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Analysis of hyperscaling in the Ising model by the high-temperature series method*†

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High-temperature series expansions are derived for $(\partial^2\chi/\partial H^2)$ for arbitrary dimension. These series and others already available are analyzed to determine if hyperscaling holds. Hyperscaling is found to hold for $d = 2$ but fails in three and four dimensions as $2\Delta - d\nu - \gamma = -0.028 \pm 0.003$ ($d = 3$), -0.302 ± 0.038 ($d = 4$). The triviality of hyper-strong-coupling, Euclidean, boson, $:\phi^4:_{3,4}$ field theory follows. The expected location of the zero of the β function in the Callan-Symanzik equation is computed in two and three dimensions.

I. INTRODUCTION AND SUMMARY

The modern renormalization-group theory of critical phenomena,[1,2] as currently practiced, implicitly assumes that hyperscaling holds. At the critical point there is singular behavior in various thermodynamic and statistical-mechanical properties. Each such singularity is characterized, near the critical point, by a critical index which specifies how it diverges $[(1 - T_c/T)^{-\phi}]$ or vanishes, as the case may be. These indices are not all independent but are related by various equations.[3] As has been recognized for a long time, these relations divide into families. The hyperscaling group of relations which we investigate here are those that depend explicitly on the spatial dimension. Some representative relations are

$$d\nu = 2 - \alpha, \quad 2 - \eta = d(\delta - 1)/(\delta + 1), \tag{1.1}$$

$$\Delta = \tfrac{1}{2}(d\nu + \gamma). \tag{1.2}$$

To give specific meaning to these possible relations we consider the usual spin-$\frac{1}{2}$ Ising model defined by the partition function

$$Z = \sum_{\{\sigma_{\vec{i}} = \pm 1\}} \exp\left(K \sum_{\substack{\text{nearest}\\ \text{neighbors}}} \sigma_{\vec{i}}\sigma_{\vec{j}} + H \sum_{\vec{i}} \sigma_{\vec{i}}\right), \tag{1.3}$$

where the sum in the exponential is over all the nearest-neighbor pairs $(\vec{i}, \vec{j})$ of a regular space lattice in d dimensions of N sites. This model has a critical point at finite temperature for $d > 1$ and a zero-temperature critical point in one dimension. At the critical point the correlation length, susceptibility, etc., diverge. By letting $\tau = 1 - T_c/T$ we have, just above the critical temperature, the behavior

$$\begin{aligned} \chi &= N^{-1}\frac{\partial^2 \ln Z}{\partial H^2} \approx A_+\tau^{-\gamma}, \\ \frac{\partial^2\chi}{\partial H^2} &\approx -B_+\tau^{-\gamma-2\Delta}, \\ M_2 &= \sum_{\vec{r}} r^2\langle\sigma_0\sigma_{\vec{r}}\rangle, \\ \xi^2 &= M_2/2d\chi \approx D_+^2\tau^{-2\nu}, \\ C_H &= N^{-1}\frac{\partial^2 \ln Z}{\partial K^2} \propto \tau^{-\alpha}, \end{aligned} \tag{1.4}$$

and exactly at $\tau = 0$,

$$M = N^{-1}\frac{\partial \ln Z}{\partial H} \propto H^{1/\delta}, \quad \langle\sigma_0\sigma_{\vec{r}}\rangle \propto 1/r^{d-2+\eta}. \tag{1.5}$$

In Eqs. (1.4) and (1.5) above, attention is directed to the critical point, and the over-all normalization of the quantities given is not always quite the standard one. However, conversion to the usual normalization is easily derived. We denote the magnetic susceptibility by χ, the spin-spin correlation function by $\langle\sigma_{\vec{a}}\sigma_{\vec{b}}\rangle$, and the second spherical moment of the correlation function by M_2. We define the correlation length in terms of this moment instead of using the true correlation length,[3] and we denote the specific heat at constant magnetic field by C_H, and the magnetization by M.

Of all the hyperscaling relations, the one best determined by methods independent of the assumptions of the renormalization group is (1.2). The best previous determinations of the critical exponents involved for the three-dimensional case are by high-temperature series methods,[5-7]

$$\begin{aligned} \Delta &= 1.563 \pm 0.003, \\ \gamma &= 1.250 \pm 0.003, \\ \nu &= 0.638^{+0.002}_{-0.001}. \end{aligned} \tag{1.6}$$

These estimates lead to

$$2\Delta - d\nu - \gamma = -0.038 \pm 0.012, \tag{1.7}$$

which suggests a possible failure of hyperscaling in three dimensions. Equation (1.7) is in accord with the general result of Baker and Krinsky[8] and Schrader,[9] who have shown that

$$d\nu + \gamma \geq 2\Delta \,. \tag{1.8}$$

In two dimensions, relation (1.2) has been proven by Kadanoff,[10] and is not in doubt.

The consequences of such a possible failure of hyperscaling are significant. For example, the, by now very large body of results derived by renormalization-group theory for critical phenomena and related areas would have to be reexamined. A clear consequence would be that those results are at best approximations to Ising theory. The possibility would, however, remain that they are correct for some physical phenomena but not others. Also, boson quantum-field theory[11] for hyperstrong coupling $\lambda : \phi^4 :_3$ theory would be necessarily trivial, i.e., free of any real scattering. By hyperstrong coupling, one means that the coupling constant goes to infinity before the ultraviolet cutoff does.

We can check on the possible failure of hyperscaling, and incidently compute a number of interest to field theory and renormalization-group theory by considering the dimensionless renormalized coupling constant

$$w = g m^{d-4} \propto -v \frac{\partial^2 \chi}{\partial H^2} \Big/ \chi^2 \xi^d$$
$$\approx [v B_+ / A_+^2 (a D_+)^d] \tau^{d\nu + \gamma - 2\Delta}, \tag{1.9}$$

which is finite at $\tau = 0$ if hyperscaling holds, and vanishes if it does not. The quantity g is the usual field-theory renormalized coupling constant, and m is the field-theory mass. We use v to denote the specific hypervolume per site, and a is the lattice constant.

In Sec. II we derive the high-temperature series for $\partial^2\chi/\partial H^2$ through order K^9 in arbitrary dimension. We use the method of Rushbrooke and Scoins.[12] This series was the only one not previously available in the literature.

In Sec. III we analyze by Padé-approximant methods the necessary series to evaluate (1.9). We find agreement with the exact results in two dimensions. In three dimensions we find $2\Delta - d\nu - \gamma = -0.028 \pm 0.003$ for our best (fcc) lattice, and a similar result but with greater uncertainty on the other three-dimensional lattices. We conclude that hyperscaling fails in three dimensions. In four dimensions, hyperscaling fails as well for $2\Delta - d\nu - \gamma = -0.302 \pm 0.038$. We remark that if we force hyperscaling in three dimensions, in the sense that we assume that ν is determined by (1.2), then the coefficient in (1.9) of τ^0 can be related to the zero of the β function in the Callan-Symanzik-equation approach of Baker *et al.*,[13] and agreement within apparent error is found with their three-dimensional results.

II. HIGH-TEMPERATURE SERIES

With one exception the series necessary to analyze the hyperscaling relation are available in the literature. In Table I we list the number of terms available, and their source. We list the number of terms available in the series for the susceptibility χ, its second partial with respect to magnetic field at zero-magnetic field $\partial^2\chi/\partial H^2$, the second spherical moment M_2, and the true correlation length (squared) ξ_T^2. We treat the lattices: square (sq), triangular (t), diamond (d), simple cubic (sc), body-centered cubic (bcc), face-centered cubic (fcc), and the hyper-simple-cubic, $d = 4$ (hsc). It will be observed from the table that there is one essential gap in the available data, namely, $\partial^2\chi/\partial H^2$ on the four-dimensional hypercubic lattice.

We have computed this series on hypercubic lattices for general dimension by the method of Rushbrooke and Scoins.[12] The usual Ising-model partition function is given by Eq. (1.3). It is convenient for our calculations to shift the zero of energy to the state where all the spins are aligned, $\uparrow\uparrow\uparrow\uparrow\cdots$. Then if Z is the partition function for that zero of energy,

$$(1/N)\ln Z_{\text{Ising}} = \tfrac{1}{2} q K + H + (1/N)\ln Z \,, \tag{2.1}$$

where q is the lattice coordination number. Since we only want finally to derive the expansions for $\partial^2/\partial H^2$ and $\partial^4/\partial H^4$ (that is, χ and $\partial^2\chi/\partial H^2$), the presence of the K and H terms are unimportant. Thus we may use the spin-aligned version.

With the notation

$$\alpha = e^{-2H}, \quad \eta = e^{-2K}, \quad f = \eta^{-2} - 1 \,, \tag{2.2}$$

we are in a position to apply the Ursell-Mayer

TABLE I. High-temperature series terms available.

Lattice	sq	t	d	sc	bcc	fcc	hsc
χ	21[a]	16[a]	22[a]	17[a]	15[a]	12[a]	11[b,c]
$\frac{\partial^2\chi}{\partial H^2}$	15[d]	10[d]	16[d]	12[d]	12[d]	8[d]	···
M_2	11[e]	8[e]	···	12[f]	12[f]	12[f]	11[c]
ξ_T^2	∞[e]	∞[e]	···	9[e]	···	···	7[g]

[a]Reference 14.
[b]Reference 15.
[c]Reference 16.
[d]Reference 5.
[e]Reference 4.
[f]Reference 7.
[g]Reference 17.

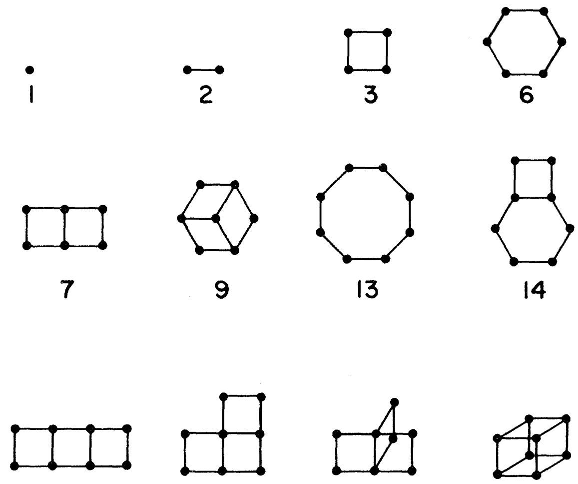

FIG. 1. Fundamental maps of Rushbrooke and Scoins which enter into construction of the series on the hyper-simple-cubic lattices through ninth order.

formalism to expand

$$\ln\Lambda(\alpha,\eta)=(1/N)\ln Z=x\left(1-\sum_k \frac{k}{k+1}\beta_k x^k\right) \tag{2.3}$$

in powers of the density of overturned spins, x.

This expansion is given in terms of the number of embeddings of multiply connected figures such that nearest neighbors occur if and only if there is a line in the embedded figure (strong system). On the hypercubic lattice one can construct an argument to show that the smallest new star graph which is strongly embedded on the d-dimensions lattice but not on the $(d-1)$-dimensional lattice has $\frac{1}{2}d(d+1)+1$ points, or 11 points for $d=4$. Since Rushbrooke and Scoins give a complete list for $d=3$–8 points, their list is also complete for all dimensions through 8 points. We show in Fig. 1 the star graphs, or maps as they call them, which have nonzero occurrence on the general hypercubic lattices, together with the Rushbrooke and Scoins map numbers. We give the number of lattice embeddings (strong system) of the maps shown in Fig. 1 for the d-dimensional hypercubic lattice:

TABLE II. Rushbrooke and Scoins N.

Map	Value
3	−38 088
6	−4896
7	−21 276
9	6093
13	−144
14	−252
15	−360
16	−360
17	−108
30	−4320

$$\begin{gathered}[1]=1,\quad [2]=d,\quad [3]=\binom{d}{2},\quad [6]=4\binom{d}{3},\quad [7]=2\binom{d}{2}+12\binom{d}{3},\quad [9]=8\binom{d}{3},\\ [13]=\binom{d}{2}+24\binom{d}{3}+168\binom{d}{4},\quad [14]=48\binom{d}{3}+192\binom{d}{4},\\ [15]=2\binom{d}{2}+36\binom{d}{3}+96\binom{d}{4},\quad [16]=4\binom{d}{2}+72\binom{d}{3}+192\binom{d}{4},\\ [17]=12\binom{d}{3}+32\binom{d}{4},\quad [30]=\binom{d}{3},\end{gathered} \tag{2.4}$$

where $\binom{d}{j}$ is the usual binomial coefficient. These embeddings were counted, where necessary, in the weak system, or taken, if available, from Fisher and Gaunt.[14] They were then converted to the strong system by standard methods[18] using the known T-matrix elements,[19] where available.

The structure of the coefficients β_k in the expansion (2.3) is

$$\beta_k=\sum_t [t]F_{t,k}(f), \tag{2.5}$$

where t ranges over the elementary maps, $[t]$ is the strong lattice-constant as in (2.4), and $F_{t,k}(t)$ is an intrinsic function associated with map t and k overturned spins. Rushbrooke and Scoins list all the information necessary to compute the required $F_{t,k}(f)$, except for the coefficient of f^9 with 8 overturned for the maps 13–30 where only a sum is given. We list in Table II the individual values for the maps which occur in this work. This coefficient is called parameter N by Rushbrooke and Scoins. We next give the value of all the Rushbrooke and Scoins parameters for the general hypercubic lattice:

$$A=4\binom{d}{2},\quad B=-40\binom{d}{2},\quad C=0,\quad D=276\binom{d}{2}+96\binom{d}{3},$$
$$E=12\binom{d}{2}+72\binom{d}{3},\quad F=-1764\binom{d}{2}-2184\binom{d}{3},\quad G=0,$$
$$H=-238\binom{d}{2}-1176\binom{d}{3},\quad K=11\,748\binom{d}{2}+27\,696\binom{d}{3}+5184\binom{d}{4}, \tag{2.6}$$
$$L=56\binom{d}{3},\quad M=2832\binom{d}{2}+13\,120\binom{d}{3}+6400\binom{d}{4},\quad N=-82\,944\binom{d}{2}-286\,200\binom{d}{3}-179\,712\binom{d}{4}.$$

From these parameters, it is a simple matter to write out the logarithm of the partition function per spin as

$$\ln\Lambda(\alpha,\eta)=-\ln(1-x)-2d\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)!}{n!\,(n-1)!}f^n x^{n+1}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(-x)^r}{n+r+1}$$
$$-\sum_{n,\,s}^{n=9}d_{n,s}f^n y^{n-s-1}x[n-s-1-(2n-2s-1)x], \tag{2.7}$$

where we have used

$$y=x(1-x), \tag{2.8}$$

or in terms of the magnetization M

$$x=\tfrac{1}{2}(1-M),\quad y=\tfrac{1}{4}(1-M^2). \tag{2.9}$$

In terms of parameters of (2.6) we can express the nonzero $d_{n,s}$ as

$$d_{4,0}=\binom{d}{2},\quad d_{5,0}=-8\binom{d}{2},\quad d_{6,0}=46\binom{d}{2}+16\binom{d}{3},\quad d_{7,1}=2\binom{d}{2}+12\binom{d}{3},$$
$$d_{7,0}=-240\binom{d}{2}-240\binom{d}{3},\quad d_{8,1}=-34\binom{d}{2}-168\binom{d}{3},\quad d_{8,0}=1230\tfrac{1}{2}\binom{d}{2}+2286\binom{d}{3}+648\binom{d}{4},\quad d_{9,2}=8\binom{d}{3}, \tag{2.10}$$
$$d_{9,1}=354\binom{d}{2}+1696\binom{d}{3}+800\binom{d}{4},\quad d_{9,0}=-6384\binom{d}{2}-18\,400\binom{d}{3}-13\,568\binom{d}{4}.$$

Thus we have the expression (2.7) for the logarithm of the partition function per spin valid through order f^9 for all values of the magnetization $-1\leq M\leq 1$.

At $M=0$ we have the formulas

$$\frac{\partial^2\ln\Lambda}{\partial x^2}=\frac{4}{\chi},\quad \frac{\partial^4\ln\Lambda}{\partial x^4}=\frac{-48\,\partial^2\chi/\partial H^2}{\chi^4}. \tag{2.11}$$

We find by direct differentiation of (2.7) that

$$\frac{\partial^2\ln\Lambda}{\partial x^2}=4\Bigg\{1-2du+2du^2-2du^3+\left[2d+8\binom{d}{2}\right]u^4-\left[2d+16\binom{d}{2}\right]u^5+\left[2d+40\binom{d}{2}+192\binom{d}{3}\right]u^6$$
$$-\left[2d+80\binom{d}{2}+480\binom{d}{3}\right]u^7+\left[2d+184\binom{d}{2}+3168\binom{d}{3}+10\,368\binom{d}{4}\right]u^8$$
$$-\left[2d+368\binom{d}{2}+8608\binom{d}{3}+27\,136\binom{d}{4}\right]u^9+\cdots\Bigg\}, \tag{2.12}$$

where for ease of comparison we have made the change of variables

$$u=\tanh K,\quad f=4u/(1-u)^2. \tag{2.13}$$

The results (2.12) have been checked directly for $d=2,3$ against those of Rushbrooke and Scoins,[12] and by computing $\chi^{-1}\chi=1$ against the results of Fisher and Gaunt[15] for $d=4$. By further differentiation we obtain

$$\left.\frac{\partial^4 \ln\Lambda}{\partial x^4}\right|_{x=1/2}=96\left\{1-6du^2+16du^3-\left[30d+72\binom{d}{2}\right]u^4+\left[48d+384\binom{d}{2}\right]u^5-\left[70d+1272\binom{d}{2}+2880\binom{d}{3}\right]u^6+\left[96d+3840\binom{d}{2}+17\,280\binom{d}{3}\right]u^7\right.$$
$$-\left[126d+11\,016\binom{d}{2}+97\,632\binom{d}{3}+217\,728\binom{d}{4}\right]u^8$$
$$\left.+\left[160d+29\,952\binom{d}{2}+451\,968\binom{d}{3}+1\,302\,528\binom{d}{4}\right]u^9+\cdots\right\}. \tag{2.14}$$

For $d=2$ and $d=3$, this result was checked against those of Essam and Hunter.[5] The series for $d=4$ is

$$\left.\frac{\partial^4 \ln\Lambda}{\partial x^4}\right|_{x=1/2}=96(1-24u^2+64u^3-552u^4+2496u^5-19\,432u^6+92\,544u^7-674\,856u^8+3\,290\,752u^9+\cdots)\quad (d=4). \tag{2.15}$$

It is worth noting that the derivatives $\partial^n \ln\Lambda/(\partial x)^n$ are all linear in the lattice constants which eased the derivation of the general expressions (2.12) and (2.14).

Computing directly from (2.11), (2.12), and (2.15) we have

$$\frac{\partial^2\chi}{\partial H^2}=-2(1+32u+584u^2+8288u^3+101\,240u^4+1\,121\,120u^5+11\,570\,360u^6+113\,293\,088u^7+1\,064\,631\,032u^8+9\,681\,082\,144u^9+\cdots)\quad (d=4). \tag{2.16}$$

which formula completes the derivation of the high-temperature series data necessary for our investigation.

III. SERIES ANALYSIS

We will employ a number of standard methods[20,21] of series analysis to study the series discussed in Sec. II. The general methods employed are those of Padé approximation.[22] We mention briefly a couple of the procedures. Since the Padé approximant is a rational fraction, it is advantageous to manipulate the function to be analyzed so that it is approximately of that form in the region of interest. One useful procedure for functions with a divergent-branch-point singularity is to form the logarithmic derivative. Thus if

$$f(x)=\sum_j f_j x^j \approx A(1-yx)^{-\psi}+B, \tag{3.1}$$

then

$$\frac{d\ln f(x)}{dx}=\frac{f'(x)}{f(x)}\approx\frac{-\psi}{x-y^{-1}} \tag{3.2}$$

near the branch point. The pole is at the branch point of $f(x)$ and the residue gives the exponent. The procedure is to form Padé approximants to the logarithmic derivative.

If, in addition to $f(x)$, we have another function

$$g(x)=\sum_j g_j x^j \approx C(1-yx)^{-\phi}+D, \tag{3.3}$$

which is singular at the same point, y^{-1}, and this singularity is the closest one to the origin, then we may use "critical-point renormalization"[7,22] to estimate the difference of the exponents for

$$h(x)=\sum\frac{f_j x^j}{g_j}\propto(1-x)^{-(\psi-\phi+1)} \tag{3.4}$$

for x near 1, and

$$\left.(1-x)\frac{d}{dx}\ln h(x)\right|_{x=1}=\phi-\psi-1. \tag{3.5}$$

The advantage here is that an estimate of the location of the singular point is not required. The procedure is to form Padé approximants to the function in (3.5).

In the analysis of series by these methods there are certain difficulties which can arise. For physical functions, these concern mainly interference by other singularities. The most severe type of interference comes from confluent singularities. By a confluent singularity one means

$$k(x)\approx(1-yx)^{-\phi}[1+A(1-yx)^{\tau}+o((1-yx)^{\tau})], \tag{3.6}$$

where τ is not an integer. Practically, one means $\tau<1$, as analytic corrections corresponding to $\tau=1$ almost always occur. The occurrence of other singularities of roughly the same strength and distance from the origin as the one of interest interfere with its accurate analysis.

Camp *et al.*[23] have found, both for the susceptibility χ and the second moment series M_2 that such confluent singularities are absent in the spin-$\frac{1}{2}$ Ising model. They also found that such singularities do occur for $s>\frac{1}{2}$. This result is one reason for selecting the $s=\frac{1}{2}$ series to study. In addition we need to examine the structure of $\partial^2\chi/\partial H^2$. A Padé analysis of $d\ln(\partial^2\chi/\partial H^2)/dK$ shows clearly a pole at K_c of residue close to the accepted values of $2\Delta+\gamma$ for all lattices. For the loose-packed lattices (sq, sc, bcc, and hsc) there is also a pole on the negative K axis considerably closer to the origin than critical-point singularity. This pole corresponds to a regular zero of $\partial^2\chi/\partial H^2$ at that point and is absent on the close-packed lattices. The only additional structure clearly visible is generally in the first and fourth quadrants in the imaginary direction and is at a greater distance from the origin than the critical point. Since the critical-point pole is isolated, we interpret this result as giving no evidence of a confluent singularity. The confluent-singularity analysis of Hunter and Baker,[21] when applied here, is strongly interfered with by the presence of the other nonconfluent singularities. Only the close-packed lattices, t and fcc, permit a conclusion by this method. The appearance is that confluent singularities do not appear. Hence the structural analysis of $\partial^2\chi/\partial H^2$ agrees with previous results for χ and M_2 in that the critical point is an isolated uncomplicated branch point.

In order to reduce the number of separately estimated quantities in (1.9) we have chosen to analyze the dimensionless quantity

$$-\frac{\partial^2\chi}{\partial H^2}\Big/\chi^2=\frac{\partial^4\ln\Lambda}{\partial x^4}\Big/3\left(\frac{\partial^2\ln\Lambda}{\partial x^2}\right)^2. \quad (3.7)$$

The analysis of this quantity leads to relatively stable Padé estimates. By choosing $\partial^2\chi/\partial H^2$ as $f(x)$ of (3.1) and χ^2 as g of (3.3), we have constructed the following estimates of $2\Delta-\gamma$ by forming Padé approximants to the function of (3.5):

$$\begin{aligned}&\text{sq}, 1.998\pm0.002;\quad \text{t}, 2.00\pm0.01;\\ &\text{sc}, 1.885\pm0.006;\quad \text{bcc}, 1.886\pm0.003; \quad (3.8)\\ &\text{fcc}, 1.8868\pm0.001;\quad \text{hsc}, 1.918\pm0.006.\end{aligned}$$

By way of illustration we give in Table III the results for the fcc, which has the most compact Padé table. The values marked with an asterisk have a close pole and zero nearer the origin than the critical point ($x=1$) and are excluded from the analysis. In addition to an examination of the variance evident in the more central part of the table as displayed, the method of Hunter and Baker[21] has been employed to estimate from small values of x the leading-order error term. The larger of

TABLE III. Padé estimates of $2\Delta-\gamma$ for the fcc lattice.

	L					
M	1	2	3	4	5	6
1	2.1383	2.0128	1.9760	2.0192*	1.8183	1.8724
2	2.0508	1.8817	1.8838	1.8866	1.8868	
3	1.6280	1.8838	1.8773*	1.8868		
4	1.8673	1.8868	1.8868			
5	1.8810	1.8868				
6	1.8839					

these two results has been used to assign the quoted errors in (3.8).

The exact result expected in (3.8) for two dimensions is two. The computed values are consistent with this answer within the quoted errors. The values quoted for three dimensions lie just within the previously established ranges[14] for $\gamma+2\beta$ (β is the magnetization index *below* T_c). For the four-dimensional case, this result by itself lies 14 times the apparent error away from the renormalization-group prediction of two. We will discuss this last result further below.

We have reanalyzed carefully the series for the spherical moment definition of ξ^2, namely,

$$\xi^2=M_2/2d\chi. \quad (3.9)$$

We have used M_2 for f of (3.1) and $2d\chi$ for g of (3.3), and $h(x)/x$ for $h(x)$ in (3.4), and have constructed the following estimates of ν by forming Padé approximants to the function of (3.5):

$$\begin{aligned}&\text{sq}, 0.98\pm0.1;\quad \text{t}, 1.0\pm0.05;\quad \text{sc}, 0.642\pm0.002;\\ &\text{bcc}, 0.6384\pm0.0007;\quad \text{fcc}, 0.6384\pm0.0006; \quad (3.10)\\ &\text{hsc}, 0.555\pm0.008.\end{aligned}$$

Since $M_2\propto u$ for small u we have also analyzed $(1+M_2)/2d\chi$. Since $1+M_2$ vanishes at a distance increasing with dimension from the origin, but closer than the critical point ($x=1$) at about $\pm60°$ from the real axis, it is only for the hsc lattice that this method gives a smaller error. This result is reported in (3.10). As an illustration of the convergence we give in Table IV the results for the fcc lattice. The asterisks mean the same as in Table III, and we have assigned apparent errors in the same manner in (3.10) as in (3.8).

We remark that $\nu=1$ is the exact answer for $d=2$. The results of (3.8) and (3.10) agree with hyperscaling within the apparent error. For $d=3$, the sharpest results are for the fcc lattice. These results are $2\Delta-\gamma-d\nu=-0.028\pm0.003$, which is a sufficiently strong result to reject the renormalization-group hypothesis of zero, in my opinion, and implies that w of (1.9) is zero. It is to be

TABLE IV. Padé estimates of 2ν for the fcc lattice.

M	L 1	2	3	4	5	6	7	8
1	1.3593	1.2819	1.2777	1.2774*	1.2764	1.2760*	1.2770	1.2769*
2	1.2823	1.2757	1.2750	1.2765	1.2767	1.2769	1.2768	
3	1.2781	1.2750	1.2756*	1.2767	1.2779	1.2768		
4	1.2792*	1.2766	1.2767	1.2769	1.2768			
5	1.2763	1.2767	1.2765*	1.2768				
6	1.2758*	1.2769	1.2768					
7	1.2770	1.2768						
8	1.2769*							

noted that Ferer[24] finds γ and ν to be universal for various continuous-spin Landau-Wilson models and equal to the spin-$\frac{1}{2}$ Ising values. The apparent errors are ± 0.012 and ± 0.008, respectively. One concludes that the renormalization group as presently implemented is unfaithful and does not describe the spin-$\frac{1}{2}$ Ising-model behavior, although it does contain a fixed point which is deceptively close in its properties to the true behavior. It could be that the assumption of hyperscaling is the aspect that automatically excludes the true Ising behavior from renormalization-group theory.

In the case of four dimensions we compute $2\Delta - \gamma - d\nu = -0.302 \pm 0.038$. We conclude that w of (1.9) is zero in four dimensions. Renormalization-group theory[1,2] has predicted that w vanishes like $1/\ln\tau$ instead of $\tau^{0.3}$, but we have not been successful in attempting to verify this hypothesis. If the renormalization-group predictions were correct, then one would have expected $(\partial^2\chi/\partial H^2)/\chi$ to be free of logarithmic terms. However, it has a more erratic Padé analysis than does $(\partial^2\chi/\partial H^2)/\chi^2$. The methods of error analysis lead to $2\Delta \simeq 3.0 \pm 0.2$, compared to $2\Delta - \gamma \simeq 1.918 \pm 0.006$ given in (3.8). Renormalization-group theory would also predict that $\chi\xi^4$ is proportional to $\tau^{-3}(\ln\tau)$. To investigate this possibility we have computed $\chi\xi^4$ by using $(1+M_2)^2$ as f in (3.1), χ as g in (3.3), and instead of using h given by (3.4) we use

$$h(x) = \sum \frac{f_j x^j}{g_j \ln(2+j)} . \tag{3.11}$$

Then we formed the Padé approximants to the function of (3.5). The factor of $\ln(2+j)$ in (3.11) was chosen to cancel the hypothesized $\ln j$ dependence of f_j/g_j. We used $\ln(2+j)$ so it would not vanish for $j=0$, or 1. We used $1+M_2$ instead of M_2 since it led to a better result in (3.10). We concluded by the same methods as before that $\gamma + 4\nu = 3.2 \pm 1$, which is much worse than the corresponding $4\nu = 2.220 \pm 0.032$ given by (3.10). While arguments of the above sort for $(\partial^2\chi/\partial H^2)/\chi$ and $\chi\xi^4$ do not demonstrate conclusively anything about the presence or absence of logarithmic correction terms, they are strongly suggestive that the singularity structure of the high-temperature series is more easily accounted for without their inclusion. Also, since we have found that hyperscaling is violated in three dimensions, we have no particular reason to believe the prediction of logarithmic corrections in four dimensions.

We give finally a table of amplitude factors. We estimate B_+/A_+^2 by the method used by Essam and Hunter,[5] namely, we form Padé approximants to

$$\left(\frac{u_c}{(1-u_c^2)(\tanh^{-1}u_c)}\right)\left(1-\frac{u}{u_c}\right)\left(\frac{\partial^2\chi}{\partial H^2}\Big/\chi^2\right)^{1/d\nu}\Bigg|_{u=u_c} = \left(\frac{B_+}{A_+^2}\right)^{1/d\nu} . \tag{3.12}$$

We have followed the method of Fisher and Burford[4] to compute D_+, namely, we have formed Padé approximants to

TABLE V. Amplitude factors.

Lattice	B_+/A_+^2	D_+	v	G_+
sq	4.72 ±0.01	0.5672 ±0.0003	a^2	14.67±0.05
t	4.683±0.002	0.526 ±0.001	$\frac{1}{2}\sqrt{3}a^2$	14.66±0.06
sc	3.17 ±0.01	0.4858 ±0.0005	a^3	27.65±0.20
bcc	3.231±0.004	0.450 34±0.0001	$4a^3/\sqrt{27}$	27.23±0.05
fcc	3.266±0.004	0.439 46±0.000 01	$a^3/\sqrt{2}$	27.21±0.04
hsc	1.221±0.002	0.400 ±0.006	a^4	48 ±3

TABLE VI. Amplitude factors with hyperscaling assumed.

Lattice	D_+	G_+
sc	0.505 ± 0.003	24.6 ± 0.5
bcc	0.469 ± 0.003	24.1 ± 0.5
fcc	0.457 ± 0.002	24.2 ± 0.4

$$\left[\ln u_c + \ln\left(\frac{M_2}{2d\chi u}\right) - 2\nu \ln\left(1 - \frac{K}{K_c}\right)\right]\Big|_{u=u_c} = 2 \ln D_+ . \tag{3.13}$$

These estimates depend on the values of the critical temperature defined by u_c and the values of ν. We have used Domb's[14] values of u_c (except for the simple cubic lattice where $1/u_c$ is misprinted on p. 426 as 4.6844 instead of 4.5844) which are

$$\text{sq}, 0.414\,213\,56;\quad \text{t}, 0.267\,949\,19;\quad \text{sc}, 0.218\,14;$$
$$\text{bcc}, 0.156\,12;\quad \text{fcc}, 0.101\,74;\quad \text{hsc}, 0.1487. \tag{3.14}$$

The value for the hsc lattice is taken from Moore.[16] In Table V we list the coefficients in

$$w = \frac{vB_+}{A_+^2 (aD_+)^d} \tau^{d\nu+\gamma-2\Delta} = G_+ \tau^{d\nu+\gamma-2\Delta}, \tag{3.15}$$

where again v is the specific volume per site and a is the lattice spacing. The two-dimensional results agree with those of Essam and Hunter[5] (B_+/A_+^2) and Fisher and Burford[4] (D_+), inside the quoted errors. In three dimensions, owing to slightly different values of the exponents, the amplitudes are shifted somewhat outside the quoted errors.

It is interesting to see the results we get for the amplitude factors in three dimensions if we impose hyperscaling. To make this hypothesis we compute $\nu = (2\Delta - \gamma)/d$. The resulting amplitude factors are much less well determined than they were in Table V. We present them in Table VI.

The coefficient G_+ given in Table VI is to be directly compared with the Callan-Symanzik–equation approach of Baker *et al.*[13] The zero of the $\beta(v)$ function should be directly related by $v^* = 9G_+/48\pi = 1.46 \pm 0.02$. They obtained 1.423 ± 0.01, which results are quite comparable. The corresponding result in two dimensions is, from Table V, given by $v^* = 9G_+/24\pi = 1.751 \pm 0.005$. In two dimensions hyperscaling is known to hold, and one would expect this result to be a valid prediction of the location of the corresponding zero of the Callan-Symanzik β function.

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[1]K. G. Wilson and J. B. Kogut, Phys. Rev. 12C, 75 (1974).
[2]E. Brézin, J. C. LeGuillou, and J. Zinn-Justin, *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, to be published), Vol. 6.
[3]M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967).
[4]M. E. Fisher and R. J. Burford, Phys. Rev. 156, 583 (1967).
[5]J. W. Essam and D. L. Hunter, J. Phys. C 1, 392 (1968); and private communication, 1976.
[6]D. S. Gaunt and G. A. Baker, Jr., Phys. Rev. B 1, 1184 (1970).
[7]M. A. Moore, D. Jasnow, and M. Wortis, Phys. Rev. Lett. 22, 940 (1969).
[8]G. A. Baker, Jr. and S. Krinsky, J. Math. Phys. (to be published).
[9]R. Schrader, Phys. Rev. B 14, 172 (1976).
[10]L. P. Kadanoff, Phys. Rev. 188, 859 (1969). This proof depends on a hypothesis shown by J. Stephenson along diagonals [J. Math. Phys. 5, 1009 (1964)], as explained by B. McCoy and T. T. Wu [*The Two-Dimensional Ising Model* (Harvard U.P., Cambridge, Mass., 1973), pp. 186–199]. I am grateful to C. Tracy for this reference.
[11]G. A. Baker, Jr., J. Math. Phys. 16, 1324 (1976).
[12]G. S. Rushbrooke and H. I. Scoins, J. Math. Phys. 3, 176 (1962).
[13]G. A. Baker, Jr., B. G. Nickel, M. S. Green, and D. I. Meiron, Phys. Rev. Lett. 36, 1351 (1976).
[14]C. Domb, *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1974), Vol. 3, p. 357.
[15]M. E. Fisher and D. S. Gaunt, Phys. Rev. 133, A224 (1964).
[16]M. A. Moore, Phys. Rev. B 1, 2238 (1970).
[17]G. A. Baker, Jr., Phys. Rev. B 9, 4908 (1974).
[18]M. F. Sykes, J. W. Essam, B. R. Heap, and B. J. Hiley, J. Math. Phys. 7, 1557 (1966).
[19]G. A. Baker, Jr., H. E. Gilbert, J. Eve, and G. S. Rushbrooke, Brookhaven National Laboratory Report No. BNL 50053 (1967) (unpublished).
[20]D. S. Gaunt and A. J. Guttmann, in Ref. 14.
[21]D. L. Hunter and G. A. Baker, Jr., Phys. Rev. B 7, 3346 (1973); 7, 3377 (1973).
[22]G. A. Baker, Jr., *Essentials of Padé Approximants* (Academic, New York, 1975).
[23]W. J. Camp and J. P. Van Dyke, Phys. Rev. B 11, 2579 (1975); W. J. Camp, D. M. Saul, J. P. Van Dyke, and M. Wortis (unpublished).
[24]M. V. Ferer (unpublished).