PHYSICAL REVIEW B VOLUME 15, NUMBER 3 1 FEBRUARY 1977

Spin-glass behavior from Migdal's recursion relations*

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(Received 29 July 1976)

The Edwards-Anderson model of a spin glass is studied by position-space renormalization-group techniques, using an inhomogeneous generalization of Migdal's approximate recursion relation. We treat the spin-1/2 Ising model with independently random nearest-neighbor interactions in dimensionalities $d = 2$, 3, and 4. The phase diagram, which is in qualitative agreement with mean-field results, exhibits paramagnetic, ferromagnetic, antiferromagnetic, and spin-glass phases. The spin-glass and paramagnetic phases meet along an extended second-order phase boundary, which terminates in two tricritical points. Critical and tricritical exponents are calculated. The spin-glass specific-heat exponent turns out to be large and negative, compatibly with recent experiments which show a rounded specific-heat anomaly.

I. INTRODUCTION

Edwards and Anderson[1] (EA) recently proposed a simple, microscopic Hamiltonian model to describe the qualitative features of the experimentally observed spin-glass transition,[2-4] a cusped susceptibility and a specific heat which appears smoothly rounded.[4] This behavior is understandable as a spin-glass critical point with exponents $\alpha_{SG} < -1$ (specific heat) and $\gamma_{SG}^F < 0$ (ferromagnetic susceptibility). The EA Hamiltonian consists of a set of classical spins with random exchange coupling, which we write, specialized to Ising spins $(\mu = \pm 1)$,

$$-\beta\mathcal{H} \equiv H[\{K\},\{h\}] = \sum_{\langle \vec{r},\vec{r}'\rangle} K(\vec{r},\vec{r}')\mu(\vec{r})\mu(\vec{r}') + \sum_{\vec{r}} h(\vec{r})\mu(\vec{r}), \tag{1}$$

where the sums run over the sites of a regular d-dimensional lattice. Each exchange bond K and magnetic field h is taken as an independently random variable with assigned probability distribution $P_1(K)$ or $P_2(h)$, respectively. In calculations below we shall specialize to nearest-neighbor coupling only and

$$\begin{aligned} P_1(K) &= p\delta(K-K_0) + (1-p)\delta(K+K_0), \\ P_2(h) &= \delta(h-h_0). \end{aligned} \tag{2}$$

The quenched random free energy per lattice site of (1) is the thermodynamic limit of the averaged partition function

$$\bar{f} = \lim_{N\to\infty} N^{-1}[\ln \mathrm{Tr}\, e^{H[\{K\},\{h\}]}]_{\mathrm{av}}, \tag{3}$$

where N is the number of lattice sites and the bracket $[\;]_{\mathrm{av}}$ denotes the configurational average $\int_{\text{bonds}} \prod dK\, P_1(K) \prod_{\text{sites}} dh\, P_2(h)$. $\bar{f}$ depends on the functions P_1 and P_2 and is often conveniently regarded as a function of their cumulant moments,[5] κ_n and μ_n, respectively. Thus, μ_1 is the analog of the pure-system magnetic field in that[6] $\partial\bar{f}/\partial\mu_1 = [\langle\mu(\vec{r})\rangle]_{\mathrm{av}}$, while a derivative with respect to μ_2 generates the spin-glass order parameter[7] (see below)

$$\begin{aligned} \frac{\partial\bar{f}}{\partial\mu_2} &= \tfrac{1}{2}[\langle\mu(\vec{r})^2\rangle - \langle\mu(\vec{r})\rangle^2]_{\mathrm{av}} \\ &= \tfrac{1}{2}\{1 - [\langle\mu(\vec{r})\rangle^2]_{\mathrm{av}}\}. \end{aligned}$$

Mean-field[1,8,9] (MF) and sphericalized[10] (SM) versions of the EA model in a uniform magnetic field have been solved exactly. A number of authors[11-13] have performed Monte Carlo calculations for a Gaussian distribution of nearest-neighbor (only) couplings K of zero mean $(\kappa_1 = 0)$. Domb[14] has discussed our model (2) (restricted to $p = \frac{1}{2}$) by series methods. Finally, there have been two renormalization-group calculations: Harris, Lubensky, and Chen[15] developed the Landau-Wilson analog of (1), found that it had a critical dimensionality at $d^* = 6$, and performed an ϵ expansion for the critical exponents in $6-\epsilon$ dimensions. Also, the Ising chain spin-glass transition was studied[16] in an exact transfer matrix formulation.

Phase diagrams are available from MF[1,9] and SM[10] calculations, which take infinite-ranged interactions of dominantly ferromagnetic sign $(\kappa_1 > 0)$. At $h = 0$ there are three phases: paramagnetic, for dominantly small K's [K_0 small in our model (2)]; ferromagnetic, for K's large and positive (K_0 large, $p \simeq 1$); and spin-glass, for large K's of mixed sign (K_0 large, $p \simeq \frac{1}{2}$). In the spin-glass phase there are long-ranged correlations; but, any individual spin is equally likely up or down in the configuration average, so the usual magnetization $[\langle\mu(\vec{r})\rangle]_{\mathrm{av}}$ vanishes, while the quantity $[\langle\mu(\vec{r})\rangle^2]_{\mathrm{av}}$ plays the role of an order parameter. Each pair of phases meets along an extended phase boundary; the three phase boundaries join at a tricritical point.[17]

Critical behavior at the paramagnetic-to-spin-glass phase boundary is of particular interest. MF[1,8,9] and SM[10] treatments show a cusped specific heat (corresponding to $\alpha_{SG} = -1$). However, Harris *et al.*[15] find $\alpha_{SG} = -1$ at $d = 6$ but $\alpha_{SG} < -1$ for $d < 6$. This latter would show up as a smoothly rounded specific heat, as is indeed observed in the Monte Carlo work[11-13] for $d = 2, 3$. The case of the (uniform ferromagnetic) susceptibility χ^F is less clear. Domb[14] proved that, provided the distribution of couplings is symmetrical $P_1(K) = P_1(-K)$ (i.e., $\kappa_{2n+1} = 0$), the paramagnetic (high-temperature) χ^F is entirely nonsingular and exactly equal to its value for uncoupled spins.[18] Extant calculations are all consistent with this result. This high-temperature behavior in conjunction with any low-temperature behavior with $\gamma_{SG}^F < 0$ would certainly produce a cusp. MF[1,8,9] and SM[10] results appear to have a horizontal tangent on the low-temperature side of the transition ($\gamma_{SG}^F = -2$?) but the Monte Carlo[11-13] data are more consistent with $-1 \leq \gamma_{SG}^F < 0$. When couplings are not symmetrical, Domb's[14] proof does not apply and one has only MF and SM results to go on. It remains unclear whether or not a non-mean-field treatment would restore the symmetry of χ^F about the transition. Finally, all methods agree in predicting that a finite uniform magnetic field rounds the susceptibility cusp, so it appears that the spin-glass transition is magnetically unstable.

We carry out in Sec. II a position-space renormalization-group treatment[19-21] of the model (1), (2) at $h_0 = 0$. To deal with a random system[22,23] by position-space methods one needs *inhomogeneous local* recursion relations[22,24] for the couplings. The quantity which transforms under the renormalization group is the joint distribution function for the coupling strengths.[22] To make the calculation tractable we make two simplifying approximations: First, (a) we employ the inhomogeneous generalization of Migdal's[25,26] approximate recursion relations. This approximation is certainly crude and gives exceedingly poor values of the pure-system critical exponents; however, it has the great advantage of being analytically tractable in all dimensionalities d. For a system with nearest-neighbor interactions the Migdal recursion relations have the additional important feature that, if the initial coupling strengths are statistically independent, then so are the renormalized coupling strengths (see Fig. 1). Thus, the full, joint probability distribution factorizes into a product over individual bonds. Second, (b) we assume that the *single-bond* coupling-strength distribution may at each iteration be adequately parametrized by the two-peaked form (2). This parametrization is always exact at pure-system fixed points and it holds for the spin-glass fixed point[16] in $d = 1$. It has been used with good numerical success in the $d = 2$ dilution problem.[27,28] Furthermore, the two variables p and K_0 suffice to parametrize the first two moments of the K distribution (κ_1 and κ_2), which serve to characterize the distributions used in other treatments.[1,8-13]

If these approximations are accepted, the remaining calculation is so simple that most of it can be carried out analytically. Our principal results are the phase diagrams for $d = 2, 3, 4$, displayed in Figs. 2–4. Our phase diagrams are quite similar to the mean-field results mentioned above. They extend the EA results symmetrically into a regime of dominantly antiferromagnetic coupling ($\kappa_1 < 0$, i.e., $p < 1/2$), where an antiferromagnetic phase and a new (para-antiferro-spin-glass) tricritical point appear. Table I gives exponents associated with the three independent critical fixed points: pure-system, spin-glass, and tricritical. While by no means quantitative, these results do exhibit many qualitatively reasonable features, such as a spin-glass specific-heat exponent $\alpha_{SG} < -1$. We hope that they will point the way to more quantitative calculations in the physical dimensionalities $d = 2, 3$.

II. FORMULATION AND RESULTS

In an inhomogeneous system the value K_l of each *individual local* coupling must be regarded as an

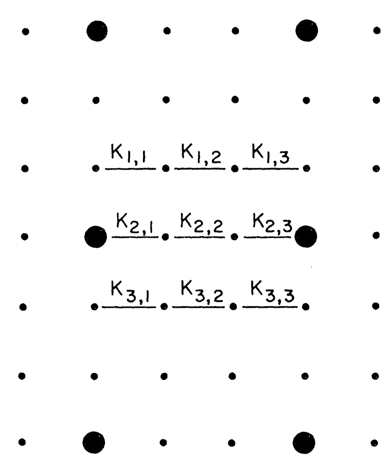

FIG. 1. Interactions which enter the Migdal recursion relation (6) in $d = 2$ with scale factor $b = 3$. The heavy dots represent lattice sites remaining after decimation. The renormalized horizontal bond between the two middle sites depends only on the labeled horizontal bonds $K_{i,j}$, $1 \leq i$, $j \leq 3$. These bonds contribute to no *other* renormalized bond in the Migdal approximation.

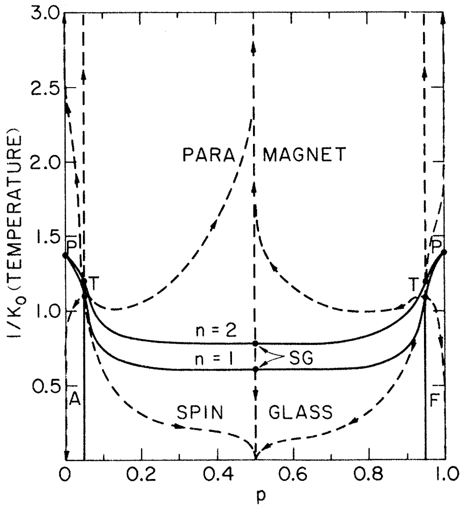

FIG. 2. Phase diagram of the nearest-neighbor Edwards-Anderson spin-glass model in $d=2$. p is the fraction of ferromagnetic couplings. The inverse coupling strength K_0^{-1} measures the temperature. Results from two different renormalization-group approximations are shown [$n=1$ and $n=2$, defined in (9)]. Phase boundaries are shown as solid lines. Light dashed lines sketch some representative renormalization-group trajectories. The four phases are paramagnetic, ferromagnetic (F), antiferromagnetic (A), and spin-glass. The phase boundaries are symmetric about $p=0.5$ because of the symmetry of the model (2) [but the representative trajectories are not drawn symmetrically]. Fixed points are indicated as pure-system (P), spin-glass (SG), and tricritical (T).

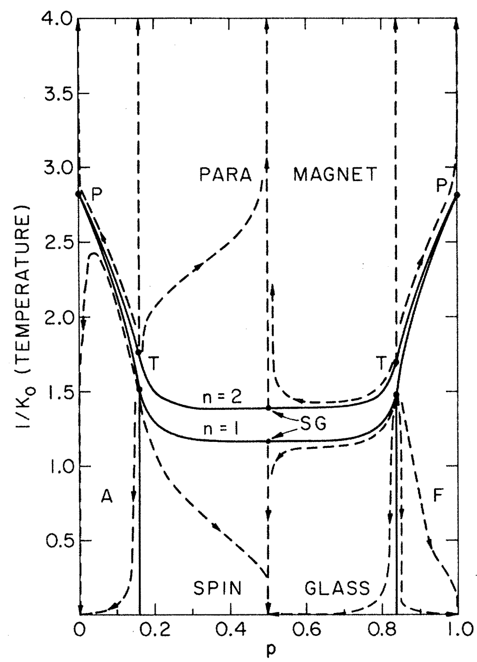

FIG. 3. Phase diagram of the nearest-neighbor Edwards-Anderson spin-glass model in $d=3$. Notation follows that of Fig. 2. Flows near the tricritical points (T) are unstable against perturbations of both p and K_0. Slopes of the phase boundaries are continuous across T. Note that the breadth of the spin-glass phase appears to decrease as d increases.

independent variable. Under renormalization-group iteration[19-21] each new, renormalized local coupling $K'_{l'}$ is a function of the full set of variables $\{K_l\}$ which we write

$$K'_{l'}=f_{l'}[K]. \tag{4}$$

In practice $K'_{l'}$ only depends appreciably on those couplings K_l located on the lattice in the near vicinity of $K'_{l'}$. In a quenched random system the probability that the couplings $\{K_l\}$ have a particular set of values is described by a joint probability distribution $\mathcal{P}[K]$. By virtue of the local recursion relations (4) the renormalized couplings are correspondingly distributed according to[22]

$$\mathcal{P}'[K']=\int \prod_l dK_l \mathcal{P}[K] \prod_{l'} \delta(K'_{l'}-f_{l'}[K]). \tag{5}$$

The fixed *distributions* $\mathcal{P}^*$ of (5) play the same role for random systems that fixed *points* play for pure systems.

In this paper we adopt [approximation (a) of Sec. I] the inhomogeneous generalization of a set of approximate recursion relations due to Migdal[25] and analyzed extensively by Kadanoff.[26] The Migdal recursion relations are closely related to "decimations"[20,33] and preserve nearest-neighbor-only interaction. (Henceforth we specialize to nearest-neighbor interaction and zero magnetic field $h_0=0$.) For Ising-like models they are exact at $d=1$ and give[34] the right $O(\epsilon)$ corrections[25,35] for the pure system in $d=1+\epsilon$. The Migdal recursion relations refer to a hypercubical lattice in d dimensions and give renormalized nearest-neighbor couplings which are each dependent on b^d original couplings according to

$$K'=\mathfrak{M}[K]\equiv\sum_{i=1}^{b^{d-1}}\tanh^{-1}\left(\prod_{j=1}^{b}\tanh K_{ij}\right), \tag{6}$$

as illustrated in Fig. 1. b is the scale factor by which the lattice spacing is increased at each iteration. In calculation we have chosen[36] $b=3$ (odd!) so as to maintain the symmetry between ferromagnetic and antiferromagnetic phases.[20] Note that a coupling K' in a given direction depends only on b^d parallel[37] couplings K. The single index l is here replaced by the pair (i,j). $i(1\leq i\leq b^{d-1})$ labels the particular "string" of b end-to-end couplings K, while the index j $(1\leq j\leq b)$ distinguishes the individual couplings along each string. Equation (6)

TABLE I. Fixed points and critical exponents for dimensionalities $d=2,3$, and 4. Results from two different renormalization-group approximations are shown [$n=1$ and $n=2$, defined in (9)]. Exact and series-based results for the pure system are shown by way of comparison. The exponent eigenvalues y_t and y_p are defined by Eq. (10). y_t is related to the specific-heat exponent α by $2-\alpha=d/y_t$. When $y_p>0$, it determines a crossover exponent ϕ by $\phi=y_p/y_t$.

	p^*	K_0^*	y_t	α	y_p	ϕ
(a) $d=2$						
Pure system (P)						
$n=1,2$	1	0.7218	0.738	−0.710	$-\infty$	
exact [a]		0.4407	1 [b]	0(ln)		
Spin glass (SG)						
$n=1$	0.5	1.6423	0.362	−3.52	$-\infty$	
$n=2$		1.2811	0.471	−2.24	$-\infty$	
Tricritical (T)						
$n=1$	0.048 95	0.9137	0.634	−1.15	0.486	0.761
$n=2$	0.95015	0.8469	0.669	−0.988		0.721
(b) $d=3$						
Pure system (P)						
$n=1,2$	1	0.3542	0.926	−1.24	$-\infty$	
series		0.2217 [c]	1.60 [b]	0.125 [d]		
Spin glass (SG)						
$n=1$	0.5	0.8558	0.665	−2.51	$-\infty$	
$n=2$		0.7218	0.738	−2.06	$-\infty$	
Tricritical (T)						
$n=1$	0.1602	0.6576	0.774	−1.88	0.739	0.955
$n=2$	0.8398	0.5822	0.816	−1.68		0.906
(c) $d=4$						
Pure system (P)						
$n=1,2$	1	0.1962	0.977	−2.10	$-\infty$	
series, etc.		0.1499 [c]	2 [b]	0(ln) [e]		
Spin glass (SG)						
$n=1$	0.5	0.5657	0.825	−2.85	$-\infty$	
$n=2$		0.4910	0.864	−2.63	$-\infty$	
Tricritical (T)						
$n=1$	0.7533	0.4985	0.860	−2.65	0.863	1.00
$n=2$	0.2467	0.4388	0.889	−2.50		0.971

[a] Reference 29.
[b] Derived via scaling.
[c] Reference 30.
[d] Reference 31.
[e] Reference 32.

is an approximation and not amenable to rigorous derivation; however, the picture behind it is as follows: Imagine decimating along a particular direction (i.e., summing out the $b-1$ spin layers intermediate between new lattice sites). If the couplings transverse to that direction were weak, it would be reasonable to calculate the new parallel coupling $K_\parallel'$ for each string according to the exact $d=1$ recursion relation (ignoring transverse couplings) $\tanh K_\parallel' = \prod \tanh K_\parallel$. Correspondingly, if the parallel couplings were strong, then the spins in each string would with high probability be aligned and it would be reasonable simply to sum the transverse couplings, $K_\perp' = \sum K_\perp$. The Migdal approximation consists in decimating successively in the d directions (for an over-all scale change of b), while always treating the transverse couplings as weak and the parallel couplings as strong. Thus, in deriving (6) we first decimate parallel to K', obtaining an effective coupling

$$\bar{K}_i = \tanh^{-1}\left(\prod_{j=1}^{b} \tanh K_{ij}\right)$$

along each string. These effective couplings are transverse to the remaining $(d-1)$ decimations, so the full coupling K' is just a sum over b^{d-1} strings

$$K' = \sum_{i=1}^{b^{d-1}} \bar{K}_i .$$

The reader is referred to Kadanoff's[26] paper for further discussion.

Generally a given coupling K_l enters into the determination of a number of distinct $K_{l'}'$'s [Eq.

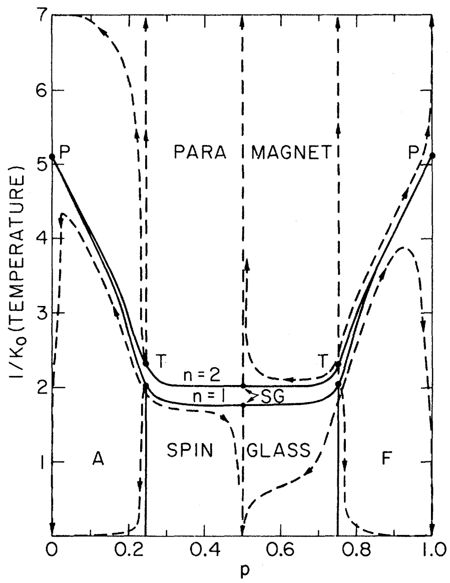

FIG. 4. Phase diagram of the nearest-neighbor Edwards-Anderson spin-glass model in $d=4$. Notation follows that of Fig. 2.

(4)]. Thus, even if all bonds were statistically independent in $\mathcal{P}[K]$, there would be correlations built into $\mathcal{P}'[K']$. It is a special feature[37] of the Migdal approximation (6) that each K_l enters into the determination of one and only one $K'_{l'}$. Therefore, it is true for the Migdal recursion relations without *additional* approximation that statistical independence of the K_l's implies statistical independence of the $K'_{l'}$'s, $\mathcal{P}'[K']$ factors, and the full content of the general transformation (5) is just

$$P'_1(K') = \int \prod_{i,j} [dK_{ij} P_1(K_{ij})] \delta(K' - \mathfrak{M}[K]). \quad (7)$$

Although (7) is enormously simpler than (5), it is still nontrivial for[16] $d>1$: Finding the fixed distributions P_1^* would, for example, require solving a highly nonlinear integral equation.[38] However, *if* $P_1(K)$ is of the form (2), it is not difficult to evaluate (7): The coupling along each string always has magnitude $\overline{K}_0 = \tanh^{-1}(\tanh^b K_0)$ and takes the value $+\overline{K}_0$ and $-\overline{K}_0$ with probabilities q_+ and q_- $(q_+ + q_- = 1)$, which for $b=3$ are $q_+ = p^3 + 3p(1-p)^2$ and $q_- = (1-p)^3 + 3p^2(1-p)$, respectively. When the b^{d-1} strings are added together [Eq. (6)], the resultant K' is an integer multiple of $\overline{K}_0$, $K' = k\overline{K}_0$, with $k = -b^{d-1}, -b^{d-1}+2, -b^{d-1}+4, \ldots, b^{d-1}$. The distribution (7) of renormalized couplings is

$$P'_1(K') = \sum_{k=-b^{d-1}}^{b^{d-1}} Q_l \delta(K' - k\overline{K}_0),$$

where (8)

$$Q_k = {}_{b^{d-1}}C_{n_+} q_+^{n_+} q_-^{n_-},$$

$k = n_+ - n_-$, and $b^{d-1} = n_+ + n_-$. ${}_{b^{d-1}}C_{n_+}$ is a binomial coefficient. Because (8) has a more complicated form than (2), we must either deal numerically[38] with (7) or adopt a second simplifying approximation. We choose the latter alternative [approximation (b) of Sec. I] and *force* (8) into the two-parameter form[27,28] (2) by writing

$$[P'_1(K')]_{\text{approx}} = p'\delta(K' - K'_0) + (1-p')\delta(K' + K'_0),$$

with[39]

$$p' = \sum_{k>0} Q_k, \quad (K'_0)^n = \overline{K}_0^n \sum_k Q_k |k|^n. \quad (9)$$

Note that $p' = p'(p)$, while $K'_0 = K'_0(p, K_0)$. These expressions define our renormalization group. The first simply assigns to $+K'_0$ all the weight in (8) at $K'>0$, while the second picks K'_0 by matching averages of $|K'|^n$. In calculations we have tried both $n=1$ and $n=2$. There are modest quantitative differences in exponents, etc. (see Table I); however, major qualitative features are comfortably insensitive to n.

Equation (9) generates renormalization-group flows in the space (p, K_0), which now characterizes the distribution function $P_1(K)$. We have plotted fixed points and phase boundaries in Figs. 2, 3, and 4, along with a few representative trajectories. Recall that p is the fraction of ferromagnetic couplings. The inverse coupling K_0^{-1} measures the temperature. Table I gives numerical values for the positions of the three independent critical fixed points (p^*, K_0^*) and their associated eigenvalue exponents[40] y,

$$b^{y_t} = \left.\frac{\partial K'_0}{\partial K_0}\right|_{p^*, K_0^*}, \quad b^{y_p} = \left.\frac{\partial p'}{\partial p}\right|_{p^*, K_0^*}. \quad (10)$$

All phase boundaries are precisely symmetrical about $p=0.5$: Because of the nearest-neighbor couplings and the hypercubical lattice structure, $P_1(K) \leftrightarrow P_1(-K)$ [$p \leftrightarrow 1-p$ in the restricted form (2)] is an exact symmetry of the model at $h=0$. Equation (9) respects this symmetry. The vertical phase boundaries are a consequence of the fact that p' is independent of K_0. Strict verticality would disappear (even in the Migdal approximation), if $P'_1(\cdot K')$ were not forced into the two-parameter form of approximation (b). The trend of the phase diagram with dimensionality is clear: At $d=1$, where the Migdal form (6) is exact and all critical behavior is at zero temperature,[16] the spin-glass phase occupies the full range $0<p<1$ and the ferromagnetic and antiferromagnetic phases reduce to points. As d increases, the width of the spin-glass phase shrinks.

The main physical content of Table I is the large *negative* value of the specific-heat exponent α_{SG} characterizing the paramagnetic-to-spin-glass transition. The predicted specific-heat curve (although singular) would appear smoothly rounded. Such behavior is compatible with present experiments[4] and quite consistent with the ϵ-expansion results,[15] which show $\alpha_{SG} = -1$ at $d=6$ and decreasing in lower dimensionality. Unfortunately this attractive conclusion is seriously weakened by the fact that our results show strongly negative α's for the pure system, as well. This represents a failure of approximation (a) [Eq. (6)]: at the pure-system fixed point $P_1(K) = \delta(K \pm K_0)$, so approximation (b) is exact! The Migdal approximation deteriorates rapidly as d departs from $d=1$. We must conclude that results for $d=3$ and $d=4$ are broadly qualitative, at best. It is encouraging, however, that α_{SG} increases between $d=2$ and $d=3$ (both absolutely and relative to the corresponding pure-system value), in agreement with the ϵ-expansion[15] trend.

It is interesting to speculate how the phase diagram might look in an approximation better than ours. For $4<d<6$ our phase diagram may be qualitatively valid; however, for $2<d<4$ there is a puzzle: It is known[41] that the pure system is unstable against a weak random perturbation, when $\alpha > 0$. This requires that the renormalization-group flows along the paramagnetic-ferromagnetic (and paramagnetic-antiferromagnetic) boundaries be reversed relative to ours. It is possible that flow simply proceeds to a tricritical point (T), which would then have only one unstable (strong) direction, instead of two; however, continuity would then suggest that the paramagnetic-spin-glass boundary should also flow towards T from the other side. Alternatively, there might be one or more additional fixed points and correspondingly more complex flows. Illumination of this point will require a substantial improvement over the level of accuracy we have been able to achieve.

Note added in proof. Dr. A. P. Young[42] and Dr. S. Kirkpatrick[43] have kindly pointed out to us an interesting problem with our method in $d=2$ (but not $d>2$). Near $T=0$ one can analyze the *exact* Migdal recursion relations approximation (a) but *not* approximation (b). In this analysis the fixed point ($p^*=0.5$, $T^*=0$ in Fig. 2) of the spin-glass phase is marginally *unstable* (thermally). The meaning of this instability is unclear. We note only that the Monte Carlo work[11] provides evidence that the model itself does have a stable spin-glass phase.

ACKNOWLEDGMENTS

We have benefited from numerous discussions with G. Grinstein. We gratefully acknowledge L. P. Kadanoff for an explication of the Migdal formula. Two of us (M.W. and J.C.) wish variously to thank C. L. Foiles and P. A. Beck for introductions to the complexities of real spin glasses. One of us (C.J.) has profited from discussions with J. Arao.

*Research supported in part by NSF Grant Nos. DMR72-03026 and DMR75-22241.

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[5] For example, if the direct moments of $P_2(h)$ are $m_n = \int dh\, h^n P_2(h)$, then $\mu_1 = m_1$, $\mu_2 = m_2 - m_1^2$, etc.

[6] In what follows $\langle\,\rangle$ denotes the usual thermal average at *fixed* coupling configuration.

[7] The formal expression for $P_2(h)$ in terms of its cumulant moments is

$$P_2(h) = \int \frac{dk}{2\pi} \exp(ikh) \exp\left(\sum_{n=1}^{\infty} \mu_n \frac{(-ik)^n}{n!}\right).$$

The general expression, $\partial \bar{f}/\partial \mu_n = 1/n!$ [nth cumulant moment of $\mu(\vec{r})$], follows by differentiating $\bar{f}$ and integrating by parts.

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[18] Domb, Ref. 14, also calculates the averaged second field derivative of the free energy. He finds this closely related to the *pure-system* Ising susceptibility.

Whether the corresponding Ising-like singularities are actually realized or whether they are preempted by different spin-glass singularities is not clear.

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[36]In $d=2$ for the pure system, the formal limit $b\rightarrow 1$ of the Migdal recursion relation gives the exact (Onsager) K_c (Refs. 25 and 26); however, the thermal exponents are as poor for $b\rightarrow 1$ as for our $b=3$ (see Table I). We need integer b to treat the inhomogeneity. Odd b allows symmetric incorporation of the antiferromagnetic phase. Our results are not strongly b dependent for $b=3,5,\ldots$; however, $b=2$ fails to give *any* spin-glass phase for $n=1,2$ [Eq. (9)].

[37]This is special to Migdal approximation. It is certainly false in general; but, it is very convenient (see below).

[38]In unpublished work on the related but somewhat simpler dilution problem, we have used Eq. (7) without parametrizing $P_1(K)$.

[39]Note that, because b is odd, k is never equal to zero.

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