Percolation thresholds in Ising magnets and conducting mixtures

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The effect of the percolation threshold on conduction in mixtures and on the magnetic ordering temperature in alloys of high- and low-T_C material is described using a real-space rescaling approximation (introduced by Kadanoff, following an idea of Migdal) which is applicable to systems of arbitrary spatial dimensionality d. We calculate exponents describing the regimes just above and just below p_c for the case of bond dilution in resistor networks or Ising ferromagnets. The results are accurate for $d = 2$ in those cases where comparison is possible and qualitatively correct for $d = 3$, but are invalid in the limit $d \to \infty$. Ising spin glasses do not show order in 2 dimensions, in this approximation, but may do so in 3 dimensions.

I. INTRODUCTION

Real-space recursion relations have been constructed for two- and three-dimensional Ising magnetic systems[1] in order to study their critical properties without the usual need to extrapolate from the critical dimensionality[2] $d_c = 4$. In this paper we shall treat the percolation threshold that results when an interacting system is randomly diluted until no long-ranged connectivity remains.[3] For this problem, $d_c = 6$,[4-6] so extrapolation down to $d = 2$ or 3 is not reliable,[6] and a real-space calculation is necessary.

Bond percolation[3] can be seen as a certain limit of the transition which occurs in the Potts model,[7] with the concentration playing the role of temperature. Several calculations of the critical exponents associated with connectedness have used this equivalence.[5,8,9] However, the Potts model formulation is difficult to generalize to models other than bond percolation and obscures the effects of dilution on other cooperative phenomena. Therefore, we shall adopt the point of view of Stinchcombe and co-workers,[10,11] who have shown that the principal effect of rescaling lengths is to modify the probability p that a bond is present, and thus that the percolation threshold can be identified as a fixed point in the rescaling of p.

Kadanoff,[12] by rederiving and generalizing a result of Migdal,[13] has recently introduced a particularly simple procedure for rescaling lattice models in real space, not restricted to two dimensions. The procedure is shown [for the two-dimensional (2D) case] in Fig. 1: First simplify the problem by shifting some of the horizontal bonds (the dashed lines) up or down into groups of b bonds, b lattice spacings apart. Next transform the chains of b vertical bonds which remain into equivalent single bonds by exact "decimation" methods developed for one-dimensional (1D) problems.[1,12,14] Repeat the process, shifting vertical bonds first, then transforming horizontal bonds. The result is an overall length rescaling by a factor of b.

Two arguments can be given in support of the apparently *ad hoc* shifting of bonds. First, the free energy calculated for magnetic models with the bonds shifted is a lower bound to the true free energy.[1,12] For the percolation problem the corresponding statement is that shifting bonds can only decrease the mean number of clusters per site.[7] Second, and more important, the error resulting from shifting bonds is small in both the high- and low-temperature limits[12] (in high- and low-concentration limits for the percolation problem). Thus this approximate transformation should have the property of interpolating between these limits.

To generalize the transformation of Fig. 1 to three dimensions (3D), shift bonds along all but one axis, leaving planes of b bonds which can be transformed into single interactions. Do this for each axis in turn. Denoting the rescaling of a 1D chain as $R_{\text{series}}(b)$ and the bond shifting as $R_{\text{parallel}}(b)$, the net transformation can be expressed as

$$R(b) = [R_{\text{parallel}}(b)]^{d-1} R_{\text{series}}(b) \ . \tag{1}$$

Since the strength of the interactions at the fixed point will depend on the order in which the transformations are applied, the resulting fixed-point Hamiltonian is anisotropic. To remove this anisotropy, one may continue $R(b)$ to values of b close to 1.[12] Writing

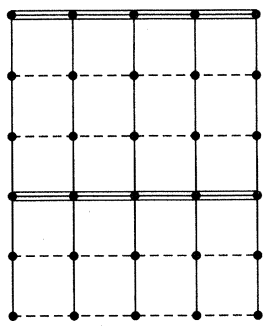

FIG. 1. First stage of the Kadanoff-Migdal transformation ($b = 3$, two dimensions).

$$R_{\text{series}}(1+\zeta) \equiv 1 + \zeta L_{\text{series}} \ , \tag{2a}$$

$$R_{\text{parallel}}(1+\zeta) \equiv 1 + \zeta L_{\text{parallel}} \ , \tag{2b}$$

where the L's are infinitesimal generators, yields

$$R(1+\zeta) = 1 + \zeta(\epsilon L_{\text{parallel}} + L_{\text{series}}) \ , \tag{3}$$

independent of sequence ($\epsilon \equiv d-1$). Calculation of approximate eigenvalues of the L's for the magnetic and percolation problems proves to be fairly straightforward, and is described below.

II. RESULTS

We first consider calculation of the percolation threshold itself. Under R_{series}, p transforms into p^b, while R_{parallel} takes $(1-p)$ into $(1-p)^b$. Letting $b \to 1+\zeta$, using $X^\zeta \sim 1 + \zeta \ln X$, and combining the two terms as in (3) yields for the effect of R acting on p

$$R[p] = p + \zeta[p \ln p - \epsilon(1-p)\ln(1-p)] \ , \tag{4}$$

and the condition that p_c be scale-invariant is the requirement that $L[p_c]$ vanish

$$p_c \ln p_c = \epsilon(1-p_c)\ln(1-p_c) \ . \tag{5}$$

For 2D ($\epsilon = 1$), (5) gives $p_c = \frac{1}{2}$, the exact result[3] for bond percolation on a square lattice. In 3D, (5) predicts $p_c = 0.16$, while the best estimate of the correct answer is 0.247.[3] The limit

$$p_c \to 1 - e^{-1/\epsilon} \quad \text{as } \epsilon \to 0 \tag{6}$$

is correct, but the opposite limit

$$p_c \to e^{-\epsilon} \quad \text{as } \epsilon \to \infty \tag{7}$$

is not correct, since it violates the rigorous bound[3]

$$p_c \geq 1/(2\epsilon + 1) \ . \tag{8}$$

Thus although it may be qualitatively correct in (3) to express the effects of increasing dimensionality as an increase in the relative importance of parallel paths, which can avoid defects, over serial paths, which cannot, (8) shows that this method overestimates the increased freedom obtained in higher dimensions.

Next we expand L, as obtained in (4), about p_c. The coefficient of the term linear in $p - p_c$ gives the correlation length exponent[1,2] ν and can be simplified using (5):

$$\begin{aligned} \nu^{-1} &= 1 + \epsilon + \ln p_c + \epsilon \ln(1-p_c) \\ &= d + (\epsilon/p_c)\ln(1-p_c) \ . \end{aligned} \tag{9}$$

For $d = 2, 3$, (9) predicts $\nu = 1.629$, 1.22. These are in fair agreement with the best known results,[6] 1.3 ± 0.1 and 0.86 ± 0.05, and show the correct trend with increasing dimensionality. As $\epsilon \to 0$, $\nu \sim 1/\epsilon$, a result Stephen[9] has obtained for the Potts model using a related method. Note, however, that upon setting $\epsilon = 0$ (1D), decimation becomes exact and expanding (4) gives the correct result $\nu = 1$. Therefore (9) cannot be regarded as an expansion in ϵ about the point $\epsilon = 0$. Finally, as $\epsilon \to \infty$, (9) predicts $\nu \sim 1 + \epsilon e^{-\epsilon}$, which disagrees with the expected diffusive limit $\nu \to \frac{1}{2}$.

The results for ν are insensitive to the value of b. For example, taking $b = 2$ in 2D gives a transformation identical to Stinchcombe *et al.*'s S1,[10,11] except that they identify the scale change as $\sqrt{2}$. Doubling their result for ν to account for the fact that our value of b is the square of theirs, we obtain $\nu(b=2, d=2) = 1.636$, which differs by less than 1% from the limiting value given in (9).

III. CONDUCTION THRESHOLDS

We consider lattice models in which the bond conductances σ are random variables. Some of them may be zero. Besides changing the effective concentration, rescaling lengths will modify the distribution and arrangement of bond conductances present in such a model. Since the present methods do not introduce correlations among bonds if there were none in the original model system it will be sufficient to consider the simplest possible description. We characterize the model by a single distribution of its conductances, of the form[11]

$$P(\sigma) = p\tilde{P}(\sigma) + \delta(\sigma) \ , \tag{10}$$

where both $P(\sigma)$ and $\tilde{P}(\sigma)$ are normalized to unity. Stinchcombe and Watson[11] find that $\tilde{P}$ may have "fixed points" under approximate renormalization-group transformations. After sufficient iteration, rescaling produces only a change of the scale of σ:

$$R[\tilde{P}(\sigma)] = \lambda \tilde{P}(\lambda\sigma) \ , \tag{11}$$

which has the effect of reducing $\langle\sigma\rangle$ by a factor of λ. Using arguments customary for analysis of tricritical points,[1,2,11] one identifies the exponent t of the conductance threshold

$$\langle\sigma(p)\rangle \propto (p - p_c)^t \ , \tag{12}$$

from

$$\lambda = b^{t/\nu} \sim 1 + \zeta(t/\nu) \ . \tag{13}$$

Mixtures of conducting and superconducting material are also of experimental interest[15] and can be modeled by resistor networks.[16] If we describe each bond by its resistance ρ the analog of (10) is

$$P(\rho) = p\delta(\rho) + (1-p)\tilde{P}(\rho) \ , \tag{14}$$

and invariance of the form of $\tilde{P}(\rho)$ at p_c will imply a second power law

$$\langle\rho(p)\rangle \propto (p_c - p)^s \ . \tag{15}$$

As Straley[16] originally pointed out (see also Ref. 17), a complete description of mixtures of strong

and weak conductors in the vicinity of p_c can be constructed once the two exponents s and t are known.

R_{series} and R_{parallel} for resistor networks simply combine b conductors in series or parallel according to the usual rules. Increasing the lattice constant by a factor of b while keeping the macroscopic conductivity the same in a uniform network requires that all the local conductances be reduced by a factor b^{d-2}. To eliminate this effect we add to R a trivial rescaling back to the original lattice constant. The result is

$$R'[\sigma] = b^{2-d}\{R_{\text{parallel}}[\sigma]\}^{d-1} R_{\text{series}}[\sigma] \ . \tag{16}$$

The analogous transform for resistances is obtained from (16) by interchanging R_{parallel} with R_{series}, and replacing p with $(1-p)$, b^{2-d} with b^{d-2}. In two dimensions, $R'[\sigma] = R'[\rho]$, and the threshold exponents above and below p_c will be identical. Thus the present approximate renormalization scheme satisfies the duality relations between random resistor and conductance models which must hold in general in 2D.[16]

R' also has a homogeneous fixed point $p=1$ at which $\tilde{P}(\sigma)$ iterates to $\delta(\sigma-\sigma^*)$ for some σ^*. As a test of the adequacy of the approximate recursion relations in treating fluctuations outside the immediate region of the critical point, we have iterated R' to convergence for several 2D random network models in which all conductances are nonzero, letting $b=2$ for numerical convenience. The fixed point is anisotropic; σ_1^*, which results from iterating (R_{parallel} R_{series}), may be much less than σ_2^*, the result of iterating (R_{series} R_{parallel}), starting from a given distribution of "bare" conductances. However, the geometric mean $(\sigma_1^*\sigma_2^*)^{1/2}$ obtained when the bonds are initially distributed uniformly over the interval $(1-A, 1+A)$, where the parameter A may take values from 0 to 1, agrees to within numerical accuracy with the predictions of effective-medium theory,[18] which is known to be accurate for this rectangular model. We also considered a model with bonds distributed according to $P_0(\sigma) = (2\sigma \ln A)^{-1}$ over the interval (A^{-1}, A). In this model, effective medium theory predicts $\sigma_{\text{eff}} = 1$ in 2D, and numerical experiment[18] (in 3D) has shown that the actual conductance agrees with σ_{eff} for $A \lesssim 100$, and is only slightly higher for $A \approx 1000$. We calculated $(\sigma_1^*\sigma_2^*)^{1/2} = 1 \pm 0.02$ for $A \leq 10$ in this model, and $= 1 \pm 0.1$ for $A = 1000$.[19] These results, besides supporting the use of the Migdal-Kadanoff transformation to study the regime near p_c, suggest that the method may provide an economical way of studying general properties of more elaborate network models of inhomogeneous conduction.

To obtain the exponents s and t, it is only necessary to calculate the effect of R' on $\langle\sigma\rangle$ and $\langle\rho\rangle$. In the limit $b \to 1$ we can consider L_{series} and L_{parallel} separately. We consider the evolution of a distribution of bonds, initially of uniform value and present with probability p_c, through one stage of each transformation. (Stinchcombe and Watson[11] have shown for $b=2$ that a reasonably accurate value of λ may be obtained from even the first step of this evolution.)

In this approximation, R_{series} takes $\langle\sigma\rangle$ into $\langle\sigma\rangle/b$, so we can identify

$$L_{\text{series}}[\langle\sigma\rangle] = -\langle\sigma\rangle \ . \tag{17}$$

R_{parallel} transforms $\langle\sigma\rangle$ into $bp\langle\sigma\rangle/[1-(1-p)^b]$. {The average conductance of from 1 to b bonds in parallel, weighted by the probability of occurrence, is found by summing

$$\sum_{n=1}^{b} n\binom{b}{n} p^n (1-p)^{b-n}\langle\sigma\rangle = p\,\frac{\partial}{\partial p}\,(p+q)^b\Big|_{q=1-p}\langle\sigma\rangle = bp\langle\sigma\rangle \ ,$$

where $\binom{b}{n}$ denotes a binomial coefficient. Then the factor $[1-(1-p)^b]$ is required to preserve normalization.} Taking the limit $b \to 1$ we obtain

$$L_{\text{parallel}}[\langle\sigma\rangle] = \langle\sigma\rangle\{1 + [(1-p)\ln(1-p)]/p\} \ . \tag{18}$$

Adding the trivial rescaling $(2-d)\langle\sigma\rangle$ to obtain R and identify λ, then using (13) gives

$$t/\nu = -\epsilon[(1-p_c)\ln(1-p_c)]/p_c \ , \tag{19a}$$

$$s/\nu = -(p_c \ln p_c)/(1-p_c) \ , \tag{19b}$$

which can be simplified via (5) to

$$s/t = p_c/(1-p_c) \ . \tag{20}$$

The numerical results are: (2D) $s=t=1.13$; (3D) $t=2.36$, $s=0.46$. The 2D results are in excellent agreement with the values of $t=1.1\pm0.1$ obtained from computer simulations,[6,16] while the 3D result for t is too high (simulations give[6] 1.6 ± 0.1) and that for s is slightly low (Straley finds[16] $s \approx 0.5-0.7$). We note that the ratio s/t found in the computer simulations is consistent with (20). Since this method in 3D gives a p_c which is too low, it is not surprising that the predicted critical region is too stretched out (t is too high) above p_c, and slightly compressed below p_c. To check the method of calculation one can obtain λ from the exact fixed point distribution at $b=2$. The results are not very sensitive to b: $t=s=1.33$ in 2D (using results of Ref. 11), and $t=2.34$, $s=0.64$ in 3D (this work).

IV. MAGNETIC THRESHOLDS

Following the same line of argument, we shall next treat threshold effects due to bond dilution in Ising models of arbitrary spatial dimension. If the interaction strength associated with the i-th bond is denoted $K_i \equiv J_i/kT$, then the series transforma-

tion reduces a set of n bonds K_i to a single bond K' according to $\tanh K' = \Pi_{i=1}^{n} \tanh K_i$, and the parallel transformation is $K' = \sum_{i=1}^{n} K_i$.

The behavior of T_c just above p_c in dilute magnetic systems is an old problem. (For recent studies, and references to the earlier papers, see Refs. 20–24.) There is also an analog in magnetic systems to the "resistive" side of the percolation threshold studied in Sec. III, which does not seem to have been discussed previously. Consider an alloy containing a fraction p of strong bonds K_s and $(1-p)$ of weak bonds K_w. If $K_s \gg K_w$, the strong bonds are quite rigid at temperatures comparable to $T_C(p=0)$, and play the same role as superconducting links in a conducting mixture. Thus one expects $T_C(p)$ to show a singular increase as $p \to p_c^-$, cut off as it reaches a value comparable to $T_C(p=1)$.

To obtain a quantitative prediction for $T_C(p \to p_c^+)$ we again study the evolution of a distribution in which the nonzero interactions initially have a unique value. The appropriate small quantity to study as T_C becomes small is[20] $k \equiv e^{-2K}$, since $\tanh K \sim 1 - 2k$ in the large K, low T limit. We shall therefore be looking for a threshold behavior of the form

$$\exp[-2J/kT_C(p)] \propto (p - p_c)^t \quad (p > p_c) . \tag{21}$$

Rescaling b bonds in series, all of which must have nonzero K, gives $k \to bk$ in the low-T limit. Thus, taking the limit $b \to 1$, we obtain

$$L_{\text{series}}[k] \sim k . \tag{22}$$

Rescaling b bonds in parallel, there is a nonzero interaction whenever one or more bonds is nonzero, and $k(nK) = k^n$. Thus,

$$\langle k' \rangle = \sum_{j=1}^{b} \binom{b}{j} k^j p^j (1-p)^{b-j} \left(\sum_{j=1}^{b} \binom{b}{j} p^j (1-p)^{b-j} \right)^{-1} . \tag{23}$$

The sums can be written

$$\langle k' \rangle = [(1 - p + pk)^b - (1-p)^b] / [1 - (1-p)^b] , \tag{24}$$

and taking the limit $b \to 1$ gives

$$L_{\text{parallel}}[k] = k[1 + p_c^{-1} \ln(1 - p_c) + O(k)] . \tag{25}$$

Combining (22) and (25) according to (3) we observe that the transformation is linear for sufficiently small k, with eigenvalue

$$t/\nu = d + (\epsilon / p_c) \ln(1 - p_c) , \tag{26}$$

but, by comparison with (9) this implies that $t = 1$, and the behavior of $T_C(p)$ as $p \to p_c^+$ is

$$\exp[-2J/T_C(p)] \propto p - p_c \quad (p > p_c) . \tag{27}$$

The result found in (27) is not new. It has been proposed by Stauffer,[20] arguing from a scaling hypothesis. Mean-field[22] and finite-cluster renormalization-group[23,28] treatments have also yielded $t = 1$. By constructing upper and lower bounds for $T_C(p)$, Bergstresser[24] has recently shown that

$$T_C(p) \sim -2J/[\ln(p - p_c)] + O([\ln(p - p_c)]^{-2}) . \tag{28}$$

Thus the present treatment is seen to be consistent with the rather stringent constraint (28). It is surprising that this method as well as the calculation in Ref. 23 should give t exactly even when ν is only approximate.

For an Ising model with a mixture of strong and weak bonds in the regime just below p_c, the obvious dual to (21) is

$$\ln T_C(p) \propto (p_c - p)^{-s} , \quad p < p_c . \tag{29}$$

To look for a dependence of this type we study the transformation properties of $\ln\tanh(K)$. Below p_c, R_{series} describes the effect of combining $n \geq 1$ weak links in a chain, the remaining $b - n$ links being treated as infinitely strong. Since for n identical links in series, $\ln\tanh(K') = n \ln\tanh(K)$, $R_{\text{series}}[\ln \tanh(K)]$ can be evaluated by following the procedure used above for $R_{\text{parallel}}[\sigma]$, with p and $1-p$ interchanged. Thus

$$L_{\text{series}}[\ln\tanh(K)] = \left(1 + \frac{p_c \ln p_c}{1 - p_c}\right) \ln\tanh(K) . \tag{30}$$

In R_{parallel}, b weak links are combined, with no shunting strong links, giving $\ln\tanh(K') = \ln\tanh(K) + \ln b$ for $K \ll 1$, or

$$L_{\text{parallel}}[\ln\tanh(K)] = 1 . \tag{31}$$

Since (30) dominates (31), we extract the eigenvalue

$$s/\nu = 1 + \epsilon \ln(1 - p_c) , \tag{32}$$

where we have made use of (5). Some numerical results are $s(2D) = \frac{1}{2}$; $s(3D) = 0.85$. The above predictions for s and the form of the threshold below p_c in this magnetic system are new, as far as we know, and should be susceptible to experimental verification.

V. SPIN GLASSES

We close on a note of caution. Although the bond-shifting transformation gives sensible results for the magnetic and conduction thresholds it may not be reliable for all inhomogeneous systems. In particular, it may estimate interaction strengths incorrectly in spin glasses.[25] For purposes of this discussion, spin glasses are random magnetic systems in which the bonds K_i are random, uncorrelated, and may take on either sign. The success of the method in treating uniform systems rests on the fact that the precise location of the individual bonds is not very important at low

temperatures.[12] While this may still be true for inhomogeneous systems in which the different interactions are additive, it is false for models, such as spin glasses, in which the interactions may have either sign. An example in which bond shifting causes spurious interference between bonds of opposite sign is given in Fig. 2. The set of interactions pictured locks the four spins into a stable arrangement at low temperatures, yet shifting any bond parallel to its axis causes a cancellation which leaves two noninteracting pairs of spins.

It is natural to seek to extend to spin glasses the present type of analysis, in which we study the transformation of some simple approximation to the fixed-point distribution of interactions. Such a distribution cannot be a single δ function, since it is possible[25] to have a spin glass phase in which $\langle K\rangle = 0$ in the bare system, and only $\langle K^2\rangle \neq 0$. Jayaprakash, Chalupa, and Wortis[26] have recently presented such a calculation, applying the Migdal transformation to a distribution $P(K) = \frac{1}{2}[\delta(K - K_0) + \delta(K + K_0)]$, and taking $b = 3$. They argue that since this is the exact fixed point distribution for 1D Ising models with random interactions of both signs[27] it should therefore be a good starting point for discussing $d = 2$ and above. However, when $d \geq 2$, interactions between two spins which proceed along different paths can interfere (think of a square like that in Fig. 2 with three bonds $+K$, and one $-K$), and generate very small effective interactions. In the remainder of this section, we carry out the Migdal transformation for a spin glass, taking the limit $b \to 1$, and show that the presence of both strong and weak interactions in $P(K)$ has important effects on the transformation.

We consider only distributions $P(K)$ such that $\langle K\rangle = 0$, $\langle K^2\rangle \neq 0$, and $P(K) = P(-K)$. Applying R_{parallel} to b random bonds to obtain K' gives $\langle K'\rangle = 0$, $\langle K'^2\rangle = b\langle K^2\rangle$. Thus

$$L_{\text{parallel}}[\langle K^2\rangle] = \langle K^2\rangle \tag{33}$$

for any such $P(K)$. The effect of R_{series} is more complicated and will depend in detail upon $P(K)$. If $P(K) = P(-K)$, the transformation

$$\tanh K' = \prod_{i=1}^{b} \tanh K_i = \prod_{i=1}^{b} \operatorname{sgn}(K_i) \tanh |K_i| \tag{34}$$

produces equal numbers of positive and negative interactions, and preserves the symmetry in K. If the magnitude of K is unique, then in the limit of low temperature, $k' = bk$ (where $k \equiv \exp(-2|K|)$) implies that $|K'| = |K| - \frac{1}{2}\ln b$. In the $b \to 1$ limit this approximation to $P(K)$ gives $L_{\text{series}}[\langle K^2\rangle] = -\frac{1}{2}$.

A linear approximation to R_{series} was used to study more general $P(K)$. At low temperatures,

$$\exp(-2|K'|) \approx \sum_{i=1}^{b} \exp(-2|K_i|) \tag{35}$$

is dominated by the smallest $|K_i|$, e.g., by the weakest link in the chain. We can incorporate this feature by approximating

$$R_{\text{series}}^{(b)}[|K|] \approx \min(|K_i|)_{i=1,\ldots,b} . \tag{36}$$

This has the effect of transforming $P(K)$ into

$$P'(K') = bP(K')\left(2\int_0^{K'} P(K)\,dK\right)^{b-1} , \tag{37}$$

or, in the limit $b \to 1$,

$$L_{\text{series}}[P(K)] = P(K)\ln 2 \int_0^{K} P(K)\,dK . \tag{38}$$

The distribution $P^*(K)$ which is invariant under $[R_{\text{series}}(b)\, R_{\text{parallel}}(b)]$ for $b = 2, 3$ was found by iteration, using (37), and is plotted (for $K > 0$) in Fig. 3. Compared to a Gaussian distribution (dotted line), P^* is more sharply peaked at small K, and has longer tails. Since the results in Fig. 3 appear

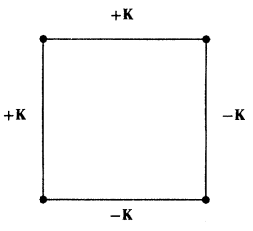

FIG. 2. Spin-glass configuration in which bond shifting underestimates the effective interaction strength even in the low-temperature limit.

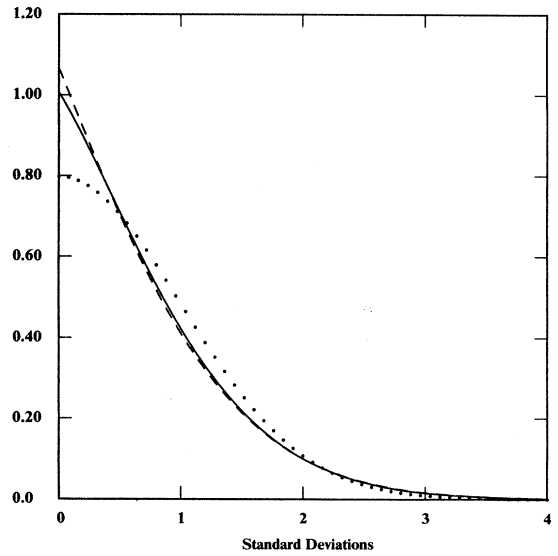

FIG. 3. Interaction strength distribution $P^*(K)$ for an Ising spin glass in 2D, found by iterating the Migdal transformation in its linear (low-T) form, using (37). The solid line is the result for $b = 2$, the dashed line the result for $b = 3$, and a Gaussian distribution is shown dotted for comparison. In both cases, the iteration converged in two to three iterations, and the result was independent of the starting distribution.

not to be sensitive to b, we have used $P^*(b=2)$ to evaluate (37), and find that the change in $\langle K^2\rangle$ is

$$L_{\text{series}}[\langle K^2\rangle] = -1.50\,\langle K^2\rangle \ . \tag{39}$$

Combining (39) with (33), we observe that in this improved approximation rescaling decreases $\langle K^2\rangle$ at low temperatures in 2D, rather than increasing it as it must if the system is to flow towards a stable $T=0$ fixed point characteristic of an ordered state. Therefore under the Migdal transformation, an Ising spin glass does not order in 2D. This conclusion is in agreement with recent calculations by Young and Stinchcombe[28] using other decimation-type transformations. Since bond shifting has unpredictable consequences for this model, we cannot draw any rigorous conclusions about 2D Ising spin glasses from the preceding calculation. For $d \geq 3$, the Migdal transformation does predict the existence of an ordered phase in spin glasses, so the phase diagrams presented in Ref. 26 for 3D and 4D may be qualitatively correct.

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