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Analytical structure of the wave-number-dependent susceptibility of many-fermion systems at low temperature and long wavelength

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The analytical structure of the wave-number-dependent magnetic susceptibility $\chi(\vec{k})$ is studied at small wave number $\vec{k}$ and low temperature. It is found to exhibit a nonanalytic behavior in contrast to commonly assumed forms for $\chi(\vec{k})$ of interacting systems. The importance and implications of these results are discussed for the cases of short-range point interactions and long-range Coulomb interactions.

I. INTRODUCTION

Fundamental information concerning the correlation of spin-density fluctuations, on a microscopic level, for a many-fermion system is contained in the wave-number- and frequency-dependent susceptibility[1,2] $\chi(\vec{k},\omega)$. For simplicity, consider a paramagnetic translationally invariant system of spin-$\frac{1}{2}$ particles to which a weak magnetic field $H(\vec{r}t)$ is applied which couples to the magnetization density $m(\vec{r},t)=\mu_B[n^{\uparrow}(\vec{r}t)-n^{\downarrow}(\vec{r},t)]$, where μ_B is the magnetic moment per particle and $n^{\sigma}(\vec{r},t)$ is the particle density of spin projection σ. The induced magnetization, in linear response, is

$$\langle m(\vec{k},\omega)\rangle=\chi(\vec{k},\omega)H(\vec{k},\omega)\,, \tag{1}$$

where

$$\chi(\vec{k},\omega)=\lim_{\eta\to 0^+}\int d^3r\,e^{-i\vec{k}\cdot\vec{r}}\int_0^{\infty}dt\,e^{i(\omega+i\eta)t}\frac{i}{\hbar}\times\langle[m(\vec{r},t),m(\vec{0},0)]\rangle\,. \tag{2}$$

The calculation of even the static ($\omega=0$) magnetic response, given by $\chi(\vec{k},\omega\to 0)\equiv\chi(\vec{k})$, is of very great interest but also presents very great difficulties in the case of realistic interacting systems. It has been generally assumed (and supported by approximate calculations) that: (i) the long-wavelength ($k\ll 2k_F$) behavior of $\chi(\vec{k})$ can be described by a regular series expansion in powers of k^2 (the interparticle interaction is taken to be rotationally invariant); (ii) $\chi(\vec{k})$ attains its maximum value at $\vec{k}=\vec{0}$; and (iii) in the strongly degenerate limit, $T\ll T_F$, the leading temperature-dependent corrections to the $T=0$ value of $\chi(\vec{k})$ are the usual minor additive corrections of order $(T/T_F)^2$ which occur for a noninteracting many-fermion system.

The above three assumptions are incorrect and the correct form of $\chi(\vec{k})$ for small $\vec{k}$ is given in this work. In Sec. II, we consider the simplified model of very weak and short-range interactions, characterized by interaction strength I and rigorously evaluate the expansion coefficient α in the leading correction term in the long-wavelength expansion

$$\chi(\vec{k})=\chi(\vec{0})-\alpha k^2+O(k^4) \tag{3}$$

to second order in the small parameter I at finite temperature $0<T\ll T_F$. It is shown that the dominant contribution to α varies as $\ln(T/T_F)$ at low temperature. It is this fact which leads to the failure of the common assumptions indicated above. The calculations are carried out using a propagator formalism for finite-temperature perturbation theory[3-5] to evaluate the Fourier transforms of the correlation functions

$$P^{\sigma\sigma'}(\vec{k},i\omega_l)=\int_0^{\beta}d\tau\,\langle\hat{T}n^{\sigma}(\vec{r},\vec{\tau})n^{\sigma'}(\vec{0},0)\rangle\,e^{i\omega_l\tau}\,, \tag{4}$$

where $\omega_l=2l\pi i/\beta$ and $\beta=(k_BT)^{-1}$. The static susceptibility requires only the $\omega_l=0$ point and is given by

$$\chi(\vec{k})=2\mu_B^2[P^{\uparrow\uparrow}(\vec{k},0)-P^{\uparrow\downarrow}(\vec{k},0)]\,. \tag{5}$$

The cases of strong short-range interactions (e.g., ^{3}He) and of the high-density electron gas with long-range Coulomb interactions (including dynamical screening) are examined in Sec. III and the corresponding expansion coefficients α are also singular at low T. Some implications (both for matters of principle and for practical calculations) of this result for the analytical structure of $\chi(\vec{k})$ are discussed. It is also shown that this logarithmic singularity does *not* appear in the corresponding response function for charge-density correlations,

$$\tilde{\chi}(\vec{k})=2e^2[P^{\uparrow\uparrow}(\vec{k},0)+P^{\uparrow\downarrow}(\vec{k},0)]. \tag{6}$$

II. ANALYSIS OF $\chi(\vec{k},0)$ FOR WEAK SHORT-RANGE INTERACTIONS

In this section, we shall explicitly determine the leading $\vec{k}$ dependence, for small $\vec{k}$, of $\chi(\vec{k},0)$ in the case of a model system which can be described by weak short-range interactions. The Hamiltonian of the translationally invariant system is $H=H_0+H_I$, where H_0 is the usual kinetic energy and the two-body interaction, in standard second quantized form is

$$H_I=\frac{1}{2}\sum_{\sigma\sigma'}\int d^3r\, d^3r'\, V(\vec{r}-\vec{r}')\times\psi^\dagger(\vec{r}\sigma)\psi^\dagger(\vec{r}'\sigma')\psi(\vec{r}'\bar{\sigma}')\psi(\vec{r}\sigma), \qquad (7)$$

where the spin indices σ,σ' can be either $\uparrow$ or $\downarrow$. In the limiting case of very short-range interactions, $V(\vec{r}-\vec{r}')\to I\delta(\vec{r}-\vec{r}')$, the Pauli principle then precludes interaction between parallel spin fermions so that Eq. (7) reduces to

$$H_I=I\int d^3r\, \hat{n}_\uparrow(\vec{r})\hat{n}_\downarrow(\vec{r}). \qquad (8)$$

This short-range interaction model has been widely used in the literature[1,2,6,7] to describe "nearly magnetic" systems.

In the present work, we shall initially assume that I is sufficiently weak to treat by second order perturbation theory and we make a consistent calculation of interaction contributions to $\chi(\vec{k},0)$ up to and including second order (which is where dynamical effects first appear). In the usual propagator formalism for finite-temperature perturbation theory,[3-5] the graphs through second order for $\chi^{\uparrow\uparrow}(\vec{k},0)$ and $\chi^{\uparrow\downarrow}(\vec{k},0)$ are shown in Figs. 1(a) and 1(b), respectively. In Fig. 1(c) are indicated some of the graphs which also contribute for a general interaction but which give exactly zero contribution, due to the combined effect of spin conservation and the Pauli principle, for this short-range model. The contributions of the graphs in Fig. 1(a) are

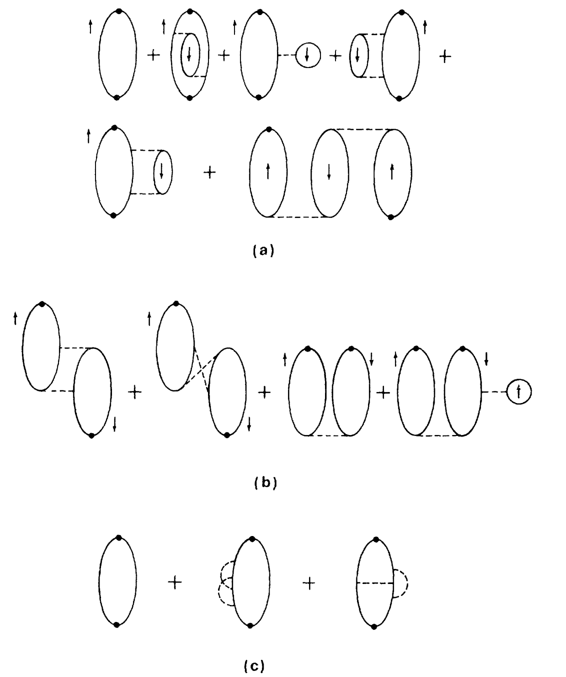

FIG. 1. (a) Lowest-order graphs which contribute to the $P^{\uparrow\uparrow}(\vec{k},0)$ term in $\chi(\vec{k})$ (see text) to order I^2; (b) contributions to $P^{\uparrow\downarrow}(\vec{k},0)$ to order I^2; (c) some of the graphs of order I^2 which do not contribute to $\chi(\vec{k})$ in the limit of short-range interactions.

$$P^{\uparrow\uparrow}(\vec{k},0)=P_0(\vec{k},0)+P_I^{\uparrow\uparrow}(\vec{k},0), \qquad (9)$$

where

$$P_0(\vec{k},0)=-\mathrm{tr}_{\vec{p}\zeta}\, g_{\vec{p}+\vec{k}/2}(\zeta_n)\, g_{\vec{p}-\vec{k}/2}(\zeta_n) \qquad (10)$$

and

$$\begin{aligned}P_I^{\uparrow\uparrow}(\vec{k},0)=\,&I^2[P_0(\vec{k},0)]^3-I^2\,\mathrm{tr}_{\vec{p}\zeta}\,\mathrm{tr}_{\vec{p}'\zeta'}[g_{\vec{p}+\vec{k}/2}(\zeta_n)g_{\vec{p}-\vec{k}/2}(\zeta_n)P_0(\vec{p}-\vec{p}',\zeta_n-\zeta_{n'})g_{\vec{p}'+\vec{k}/2}(\zeta_{n'})g_{\vec{p}'-\vec{k}/2}(\zeta_{n'})]\\&-I^2\,\mathrm{tr}_{\vec{p}\zeta}\,\mathrm{tr}_{\vec{p}'\zeta'}\{[g_{\vec{p}}(\zeta_n)]^2g_{\vec{p}'}(\zeta_{n'})P_0(\vec{p}-\vec{p}',\zeta_n-\zeta_{n'})[g_{\vec{p}+\vec{k}}(\zeta_n)+g_{\vec{p}-\vec{k}}(\zeta_n)]\}\\&+\mathrm{tr}_{\vec{p}\zeta}\{[g_{\vec{p}}(\zeta_n)]^2\delta\mu[g_{\vec{p}+\vec{k}}(\zeta_n)+g_{\vec{p}-\vec{k}}(\zeta_n)]\}. \end{aligned}\qquad (11)$$

The single-particle propagator is denoted by $g_{\vec{p}}(\zeta_n)=1/(\zeta_n-\tilde{\epsilon}_p)$ where $\zeta_n=(2n+1)\pi i/\beta$ is the usual "frequency," $\tilde{\epsilon}_p=\hbar^2\vec{p}^2/2m-\mu_0$, and

$$\mathrm{tr}_{\vec{p}\zeta}(\cdots)=[(2\pi)^3\beta]^{-1}\int d^3p\sum_n(\cdots).$$

The chemical potential for noninteracting particles is denoted by μ_0 and the effect of interactions on the exact (to order I^2) chemical potential, $\mu=\mu_0+\delta\mu$, have already been explicitly accounted for by the presence of the last term in Eq. (11). [The Hartree graphs in Fig. 1(a) contribute a trivial shift in chemical potential and are of no consequence.] Similarly, the graphs of Fig. 1(b) contribute

$$\begin{aligned}P^{\uparrow\downarrow}(\vec{k},0)=\,&-I[P_0(\vec{k},0)]^2+\tfrac{1}{2}I^2\,\mathrm{tr}_{\vec{p}\zeta}\,\mathrm{tr}_{\vec{p}'\zeta'}\,g_{\vec{p}+\vec{k}/2}(\zeta_n)g_{\vec{p}-\vec{k}/2}(\zeta_n)g_{\vec{p}'+\vec{k}/2}(\zeta_{n'})g_{\vec{p}'-\vec{k}/2}(\zeta_{n'})\\&\times\mathrm{tr}_{\vec{q}\xi}[g_{\vec{p}+\vec{q}}(\zeta_n+\zeta_m)+g_{\vec{p}-\vec{q}}(\zeta_n-\xi_m)][g_{\vec{p}'+\vec{q}}(\zeta_{n'}+\xi_m)+g_{\vec{p}'-\vec{q}}(\zeta_{n'}-\xi_m)]. \end{aligned}\qquad (12)$$

The expansion of Eqs. (10) to (12) in powers of $\vec{k}$ is easily carried out by using relations such as

$$g_{\vec{p}+\vec{k}}(\zeta)=g_{\vec{p}}(\zeta)+(\epsilon_{\vec{p}+\vec{k}}-\epsilon_{\vec{p}})\frac{\partial g_{\vec{p}}(\zeta)}{\partial\mu_0}+\tfrac{1}{2}(\epsilon_{\vec{p}+\vec{k}}-\epsilon_{\vec{p}})^2\frac{\partial^2 g_{\vec{p}}(\zeta)}{\partial\mu_0^2}+\cdots$$

The algebraic details of this expansion are identical to those already given in the corresponding problem of charge density correlations[8,9] and need not be repeated. Collecting all contributions, we have

$$\delta P^{\uparrow\uparrow}(\vec{k},0)\equiv P^{\uparrow\uparrow}(\vec{k},0)-P^{\uparrow\uparrow}(\vec{k}\to\vec{0},0)$$

given by

$$\delta P^{\uparrow\uparrow}(\vec{k},0)=-b^{\uparrow\uparrow}k^2+O(k^4), \tag{13}$$

where $b^{\uparrow\uparrow}=b_0+b_I^{\uparrow\uparrow}$, with Eq. (10) contributing $b_0=(\hbar^2/24m)\partial^2 n/\partial\mu_0^2$ and Eq. (11) contributing $[P_0(\vec{k}\to\vec{0},0)=N_0(0)=mk_F/2\pi^2\hbar^2]$,

$$b_I^{\uparrow\uparrow}=3[IN_0(0)]^2b_0+\delta\mu_I\frac{\hbar^2}{24m}\frac{\partial^3 n}{\partial\mu_0^3}-\frac{\hbar^2}{2m}I^2\operatorname{tr}_{\vec{p}\zeta}\operatorname{tr}_{\vec{p}'\zeta'}P_0(\vec{p}-\vec{p}',\zeta_n-\zeta_{n'})$$
$$\times\left[g_{\vec{p}'}(\zeta_{n'})\left(\frac{1}{3}\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta_n)-\tfrac{1}{9}\epsilon_{\vec{p}}\frac{\partial^4}{\partial\mu_0^4}g_{\vec{p}}(\zeta_n)\right)+\left(\frac{\partial}{\partial\mu_0}g_{\vec{p}'}(\zeta_{n'})\right)\left(\frac{1}{2}\frac{\partial^2}{\partial\mu_0^2}g_{\vec{p}}(\zeta_n)-\tfrac{1}{9}\epsilon_{\vec{p}}\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta_n)\right)\right]. \tag{14}$$

Similarly, we obtain

$$b^{\uparrow\downarrow}=-2[IN_0(0)]b_0-\frac{\hbar^2}{2m}I^2\operatorname{tr}_{\vec{p}\zeta}\operatorname{tr}_{\vec{q}\xi}\left(\frac{1}{4}\frac{\partial^2}{\partial\mu_0^2}g_{\vec{p}}(\zeta)-\tfrac{1}{18}\epsilon_{\vec{p}}\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta)\right)$$
$$\times[g_{\vec{p}+\vec{q}}(\zeta_n+\xi_m)+g_{\vec{p}-\vec{q}}(\zeta_n-\xi_m)]\frac{\partial}{\partial\mu_0}P_0(\vec{q},\xi_m). \tag{15}$$

At finite temperature, all contributions to Eqs. (14) and (15) are well defined. However, in the limit of $T\to 0$, both $b^{\uparrow\uparrow}$ and $b^{\uparrow\downarrow}$ develop singular behavior of the form $\ln(k_BT/\mu_0)$. The physical significance of this behavior will be discussed below and, for the present, we shall isolate and evaluate those terms in Eqs. (14) and (15) which are responsible for the singularity. As will be verified by the following analysis, the singular contributions arise in terms with the largest number of confluent propagators and are contained in

$$b_S^{\uparrow\uparrow}=\frac{\hbar^2}{18m}I^2\operatorname{tr}_{\vec{p}\zeta}\operatorname{tr}_{\vec{q}\xi}P_0(\vec{q},\xi_m)\epsilon_{\vec{p}}\frac{\partial}{\partial\mu_0}\left(g_{\vec{p}+\vec{q}}(\zeta_n+\xi_m)\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta_n)\right) \tag{16}$$

and

$$b_S^{\uparrow\downarrow}=\frac{\hbar^2}{18m}I^2\operatorname{tr}_{\vec{p}\zeta}\operatorname{tr}_{\vec{q}\xi}\left(\frac{\partial}{\partial\mu_0}P_0(\vec{q},\xi_m)\right)\epsilon_{\vec{p}}\left(g_{\vec{p}+\vec{q}}(\zeta_n+\xi_m)\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta_n)\right). \tag{17}$$

We can rewrite Eq. (16) as

$$b_S^{\uparrow\uparrow}=-b_S^{\uparrow\downarrow}+\frac{\hbar^2}{18m}I^2\frac{\partial}{\partial\mu_0}\operatorname{tr}_{\vec{p}\zeta}\operatorname{tr}_{\vec{q}\xi}P_0(\vec{q},\xi_m)\epsilon_{\vec{p}}g_{\vec{p}+\vec{q}}(\zeta_n+\xi_m)\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta_n). \tag{18}$$

Consequently, the singular contribution to the expansion coefficient $\alpha=2\mu_B^2(b^{\uparrow\uparrow}-b^{\uparrow\downarrow})$ in Eq. (2) is given by

$$\alpha_S=-4\mu_B^2 b_S^{\uparrow\downarrow} \tag{19}$$

since the second term on the right-hand side of Eq. (18) will be shown (see below) to be finite.

It is useful to have an explicit representation for the function $\partial P_0(\vec{q},\xi)/\partial\mu_0$ which enters Eq. (17). Returning to Eq. (10), the ζ_n sum and the angular integration are straightforward and yield, for the μ_0 derivative,

$$\frac{\partial P_0(\vec{q},\xi)}{\partial\mu_0}=-\frac{m}{2\pi^2\hbar^2 q}\operatorname{Re}\int_0^\infty dp\,pf'(\bar{\epsilon}_p)\left[\ln\left(\epsilon_q+\frac{\hbar^2pq}{m}+\xi\right)-\ln\left(\epsilon_q-\frac{\hbar^2pq}{m}+\xi\right)\right]. \tag{20}$$

This need not be evaluated in full generality as we are interested only in the strongly degenerate limit. In Appendix A, we give an approximate evaluation of Eq. (20) which is valid for $0\leq T\ll T_F\equiv\epsilon_F/k_B$. The result [see Eq. (A5)], for ξ on the positive imaginary axis, is

$$\frac{\partial P_0(\vec{q},\xi)}{\partial\mu_0}\to\frac{m^2}{2\pi^2\hbar^4 q}\operatorname{Re}\left[\psi\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi iq}(\epsilon_q+\hbar v_F q+\xi)\right)-\psi\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi iq}(\epsilon_q-\hbar v_F q+\xi)\right)\right], \tag{21}$$

where $v_F = \hbar k_F/m$ and $\psi(Z) = (d/dZ)\ln\Gamma(Z)$ is the usual digamma function. Note that the above is a regular function of q and ξ for $T \neq 0$, as expected on general grounds. It may be verified that Eq. (21) reproduces exactly the $T \to 0$ limit of $P_0(\vec{q}, \xi)/\partial\mu_0$ and of its derivatives. Also note that Eq. (21) is logarithmically divergent $[\psi(Z) \to \ln Z + \cdots$ for $|Z| \to \infty]$ for $T \simeq 0$, $q \simeq 2k_F$, and $\xi \simeq 0$. It is this fact, coupled with the repeated propagators in Eq. (17), which leads to the $\ln T$ divergence in $b^{\uparrow\downarrow}$ for $T \to 0$. Using the methods which led to Eq. (21), the sums over the propagators in Eq. (17) may be expressed in the same spirit. From Eq. (A9), the relevant singular contributions are contained in (again, ξ_l is on the positive imaginary axis)

$$\mathrm{tr}_{\vec{p}\zeta}\epsilon_{\vec{p}}\, g_{\vec{p}+\vec{q}}(\zeta_n+\xi_l)\frac{\partial^3}{\partial\mu_0^3}\, g_{\vec{p}}(\zeta_n) + \text{c.c.} \to \frac{m^2\mu_0}{2\pi^2\hbar^4 q}\ \mathrm{Re}\sum_{\pm}\left[C_{\pm}\psi'\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi i q}(\epsilon_q \pm \hbar v_F q + \xi_l)\right) + D_{\pm}\psi''\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi i q}(\epsilon_q \pm \hbar v_F q + \xi_l)\right)\right], \tag{22}$$

where we used the fact that the contributions to Eq. (17) from $\xi_l/i < 0$ are just the complex conjugates of those for $\xi_l/i > 0$. The coefficients $C_\pm$ and $D_\pm$ are given in Eqs. (A10) and (A11). Having explicit representations for the quantities which enter Eq. (17), the evaluation of $b_S^{\uparrow\downarrow}$ can be carried out. The calculation is lengthy and the details are given in Appendix A. The final result [see Eq. (A22)] is

$$b_S^{\uparrow\downarrow} = -(m^2 k_F I^2/1152\pi^6\hbar^6)\ln(k_B T/\epsilon_F). \tag{23}$$

From Eqs. (3), (19), and (23), the coefficient of the $\vec{k}^2$ term in the expansion of $\chi(\vec{k})$ is given by

$$\alpha = \alpha' + \alpha'' \ln(k_B T/\epsilon_F), \tag{24}$$

where

$$\alpha'' = \mu_B^2 m^3 k_F I^2/288\pi^6\hbar^6, \tag{25}$$

and α' contains all regular contributions for $T \to 0$. For small I, α' is simply given by $\alpha' = \alpha_0 + O(I^2)$, where $\alpha_0 = \mu_B^2 m/12\pi^2\hbar^2 k_F$ is given by the Lindhard function [Eq. (10)]. In the limiting case of a weak interaction, which can be treated by perturbation theory and which does not destroy the Fermi surface, the above analytic structure will remain to all orders in the interaction. When I is not weak enough to treat by a simple form of perturbation theory to finite order (e.g., ^{3}He, nearly ferromagnetic metals such as Pd), the above conclusions are no longer rigorous. Nevertheless, we have seen no way that higher-order terms could conspire to cancel the $\ln(T/T_F)$ in Eq. (24), provided the system remains normal, and we thus expect our conclusion to extend to interactions of arbitrary strength (see Sec. III). In fact, such cancellation is also highly unlikely for interactions having finite range (as opposed to point interactions). In particular, the electron gas will also exhibit the structure implied by Eq. (24). In Sec. III, we discuss the implications of these results for paramagnetic many-fermion systems.

III. IMPLICATIONS AND DISCUSSION

A. Short-range interactions

First, we focus attention on the case of weak interactions. From the fact that the expansion coefficient α [Eq. (24)] is logarithmically divergent for $T \to 0$, we may conclude that the low-temperature and long-wavelength structure of $\chi(\vec{k})$ is given by

$$\chi(\vec{k}) \simeq \chi(\vec{0}) - k^2[\alpha' + \alpha'' \ln(T/T_F + k^2/k_0^2)], \tag{26}$$

where k_0 is of order k_F. The immediate implication is that $\chi(\vec{k})$ is not an analytic function of $\vec{k}$ at zero temperature.

In numerous applications, the long-wavelength behavior of $\chi(\vec{k})$ is important and it is assumed that $\chi(\vec{k})$ can be expanded in powers of $\vec{k}^2$. From Eq. (26), this is clearly valid only for $k^2/k_0^2 < T/T_F$. How important is this fact? To obtain the above structure for $\chi(\vec{k})$, it was assumed that the system remains in a normal paramagnetic state. For many-fermion systems which do remain normal and paramagnetic to very low T (e.g., the electron gas, see below) the usual expansions of $\chi(\vec{k})$ at $T=0$ are clearly invalid. On the other hand, for systems which exhibit a transition to a nonparamagnetic state at finite T_c (e.g., ^{3}He) the $\vec{k}^2$ expansion is valid for $T > T_c$, but the expansion coefficients will inevitably show a strong temperature dependence for T near T_c.

Further scrutiny of Eq. (26) reveals an additional interesting feature of $\chi(\vec{k})$. Owing to the fact that both α' and α'' are positive, there is a finite temperature T_0, below which $\chi(\vec{k})$ has its maximum value occuring at some *finite* k, with this temperature given by

$$T_0 \approx T_F\, e^{-\alpha'/\alpha''}. \tag{27}$$

Thus, below T_0, the system is more susceptible to magnetic perturbations of finite wave number than it is to uniform perturbations. In this respect, the situation may appear to be "reminiscent" of a spin-density wave state[1] (Sec. III B below). However, it must be emphasized that the present analysis is restricted to $k \ll k_F$ and conclusions concerning the region of $k \sim 2k_F$ are not to be drawn.

What is the origin of this surprising nonanalytic behavior of $\chi(\vec{k})$? Mathematically, a careful inspection of Appendix A reveals immediately that the result is *independent* of the form of the interaction. It is strictly a consequence of the dynamics arising from the creation of electron-hole pairs around a *sharp* Fermi surface; the electron-hole scattering events contributing to this singularity are those on opposite sides of the Fermi surface. It is worth pointing out that an instability toward a superconducting state, also independent of the interaction and due to the sharpness of the Fermi surface, has been noted by Kohn and Luttinger[10] in the electron-electron scattering function. The peculiar analytic behavior at small $\vec{k}$, which we note is different in that if any instability should be associated with it (certainly not demonstrated by the above results), it would be towards a nonparamagnetic state rather than a superconducting one. Let us now turn to the case of interactions which are not weak. Certainly, contributions of higher orders in I have to be included in the calculation of $\chi(\vec{k})$. These will modify the coefficient α'' in Eq. (26) but we see no way there can be cancellation of this term provided the Fermi surface remains sharp. It would be interesting to speculate on the effect of these higher-order corrections on α''. A common approach, in ^{3}He, for example, is to include those effects via the virtual excitations of longitudinal and transverse paramagnons. For the sake of brevity, we will consider only the contribution of the longitudinal paramagnons to lowest order shown in Fig. 2(a). The transverse paramagnons [Fig. 2(b)] give similar contributions. Straightforward calculation of the $\vec{k}^2$ coefficient contributed by Fig. 2(a) gives Eq. (14) with I^2P_0 replaced by $I[1/(1-IP_0)-1]$ where we have neglected the regular contributions of the charge fluctuations associated with the longitudinal paramagnons $\{$i.e., $I[1-1/(1+IP_0)]\}$.

Following the analysis of Sec. II and Appendix A, we find that $\chi(\vec{k})$ has the structure given in Eq. (26) with α'' replaced by

$$\alpha'' = \frac{\mu_B^2 m^3 k_F I^2}{576\pi^6\hbar^6}\,\frac{1}{[1-IP_0(2k_F,0)]^2}\,. \tag{28}$$

As might be expected, the higher-order contributions are reflected through an enhancement factor. However, it is interesting to note the following observations. While the inclusion of paramagnons has a very significant effect in many properties,[6,7] due to the usual Stoner enhancement, this is not the case for the nonanalytic contribution of $\chi(\vec{k})$ investigated here. The difference lies in the fact that the Stoner enhancement $[1-IP_0(0,0)]^{-1}$, is very large since $IP_0(0,0)$ is almost unity for ^{3}He or Pd while the "enhancement" factor in Eq. (28) is $[1-IP_0(2k_F,0)]^{-1}\approx 2$. This again emphasizes that our second-order analysis is unlikely to be drastically modified by higher-order corrections in real systems.

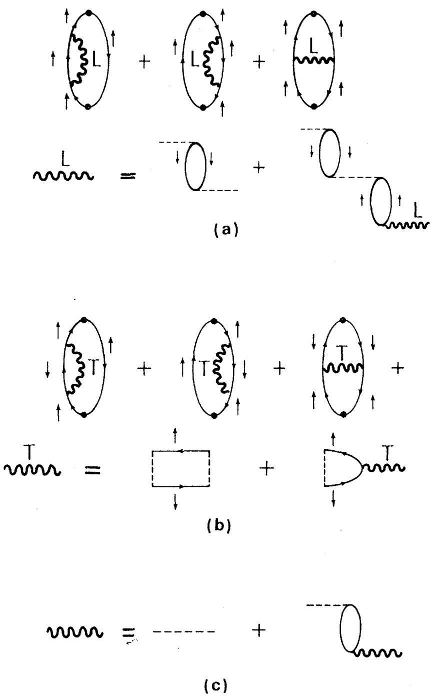

FIG. 2. (a) Lowest-order contributions to the susceptibility, due to virtual excitation of a single longitudinal paramagnon L; the dashed line represents the point interaction I; (b) the same as (a) for the transverse paramagnons T; (c) the usual dynamically screened Coulomb interaction (dashed line) $\equiv 4\pi e^2/q^2$.

Although our interest in this work is the magnetic susceptibility, an interesting observation can be made of the charge-density response (or screening) function. From Eqs. (6), (13), and (18), it is clear that the singular terms which contribute to $\chi(\vec{k})$ cancel exactly in the charge-density response function. We thus see that the screening function (to second order in I) does not exhibit the nonanalyticity of $\chi(\vec{k})$ at small $\vec{k}$ in accord with Ref. 9. Careful investigation of the higher order paramagnon corrections discussed above leads to the same conclusion.

The screening function is related to the energy of a nonuniform density Fermion system[8,9,11] while $\chi(\vec{k})$ refers to the nonuniform magnetic Fermi system[1,12-17] (see also below). If the system should show any preference to nonuniformity (see above), we expect it to prefer a magnetically ordered state.[13] It is thus important and consistent to have these nonanalyticities cancel in the screening function.

B. Electron gas

We next turn to a system whose interaction is long range; the opposite extreme of the point interaction considered above. With a finite-range interaction, the terms in Fig. 1(c) make a contribution and the simplicity of the point interaction is lost. For Coulomb interaction ($I \to 4\pi e^2/\vec{q}^2$), further difficulties arise at small $\vec{q}$. The latter are removed, in the usual way, by always replacing these bare interactions by dynamically screened ones [Fig. 2(c)].

A common set of graphs used for calculating the susceptibility $\chi(\vec{k})$ of systems with long-range interactions are those in the random-phase approximation.[16-17] Their lowest-order contributions are given in Fig. 2(a) with the wiggly line replaced by Fig. 2(c). (Note that there are no longer spin preferences in the interaction and a usual multiplicative factor of 2 accounts automatically for the two spins.)

The lowest-order contributions in Fig. 2(a) represent rigorously the high-density limit (HDL) for $\chi(\vec{k})$ of the electron gas. We thus confine our discussion for the HDL of the electron gas. The required analysis again closely follows that in Sec. II and Appendix A.

Define $V(\vec{q}, \xi_m)$ as

$$V(\vec{q}, \xi) = v(\vec{q})/[1 + v(\vec{q})\,\pi_0(\vec{q}, \xi)], \tag{29}$$

where $v(\vec{q}) = 4\pi e^2/\vec{q}^2$ and $\pi_0(\vec{q}, \xi) = 2P_0(\vec{q}, \xi)$ taking account of both spins. Next define $\overline{V}(\vec{q}, \xi_m) = V(\vec{q}, \xi_m) - v(\vec{q})$, since the dynamics of electron-hole scattering first appears in $\overline{V}(\vec{q}, \xi_m)$. We now replace in Eq. (14) the term $I^2 P_0(\vec{q}, \xi_m)$ by $\overline{V}(q, \xi_m)$. After some algebra we get for α'' in Eq. (26),

$$\alpha'' = \frac{\mu_B^2 e^4 m^3}{288\pi^4 \hbar^6 k_F^3}\,\frac{1}{(1 + e^2 m/\hbar^2 2\pi k_F)^2}\,. \tag{30}$$

In addition to the singular contribution α'', we can also give the regular coefficient α' rigorously in the HDL. Write $\alpha' = \alpha_0 + \alpha'_{\rm HD}$ and $\alpha'_{\rm HD} = \alpha'_{\rm ex} + \alpha'_c$, where α_0 is the Lindhard contribution (Sec. II), $\alpha'_{\rm ex}$ and α'_c are the exchange and correlation contributions, respectively, in the HDL. To get $\alpha'_{\rm ex}$, simply replace the wiggly line in Fig. 2(a) by the bare Coulomb interaction. It is given in Ref. 9 and is

$$\alpha'_{\rm ex} = \mu_B^2 5 m^2 e^2/72\pi^3 k_F^2 \hbar^4 \tag{31}$$

Some rearrangement of terms in Ref. 9 gives α'_c as $\alpha'_c = \mu_B^2(b' + b'' - b_2^3)$, where b' and b'' are given in Ref. 8 and b_2^3 in Ref. 9. Collecting terms yields

$$\alpha'_c = -\mu_B^2[0.0714/(2\pi)^3]\,m^2 e^2/k_F^2 \hbar^2\,. \tag{32}$$

These results complete the rigorous evaluation of both the regular and singular contributions to $\chi(\vec{k})$, of the electron gas, in the HDL and low temperature and long wavelength. The peculiar nonanalytical behavior of $\chi(\vec{k})$ at small $\vec{k}$, is thus also present in the spin susceptibility of the electron gas.

We observe that again, due to the sign of α'', for low T, the spin susceptibility reaches a maximum at some finite $\vec{k}$ for $T < T_0$. It is instructive to estimate this T_0. Using Eq. (27) and the above results, we get $T_0/T_F \sim 10^{-400}$ for $r_S = 1$. The degree of preference, in the electron gas, towards a state with "anomalous" spin correlations of finite $\vec{k}$ is clearly an academic one as regards practical realizable systems.

The mathematical origin of the singular structure of $\chi(\vec{k})$ for the electron gas is identical to that for the short-range case. At this juncture, we wish to speculate on its physical origin. The tendency of a system to be more susceptible to magnetic perturbations of finite $\vec{k}$ may be suggestive of spin-density waves. These are known to exist in the *ground state* of the electron gas within the Hartree-Fock approximation.[13] The addition of correlation is known to strongly inhibit (and likely remove[14,15]) the instability of the paramagnetic state.

The singular behavior of $\chi(\vec{k})$ at small $\vec{k}$ and at very low temperature shows that even with correlation included, the ground state of the electron gas still reflects "anomalous" correlations. However, it must be emphasized that the present analysis makes no prediction about the position or magnitude of the maximum, at finite $\vec{k}$, of $\chi(\vec{k})$. *If* this maximum should develop into a singularity, one could anticipate that, with decreasing temperature, a transition to a nonparamagnetic state might occur for the fully correlated electron gas. Further discussion of this point would be speculative and is beyond the scope of the present work.

In any event, even at temperatures where $\chi(\vec{k})$ is a decreasing function at small $\vec{k}$, the structure of $\chi(\vec{k})$ has important consequences both for fundamental reasons and practical ones, since *rigorously* the commonly assumed power expansions of $\chi(\vec{k})$ are invalid. We finally note that, as in the point-interaction case, the screening function can be shown *not* to exhibit these nonanaliticities at small $\vec{k}$.

We conclude by choosing an example to demonstrate some implications of the above results. Consider a system with large density variation (e.g., a surface of metal) to which we apply a weak magnetic field $H(\vec{r})$. From the extension of the functional density formalism to a paramagnetic system in the presence of a magnetic field, the energy of the system is given by[12]

$$E=\int[v(\vec{r})n(\vec{r})-H(\vec{r})m(\vec{r})]\,d\vec{r}+\tfrac{1}{2}e^2\int\frac{n(\vec{r})n(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}\,d\vec{r}'+G[n,m]\,, \tag{33}$$

where $G[n,m]$ is a universal functional of the density $n(\vec{r})$ and the magnetization $m(\vec{r})$ and the correct $m(\vec{r})$, $n(\vec{r})$ make Eq. (33) a minimum. In Eq. (33), $v(\vec{r})$ is some external potential coupling only to the density. For small $H(\vec{r})$, expand $G[n,m]$ in the form

$$G[n,m]=G[n]+\tfrac{1}{2}\int d\vec{r}\,d\vec{r}'G(\vec{r},\vec{r}';n)m(\vec{r})m(\vec{r}')\,, \tag{34}$$

where $G[n]$ is the kinetic and exchange-correlation functional in the absence of $H(\vec{r})$. The exchange-correlation contribution has been successfully approximated by a gradient expansion of $n(\vec{r})$.[8,9,11] We can similarly split $G(\vec{r},\vec{r}';n)$ into the kinetic contribution $G_S(\vec{r},\vec{r}';n)$ and exchange-correlation $G_{xc}(\vec{r},\vec{r}';n)$. $G_S(\vec{r},\vec{r}';n)$ can, in principle, be handled exactly. As in the case of $G[n]$, the fundamental problem resides in $G_{xc}(\vec{r},\vec{r}';n)$. Taking the limit of a uniform density, i.e., $G_{xc}(\vec{r},\vec{r}',n(\vec{r}))\approx G_{xc}(\vec{r}-\vec{r}',n_0)$, G_{xc} is given by

$$G_{xc}(\vec{k},n_0)=1/\chi(\vec{k})-1/\chi_0(\vec{k})\,, \tag{35}$$

where $G_{xc}(\vec{k},n_0)$ is the Fourier transfer of $G_{xc}(\vec{r}-\vec{r}',n_0)$ and $\chi_0(\vec{k})$ is the susceptibility of noninteracting electrons. Following the gradient expansion of $n(\vec{r})$, we attempt to approximate the exchange and correlation part of the second term in Eq. (3) by

$$\int[A_H^{xc}(n(\vec{r}))+B_H^{xc}(n(\vec{r}))|\nabla m(\vec{r})|^2]\,d\vec{r}\,. \tag{36}$$

Using Eqs. (26), (31), and (32), the expansion of Eq. (35) at small $\vec{k}$ is given by

$$\mu_B^{-2}k^2\left[\frac{e^2\pi}{k_F^4}\;\frac{-7-0.0714\times 9}{72}+\frac{\hbar^4\pi^4}{m^2k_F^2}\,\alpha''\ln\left(\frac{T}{T_F}+\frac{k^2}{k_0^2}\right)\right], \tag{37}$$

where α'' is given in Eq. (30). Comparison of Eqs. (36) and (37) shows that, unlike the gradient expansion of the $G[n]$ for the density response, the logarithmic term in Eq. (37) makes such a *rigorous* expansion in $m(\vec{r})$ impossible. Infact, $B_H^{xc}(n(\vec{r}))$ in Eq. (36) includes only the first term in Eq. (37), i.e.,

$$B_H^{xc}(n)=(e^2\pi/\mu_B^2k_F^4)\tfrac{1}{144}(-7-0.0714\times 9)\,. \tag{38}$$

In addition, we have to include the logarithmic contribution separately in Eq. (37). It is interesting to estimate the magnitude of this term for reasonable temperatures. Setting $T=1\,^\circ$K and $r_s=1$, and using Eq. (30), we find it to make a correction of only 5% to $B_H^{xc}(n(\vec{r}))$; at $T=100\,^\circ$K, the correction is $\approx 2.5\%$. The gradient expansion in $m(\vec{r})$ is thus not rigorous but it is adequate (to within $\simeq 5\%$) for applications to problems at reasonable temperatures.

In conclusion, we have rigorously demonstrated that the expansion in powers of $\vec{k}^2$ of the wave-number-dependent susceptibility $\chi(\vec{k})$ develops nonanalytical structure at sufficiently low temperature and that $\chi(\vec{k})$ attains its maximum value at *finite* $\vec{k}$. These facts contradict previously assumed behavior of $\chi(\vec{k})$, based on approximate calculations and they have several important implications (both in principle and for practical calculations) for the treatment of low-temperature many-fermion systems.

APPENDIX A

In this appendix, we provide some details of the evaluation of the singular contributions to $b^{\uparrow\downarrow}$ [see Eq. (17)] in the low-temperature limit. We shall first describe the derivation of Eq. (21) from Eq. (20) which we rewrite

$$\frac{\partial P_0(\vec{q},\xi)}{\partial\mu_0}=-[A(\vec{q},\xi)+\text{c.c.}]\,, \tag{A1}$$

where

$$A(\vec{q},\xi)=\frac{m}{4\pi^2\hbar^2q}\int_0^\infty dp\,p f'(\bar{\epsilon}_p)\left[\ln\left(\epsilon_q+\frac{\hbar^2pq}{m}+\xi\right)-\ln\left(\epsilon_q-\frac{\hbar^2pq}{m}+\xi\right)\right]. \tag{A2}$$

At low T, the integrand of Eq. (A2) is sharply peaked at $p=k_F$ and suggests that we introduce the simplifying approximation of replacing $\hbar^2pq/m$ by $(q/k_F)\bar{\epsilon}_p+\hbar v_Fq$ which correctly reproduces the integrand of Eq. (A2), and its derivative with respect to p, at $p=k_F$. We may then integrate over $\bar{\epsilon}_p$ and extend the lower limit to $-\infty$ with negligible error. This defines the approximant

$$A(\vec{q},\xi)\to\tilde{A}(\vec{q},\xi)\equiv\frac{m^2}{4\pi^2\hbar^2q}\int_\infty^\infty d\bar{\epsilon}_p\,f'(\bar{\epsilon}_p)\sum_\pm\ln\left(\epsilon_q+\xi\pm\frac{q\bar{\epsilon}_p}{k_F}\pm\hbar v_Fq\right)$$
$$=-\frac{m^2}{4\pi^2\hbar^4q}\left(\pi i+\int_{-\infty}^\infty d\bar{\epsilon}_p\,f(\bar{\epsilon}_p)\sum_\pm\frac{1}{\pm\bar{\epsilon}_p+(\epsilon_q\pm\hbar v_Fq+\xi)k_F/q}\right), \tag{A3}$$

where ξ is taken to be in the upper half of the complex ξ plane. The integral may be evaluated in terms of the diagamma function $\psi(z)=(d/dz)\ln\Gamma(z)$, where $\Gamma(z)$ is the usual gamma function, by deforming the contour to enclose only poles of the Fermi function in the complex $\bar{\epsilon}_p$ plane. Taking care with convergence questions, we find

$$\tilde{A}(\vec{q},\xi)=-\frac{m^2}{4\pi^2\hbar^4 q}\sum_{\pm}(\pm)\psi\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi i q}Z_{\pm}\right), \quad \text{(A4)}$$

where $Z_{\pm}=\epsilon_q \pm \hbar v_F q+\xi$. It is important to note that $A(\vec{q},\xi)$ and similar quantities which enter are regular functions of $Z_{\pm}$ for $T\neq 0$. The virtue of having explicit forms, such as Eq. (A4), is that required limiting cases when either or both of $Z_{\pm}$ or k_BT are small can be treated. For this purpose, Eq. (A1) may be replaced by

$$\frac{\partial\tilde{P}_0(\vec{q},\xi)}{\partial\mu_0}\equiv\frac{m^2}{2\pi^2\hbar^4 q}\operatorname{Re}\sum_{\pm}(\pm)\psi\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi i q}Z_{\pm}\right). \quad \text{(A5)}$$

These methods also provide a suitable approximant to the sums over repeated propagators which appear in $b^{\uparrow\downarrow}$. From Eq. (17), we require

$$J^{+}(\vec{q},\xi)\equiv \operatorname{tr}_{\vec{p}\zeta}\epsilon_{\vec{p}}\, g_{\vec{p}+\vec{q}}(\zeta_n+\xi)\frac{\partial^3}{\partial\mu_0^3}g_{\vec{p}}(\zeta_n)$$
$$=\operatorname{tr}_{\vec{p}}\left(\frac{f'''(\bar{\epsilon}_{\vec{p}})}{\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}+\xi}+\frac{3f''(\bar{\epsilon}_{\vec{p}})}{(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}+\xi)^2}+\frac{6f'(\bar{\epsilon}_{\vec{p}})}{(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}+\xi)^3}+6\frac{f(\bar{\epsilon}_{\vec{p}})-f(\bar{\epsilon}_{\vec{p}+\vec{q}})}{(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}+\xi)^4}\right)\epsilon_{\vec{p}}. \quad \text{(A6)}$$

It is important to note that the integrand of Eq. (A6) is not singular, even for $\xi=0$ and $\epsilon_{\vec{p}+\vec{q}}\to\epsilon_{\vec{p}}$, for $T\neq 0$, so the $\xi=0$ point can be isolated (and has zero weight) in sums such as

$$\operatorname{tr}_{\xi}F_T(\xi)=\beta^{-1}\sum_{m=-\infty}^{\infty}F_T(\xi_m)$$
$$=\frac{F_T(0)}{\beta}+\beta^{-1}\sum_{m=1}^{\infty}\left[F_T(\xi_m)+F_T(-\xi_m)\right]. \quad \text{(A7)}$$

Consequently, we require only $J(\vec{q},\xi)=J^{+}(\vec{q},\xi)+\text{c.c.}$ for ξ on the positive imaginary axis. The contributions to $J(\vec{q},\xi)$ which are dominant for $q\simeq 2k_F$ and $T\ll T_F$ arise from the first three terms in Eq. (A6). Integrating over angles and using Eq. (A2), we find the singular contributions to be given by

$$J_S(\vec{q},\xi)=2\mu_0\operatorname{Re}\left(\frac{\partial^2}{\partial\mu_0^2}+3\frac{\partial^2}{\partial\xi\,\partial\mu_0}+3\frac{\partial^2}{\partial\xi^2}\right)A(\vec{q},\xi). \quad \text{(A8)}$$

Using Eq. (A4), the approximant to $J_S(q,\xi)$ is then given by

$$\tilde{J}_S(\vec{q},\xi)=\frac{m^2\mu_0}{2\pi^2\hbar^4 q}\operatorname{Re}\sum_{\pm}\left[C_{\pm}(\vec{q},\xi)\left(\frac{\beta k_F}{2\pi i q}\right)\psi'\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi i q}Z_{\pm}\right)+d_{\pm}(\vec{q},\xi)\left(\frac{\beta k_F}{2\pi i q}\right)^2\psi''\left(\tfrac{1}{2}+\frac{\beta k_F}{2\pi i q}Z_{\pm}\right)\right], \quad \text{(A9)}$$

where

$$C_{\pm}(\vec{q},\xi)=\pm(m/\hbar^2k_F^2)(\nu_{\pm}-3\mp q/k_F) \quad \text{(A10)}$$

and

$$d_{\pm}(q,\xi)=\mp\left[\nu_{\pm}^2+(3\pm 2q/k_F)\nu_{\pm}+3(1\pm q/k_F)+q^2/k_F^2\right], \quad \text{(A11)}$$

with $\nu_{\pm}=(m/\hbar^2k_F^2)Z_{\pm}$. It can be verified that all of the above results correctly reproduce the zero temperature results [the regular terms not written explicitly in Eq. (A8) can also be included] on making use of asymptotic expansions of the polygamma functions such as $\psi(Z)\to\ln Z+\cdots$ and $\psi'(Z)\to 1/Z+\cdots$ for $|Z|\to\infty$.

Using the above results, we can now evaluate $b_S^{\uparrow\downarrow}$ at low T. The singular contributions to Eq. (17) arise from the vicinity of $q\simeq 2k_F$ as can be anticipated from the structure of Eqs. (A5) and (A9). Collecting various factors, we have

$$b_S^{\uparrow\downarrow}=(m^2k_F^2I^2/576\pi^7\hbar^6)K_S^{\uparrow\downarrow}, \quad \text{(A12)}$$

where

$$K_S^{\uparrow\downarrow}=\operatorname{Re}\int^{*}dq\int_0^{\infty}d\phi\,\Theta_q(\phi)\sum_{\pm}\left[C_{\pm}(2k_F,0)\frac{\beta}{4\pi i}\psi'(\gamma_{\pm})+d_{\pm}(2k_F,0)\left(\frac{\beta}{4\pi i}\right)^2\psi''(\gamma_{\pm})\right]. \quad \text{(A13)}$$

The asterisk indicates that the q integration is limited to $|q-2k_F|\leq q_C$ where q_C/k_F is small but finite and independent of temperature. Notation has been condensed slightly by defining $\xi=i\phi$,

$$\gamma_{\pm}=\tfrac{1}{2}+\beta(\epsilon_q\pm\hbar v_F q)/4\pi i+\beta\phi/4\pi,$$

and $\Theta_q(\phi)=\operatorname{Re}[\psi(\gamma_+)-\psi(\gamma_-)]$. We have set $q=2k_F$ and $\xi=0$ at various points [e.g., in the coefficients

$C_\pm(q,\xi)$ and $d_\pm(q,\xi)$] where there is manifestly no singular behavior. Also, we have indicated an explicit procedure for the evaluation, at low T, of sums such as Eq. (A7) in cases where $F_T(\xi)$ does not have a regular function of ξ as its $T\to 0$ limit.

$$2\beta^{-1}\sum_{m=1}^{\infty} F_T(\xi_m) \to \frac{1}{\pi}\int_0^\infty d\phi\, F_T(i\phi). \tag{A14}$$

This is a well-known procedure[3] in cases where $\lim F_T(i\phi)$ as $T\to 0$ is regular for all ϕ or develops, *at most*, a simple pole as a function of ϕ. In the case of present interest, higher order singularities are developed at low T. However, Eq. (A14) is still valid for determination of the leading singularity *provided* the $T\to 0$ limit is considered only at an appropriate point *after* integration over ϕ. The importance of having constructed *explicit* representations for the functions involved should not be underestimated as it permits one to follow through the above limiting procedure in detail.

Returning to Eq. (A13), we first integrate by parts over ϕ to get

$$K_S^{\uparrow\downarrow} = \int^* dq \sum_\pm \left[-C_\pm(2k_F,0)\Theta_q(0)\,\mathrm{Im}\psi\left(\tfrac{1}{2} + \frac{\beta}{4\pi i}(\epsilon_q \pm \hbar v_F q)\right) + d_\pm(2k_F,0)\Theta_q(0) - \frac{\beta}{4\pi}\mathrm{Re}\psi'\left(\tfrac{1}{2} + \frac{\beta}{4\pi i}(\epsilon_q \pm \hbar v_F q)\right) \right.$$
$$\left. - \int_0^\infty d\phi\, \Theta_q'(\phi)\left(C_\pm(2k_F,0)\,\mathrm{Im}\psi(\gamma_\pm) - d_\pm(2k_F,0)\frac{\beta}{4\pi}\mathrm{Re}\psi'(\gamma_\pm)\right)\right]. \tag{A15}$$

The first term above is finite since $\mathrm{Im}\psi$ is bounded (and an odd function of $q-2k_F$) while $\Theta_q(0)$ has only an integrable logarithmic singularity for $T\to 0$. However, the second term above will have a large contribution since

$$\lim_{T\to 0}\frac{\beta}{4\pi}\mathrm{Re}\psi'\left(\tfrac{1}{2} + \frac{\beta}{4\pi i}(\epsilon_q - \hbar v_f q)\right) = \frac{\pi}{\hbar v_F}\delta(q-2k_F). \tag{A16}$$

The corresponding quantity with $\epsilon_q - \hbar v_F q$ replaced by $\epsilon_q + \hbar v_F q$ has zero as its $T\to 0$ limit. Noting that

$$\Theta_q(0) \xrightarrow[q\to 2k_F]{} \mathrm{Re}\left[\psi\left(\tfrac{1}{2} + \frac{2\beta\mu_0}{\pi i}\right) - \psi(\tfrac{1}{2})\right] \xrightarrow[T\ll T_F]{} \ln\beta\mu_0 + \cdots, \tag{A17}$$

we find that the net contributions of the first two (partially integrated) terms in Eq. (A15) is

$$(K_S^{\uparrow\downarrow})_1 = (\pi/\hbar v_F)\ln\beta\mu_0. \tag{A18}$$

Turning to the third term in Eq. (A15), we again note that $\mathrm{Im}\psi(\gamma_\pm)$ is bounded, so that its contribution is of order

$$\int^* dq \int_0^\infty d\phi\,\Theta_q'(\phi) = \int^* dq\,[\Theta_q(\infty) - \Theta_q(0)],$$

which is finite since the final q integrand develops only a logarithmic divergence for $T\to 0$. The same conclusion applies to the $d_+(2k_F,0)$ part of the fourth term in Eq. (A15) since $(\beta/4\pi)\mathrm{Re}\psi'(\gamma_+)$ is also bounded for all $\phi\geq 0$ and T. Thus the only remaining contribution to consider is

$$(K_S^{\uparrow\downarrow})_2 = \int^* dq \int_0^\infty d\phi \frac{\beta}{4\pi}\mathrm{Re}\psi'\left(\tfrac{1}{2} + \frac{\beta\Delta^- q}{4\pi i} + \frac{\beta\phi}{4\pi}\right)\frac{\beta}{4\pi}\mathrm{Re}\left[\psi'\left(\tfrac{1}{2} + \frac{\beta\Delta_q^+}{4\pi i} + \frac{\beta\phi}{4\pi}\right) - \psi'\left(\tfrac{1}{2} + \frac{\beta\Delta_q^-}{4\pi i} + \frac{\beta\phi}{4\pi}\right)\right], \tag{A19}$$

where we have used the fact [see Eq. (A11)] that $d_-(2k_F,0)=1$ and have defined $\Delta_q^\pm = \epsilon_q \pm \hbar v_F q$. The above has been written in detail to permit easy verification of the fact that, for $T\neq 0$, the q integrand (after integration over ϕ) is well defined even for $q=2k_F$. Once again, for reasons given above, the contribution from the term involving Δ_q^+ has no singularity. On the other hand, the contribution from the term involving the product of ψ' functions with argument containing Δ_q^- has a delicate singularity which may be appreciated by writing it in the form

$$(K_S^{\uparrow\downarrow})_2 = -\frac{2}{\hbar v_F}\int_0^{S_{\max}} dS \int_0^\infty dt\,[\mathrm{Re}\psi'(\tfrac{1}{2} + t + iS)]^2, \tag{A20}$$

where $S_{\max} = \beta\hbar v_F q/4\pi$ and variables of integration have been scaled by $\beta\phi/4\pi = t$ and $\beta\Delta_q^-/4\pi = S$ and all temperature dependence has been isolated in the upper limit of the S integration. Since this upper limit is large and since the S integrand varies as $1/S$ for large S, Eq. (A20) can be evaluated by elementary means using the asymptotic form $\psi'(Z)\to 1/Z$ for $|Z|\to\infty$, and we can readily obtain the leading term

$$(K_S^{\uparrow\downarrow})_2 = -(\pi/2\hbar v_F)\ln(\beta\hbar v_F k_F/4\pi). \tag{A21}$$

Combining Eqs. (A12), (A18), and (A21), we obtain

$$b^{\uparrow\downarrow} = (m^3 k_F I^2/1152\pi^6\hbar^6)\ln(\mu_0/k_B T) + \mathrm{const.} \tag{A22}$$

Finally, it should be pointed out that the above methods may be applied to verify explicitly that the terms dropped in the derivation of Eq. (A22) are indeed finite for $T\to 0$.

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[1]An extensive review of questions of principle and of a variety of calculations for many-electron systems [particularly of the static susceptibility $X(\vec{k}, \omega \to 0)$] is given by C. Herring, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic, New York, 1966), Vol. IV, p. 1.
[2]W. Marshall and S. W. Lovesey, *Theory of Thermal Neutron Scattering* (Oxford U. P., Oxford, 1971).
[3]J. M. Luttinger and J. C. Ward, Phys. Rev. 118, 1417 (1960).
[4]A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, N. J., 1973).
[5]C. Bloch, in *Studies in Statistical Mechanics*, edited by J. deBoer and G. E. Uhlenbeck (North-Holland, Amsterdam, 1965), Vol. III, p. 1.
[6]S. Doniach and S. Engelsberg, Phys. Rev. Lett. 17, 750 (1966).
[7]N. F. Berk and J. R. Schrieffer, Phys. Rev. Lett. 17, 433 (1966); see also J. R. Schrieffer, J. Appl. Phys. 39, 642 (1968).
[8]S. Ma and K. Brueckner, Phys. Rev. 165, 18 (1968).
[9]D. J. W. Geldart and M. Rasolt, Phys. Rev. B 13, 1477 (1976).
[10]W. Kohn and J. M. Luttinger, Phys. Rev. Lett. 15, 524 (1965); see also J. M. Luttinger, Phys. Rev. 150, 202 (1966).
[11]P. Hohenberg and W. Kohn, Phys. Rev. 136, B864 (1964).
[12]W. Kohn and L. J. Sham, Phys. Rev. 140, A1133 (1965).
[13]A. W. Overhauser, Phys. Rev. 128, 1437 (1962).
[14]D. R. Hamann and A. W. Overhauser, Phys. Rev. 143, 183 (1966).
[15]P. A. Fedders and P. C. Martin, Phys. Rev. 143, 245 (1965).
[16]R. Dupree and D. J. W. Geldart, Solid State Commun. 9, 145 (1971).
[17]U. von Barth and L. Hedin, J. Phys. C 5, 1629 (1972).