\beta)(1 + \tanh\beta\tau) , \quad (5.12)$$

where we have used the initial condition that the soliton be at rest for $\tau \to -\infty$ (far from the jump in ω_0^2) in the frame moving with v. We see that in this frame the soliton achieves a *negative* velocity

$$\beta' = -\alpha/2\beta , \quad (5.13)$$

or in the lab frame

$$\beta'_{\mathrm{lab}} \cong \beta - \tfrac{1}{2}\alpha(1/\beta - \beta) . \quad (5.14)$$

Since $1/\beta > \beta$, we see that the soliton slows down as it enters the medium with larger ω_0^2. Physically, the soliton converts some of its kinetic energy into extra "potential energy" (rest energy) associated with a contracted width in the new medium. We note that Eq. (5.14) is consistent with what one expects on the basis of energy conservation

$$Mc_0^2/(1-\beta^2)^{1/2} = M'c_0^2/(1-\beta'^2_{\mathrm{lab}})^{1/2} , \quad (5.15)$$

where

$$M'/M = (1+\alpha)^{1/2} . \quad (5.16)$$

This leads to

$$\beta'^2_{\mathrm{lab}} = \beta^2(1 - \alpha/\beta^2 + \alpha) , \quad (5.17)$$

which agrees with (5.14) when α is small.

We now examine the continuum contributions to $\phi(z,\tau)$. Substitution of (5.10) into (5.9b) leads to the following result for $\phi(k,\tau)$:

$$\phi(k,\tau) = -\frac{\alpha i}{2}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\bar{\omega}_k}\left[\frac{-2}{\cosh\frac{1}{2}\pi k} + \left(1 - \frac{1}{\beta^2}\right)\left(\frac{e^{i\bar{\omega}_k\tau}}{\cosh\frac{1}{2}\pi\beta(\beta k + \bar{\omega}_k)} + \frac{e^{-i\bar{\omega}_k\tau}}{\cosh\frac{1}{2}\pi\beta(\beta k - \bar{\omega}_k)}\right) + e^{-i\beta k\tau}\frac{2}{\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n e^{-(2n+1)\beta\tau}\{\bar{\omega}_k^2 - [k - i(2n+1)]^2\}}{[(2n+1)i - k]\{\beta^2[k - i(2n+1)]^2 - \bar{\omega}_k^2\}}\right] \quad (\tau > 0) \quad (5.18a)$$

and

$$\phi(k,\tau) = -\frac{\alpha}{(2\pi)^{1/2}}\frac{1}{\bar{\omega}_k}e^{-i\beta k\tau}\sum_{n=1}^{\infty}\frac{(-1)^n e^{(2n-1)\beta\tau}\{\bar{\omega}_k^2 - [k + i(2n-1)]^2\}}{[k + (2n-1)i]\{\beta^2[k + i(2n-1)]^2 - \bar{\omega}_k^2\}} \quad (\tau < 0) . \quad (5.18b)$$

It can be shown upon substitution of (5.18) into (5.8) that the time-dependent continuum contributions occur only near $\tau = 0$, i.e., when the soliton is near the jump in ω_0^2. There is a *time-independent* contribution as well for $\tau > 0$:

$$\phi_0(z) = \alpha z/\cosh z \quad (\tau > 0)\,. \tag{5.19}$$

This contribution is localized about the soliton center ($z = 0$) *after* the soliton has passed into the new medium ($\tau > 0$). To see what this contribution corresponds to physically we note that in the new medium $[\omega_0'^2 = (1+\alpha)\,\omega_0^2]$ a soliton at rest would have the waveform

$$\psi_\alpha(z) = 4\tan^{-1}\{\exp[(1+\alpha)^{1/2}z]\}\,. \tag{5.20}$$

The difference between this and a soliton with $\alpha = 0$ is

$$\psi_\alpha(z) - \psi_{\alpha=0}(z) = \alpha z/\cosh z + O(\alpha^2)\,. \tag{5.21}$$

Thus, the time-independent continuum contribution (5.19) yields, to first order in α, *precisely* the correction needed to give the correct shape to the soliton in the new medium. It is worth emphasizing that unlike the impurity situation (Sec. III), there are no residual perturbations in the transition region when the soliton is far away. This can be easily understood if one notes that if the soliton is far from the transition region the system has nothing to gain energetically by allowing ψ to vary where ω_0^2 is changing since the local potential energy density $[\propto(1-\cos\psi)]$ is already a minimum for $\psi = 0$ (mod 2π).

It is a straightforward matter to determine the effect of a localized defect of width $2a$ in ω_0^2 of the form

$$g(\gamma(z+\beta\tau)) = \alpha\,[\Theta(\gamma(z+\beta\tau)+a) - \Theta(\gamma(z+\beta\tau)-a)] \tag{5.22}$$

by a simple superposition of results for the case (5.10) with shifted arguments of g. The principle result is that the soliton acquires a net phase shift $\delta = \alpha a/2\beta^2\gamma^2$. If the soliton is moving too slowly, however, it can become trapped or repelled depending on the sign of α. In this context it should be noted that for the single-step perturbation (5.10), Eq. (5.17) determines a critical velocity for the soliton to overcome the barrier, i.e.,

$$\beta_c = \alpha/(1+\alpha)\,. \tag{5.23}$$

If $\beta < \beta_c$, the soliton will eventually stop and turn around to travel in the negative direction.

VI. SUMMARY, REMARKS, AND CONCLUSIONS

In this paper we have described a simple method for determining the effect of weak external perturbations on the motion of solitary-wave solutions of nonlinear wave equations. The method is based on a linear perturbation theory which makes use of the fact that small changes in the waveform may be expanded in terms of a complete set of eigenfunctions of a self-adjoint differential operator [e.g., Eq. (2.13)]. Of particular importance is the existence of a "translation mode" solution in the eigenspectrum of this operator, since it is intimately connected with the motion of the solitary wave in the presence of perturbations via Newton's law [e.g., Eqs. (3.12a), (3.23), (4.8a), and (5.9a)].

Our results for the specific example of sine-Gordon solitons can be summarized as follows. In Sec. III we found that low-velocity solitons can be trapped (repelled) by an attractive (repulsive) model impurity potential while high-velocity solitons pass through the impurity region suffering only a phase shift. In the trapped configuration, the soliton executes oscillatory motion determined by Newton's law. In Sec. IV, we found that in the presence of damping and a constant "force" the soliton achieves a terminal velocity, in agreement with the numerical results of Ref. 6. Finally, in Sec. V we found that spatial variations in the characteristic frequency (ω_0) cause the soliton to adjust its velocity and shape accordingly. In all these cases, the soliton maintains its integrity and behaves as a classical "extended particle" whose "center-of-mass" motion obeys Newton's law.

We are currently conducting numerical simulation studies of the situations described in Secs. III and V in order to test the ideas and results put forth in this paper. The results of these numerical studies verify the basic conclusions obtained here and will be presented in detail in a future publication.[14]

We now comment on the range of applicability of our method, apart from the general applicability to single solitary-wave solutions. In most systems of interest, more than one solitary wave will be present. In these cases our method will work only when the solitary waves do not overlap significantly (i.e., for dilute densities of solitary waves). Fortunately, at low temperatures compared to the rest energy of the solitary wave, the density will be low. We remark here that our method will not be useful for examining perturbations of the so-called "doublet" or "breather" solution to the sine-Gordon equation,[2] since the potential entering Eq. (2.9) would be time-dependent. However, for soliton-bearing equations, it is possible to examine the effect of perturbations using the techniques of inverse scattering theory,[15] which are not limited to single-soliton solutions.

The procedure to be used in applying our method to equations other than sine-Gordon equations is to find the equation similar to Eq. (2.9) and solve for its eigenvalues and associated eigenfunctions. The $\omega^2=0$ solution will be the translation mode for the solitary wave. There may be additional "bound states" (e.g., in the ϕ^4 problem[2]) which correspond to internal degrees of freedom of the solitary wave. All of the bound states, together with the continuum solutions, form a complete set of functions which may be used to expand perturbations about the solitary wave in a manner similar to that discussed in this paper.

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