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Tricritical dynamics near four dimensions

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Phenomenological models of dynamic tricritical and λ-line behavior, with an emphasis on ^{3}He-^{4}He mixtures, are found to deviate from mean-field theory below $d = 4$ and are studied to first order in $\epsilon = 4 - d$. At the tricritical point in the mixtures, the mass diffusion coefficient D varies as κ^x, where $x = (1/3)\epsilon + O(\epsilon^2)$, in disagreement with previous mode-coupling theories. The dynamic crossover scaling functions are determined, and are found to exhibit *three* distinct varieties of critical behavior. Although these results are consistent with present experiments, our model suffers from ambiguities similar to those discussed by Halperin, Hohenberg, and Ma for their model C. The dynamic properties of magnetic systems with tricritical points, such as FeCl$_2$, are also treated to order ϵ. Ambiguities inherent in the calculations for ^{3}He-^{4}He mixtures are not present in this case.

I. INTRODUCTION

The tricritical points which occur in ^{3}He-^{4}He mixtures and in Ising antiferromagnets have come under intense theoretical and experimental scrutiny in recent years. The static properties of these isolated multicritical points are now understood in terms of phenomenological scalings laws[1] and renormalization-group theory.[2] The most recent experiments, both in ^{3}He-^{4}He mixtures,[3-7] and in FeCl$_2$,[8,9] and dyprosium aluminum garnet,[10] appear to be in reasonable agreement with these theories. It has also become possible to calculate the universal crossover scaling functions associated with tricritical phenomena.[11,12] Again, the available experimental evidence[13,14] seems to be consistent[15] with the theory.

Dynamic properties at the tricritical point and along the λ line in ^{3}He-^{4}He mixtures have been investigated by several authors[16,17] using mode-coupling theories. The bulk of this paper will describe an analysis of a phenomenological model for the dynamics of ^{3}He-^{4}He mixtures by renormalization-group methods. Although we find agreement with earlier theoretical work in the vicinity of the λ line, our results disagree at the tricritical point.[18] We have also determined (to first order in $\epsilon = 4 - d$) the scaling functions for the transport coefficients, which describe the change over from tricritical to λ-line behavior. These functions are quite complicated and display over an intermediate region a new variety of critical behavior distinct from that found on either the λ line or at the tricritical point.

The discrepancy between our work and previous analysis should not be attributed to the mode-coupling calculational scheme itself, but to the incorrect assumption that the static fluctuations described by the free energy are Gaussian and hence uncoupled. We will demonstrate in Sec. IV that the tricritical fixed point studied previously in the mode-coupling approximation is unstable with respect to the dissipative mode couplings induced when the correct free energy is used.

Although this coupling leads to new and quite intriguing tricritical behavior, it also leads to problems. Both the mode-coupling theories[16] and the present work predict that a certain ratio of transport coefficients approaches unity, both along the λ line and at the tricritical point. A consequence is that the thermal conductivity remains finite in agreement with the experiments of Ahlers[19] near the λ line.[20] However, an additional consequence is that one of the diffusive modes relaxes more slowly than the order parameter. The existence of this mode, together with a dissipative mode coupling present for $\alpha > 0$, makes the model for ^{3}He-^{4}He mixtures resemble a dynamical system discussed by Halperin, Hohenberg, and Ma[21]—their "model C" with $2 < n < 4$.

This resemblance is disturbing, since the analysis of model C is plagued by ambiguities.[21] Similar difficulties appeared when we attempted to analyze the $O(\epsilon^2)$ corrections to the results presented here. These corrections are not obviously small, and may change our $O(\epsilon)$ results qualitatively. An optimistic view is that the naive exponents and scaling functions obtained in Sec. IV for ^{3}He-^{4}He mixtures may be correct for transport coefficients evaluated at zero frequency and wave number, but might not describe the finite $\vec{k}$ and ω properties correctly. A more pessimistic view is that, although previous mode-coupling work is likely to be incorrect at the tricritical point, our results may also be in error even to $O(\epsilon)$.
These difficulties are disturbing because the

conservation laws and "Poisson-bracket relations"[22] which control the dynamics of ^{3}He-^{4}He mixtures are presumably well understood and this system has been and can be extensively investigated by experiment. To the extent that the problems encountered here are more than formal, whatever new effects they lead to are expected to be larger in helium mixtures than in systems described by model C.

We have also studied the dynamics generated by models of magnetic tricritical behavior. The ordered phases of these systems are Ising-like, and the propagating modes which characterize the superfluid phase of ^{3}He-^{4}He mixtures are absent. Not surprisingly, the tricritical dynamics is significantly different[23] than that found for the mixtures.

Conservation of magnetization and energy in these systems can only be approximate, because of spin-lattice interactions.[21] Thus it is difficult to determine the precise model applicable to a given substance in the temperature range of interest. The analysis of Sec. V suggests that most systems will be described by either model A or model C of Ref. 21.

Fortunately at the tricritical point it should be possible to decide if a system has model A or model C dynamics experimentally because the two models are expected to have quite different exponents. The critical exponent z describing the critical slowing down of the order parameter at the tricritical point is[24]

$$\begin{aligned} &z=2, \quad \text{model A} \\ &z=2+\alpha_t/\nu_t=3, \quad \text{model C}. \end{aligned} \tag{1.1}$$

The dynamic tricritical crossover scaling functions for both models A and C are discussed in Sec. V. None of the Ising-like ($n=1$) models we have considered suffer from the difficulties inherent in the analysis of ^{3}He-^{4}He mixtures.

The organization of this paper is as follows. In Sec. II, we describe our model of ^{3}He-^{4}He mixtures in detail. The underlying static recursion relations and scaling functions are summarized in Sec. III. The dynamic recursion relations for the model are studied in Sec. IV. The appropriate dynamic crossover scaling functions are determined as well. Section V is devoted to magnetic tricritical systems. Section VI gives the experimental implications of the $O(\epsilon)$ results of Sec. IV and explains why they must be incorrect when higher-order terms are considered.

II. PHENOMENOLOGICAL MODEL

Although the phenomenological Langevin equations that are used here to describe ^{3}He-^{4}He mixtures around the tricritical point could be written down immediately in analogy with earlier work on pure helium and binary fluids,[25-27] it is necessary to make contact with thermodynamics and linearized hydrodynamics in order to compare with experiment. To accomplish this, we first use thermodynamic relations to determine linear combinations of the conserved densities whose equilibrium long-wavelength fluctuations are independent and Gaussian. Various susceptibilities which appear in our model can then be identified as thermodynamic derivatives. Next, we establish the form of the Josephson coupling between the order parameter and the conserved densities by comparison with linearized hydrodynamics below T_c. Finally, a comparison with linearized hydrodynamics *above* T_c will allow an identification of the three independent transport coefficients that characterize ^{3}He-^{4}He mixtures. The complete nonlinear equations which define the model can then be written down, and all experimentally accessible parameters identified.

The thermodynamic state of ^{3}He-^{4}He mixtures in the normal phase can be specified in terms of the pressure p, the entropy per unit mass σ, and the mass concentration c, where we have used the conventions of Khalatnikov.[28] We consider a thermodynamic potential W, such that the probability of fluctuations in equilibrium is proportional to e^{-W}, and

$$dW=\rho^{-1}dp+T\,d\sigma+\mu\,dc\,. \tag{2.1}$$

The chemical potential μ, conjugate to c, was denoted by Z/ρ in Ref. 28. The hydrodynamic equations near T_c imply that fluctuations in the pressure relax at the first sound frequency, which makes them irrelevant to the critical dynamics. The conserved densities σ and c relax much more slowly at long wavelengths, permitting us to work at constant pressure, $dp=0$. We have not included the conserved transverse momentum density, because its coupling to the other modes is irrelevant in the critical region.

The probability of small fluctuations about equilibrium can be found from (2.1) by expanding T and μ about their equilibrium values, where σ and c are intended to represent deviations from the equilibrium entropy and concentration. On defining

$$q(x)\equiv\rho_0\sigma(x)+\rho_0\frac{\partial\mu}{\partial T}\bigg|_{c,p}c(x)\,, \tag{2.2}$$

and allowing for spatially dependent fluctuations we obtain a contribution to the fluctuating part of W;

$$W_1=\frac{1}{2}\int d^dx\left(\rho_0^{-1}\frac{\partial T}{\partial\sigma}\bigg|_{c,p}q^2(x)+\rho_0\frac{\partial\mu}{\partial c}\bigg|_{T,p}c^2(x)\right). \tag{2.3}$$

The equilibrium values of quantities such as ρ, σ, and c will be denoted ρ_0, σ_0, etc. Energy is measured in units of $k_B T_c$. We note that the fluctuations of $T(x)$ and $\mu(x)$ can be obtained from (2.3) by functional differentiation[29]:

$$\mu(x)=\rho_0^{-1}\frac{\delta W_1}{\delta c(x)}+\frac{\partial\mu}{\partial T}\bigg|_{c,p}\frac{\delta W_1}{\delta q(x)}, \tag{2.4a}$$

$$T(x)=\frac{\delta W_1}{\delta q(x)}. \tag{2.4b}$$

Higher-order terms in $q(x)$ and $c(x)$ as well as their gradients in (2.3) turn out to be irrelevant variables near the superfluid transition. We have not yet introduced the part of W which describes the order parameter.

The form of the coupling between $q(x,t)$ and $c(x,t)$ and the order parameter in the equations of motion will now be determined below T_c where it is a consequence of the linearized hydrodynamics.[28] From Khalatnikov's equations[28] (24.65) we can construct the slow mode that responds at the second sound frequency, and verify that the pressure does indeed relax at the frequency of first sound. Although we shall avoid the simplifying assumption $C_p/C_V=1$, the analysis is rather straightforward, since in the critical regime we need only work to lowest nontrivial order in the small parameter ρ_s/ρ_0. Neglecting dissipation, the hydrodynamic equations are

$$\frac{\partial^2\rho}{\partial t^2}=\nabla^2 p, \tag{2.5a}$$

$$\frac{\partial q}{\partial t}=\overline{\sigma}_0\rho_s\vec{\nabla}\cdot\vec{v}_s, \tag{2.5b}$$

$$\frac{\partial c}{\partial t}=(c_0\rho_s/\rho_0)\vec{\nabla}\cdot\vec{v}_s, \tag{2.5c}$$

$$\frac{\partial\vec{v}_s}{\partial t}=c_0\vec{\nabla}\mu+\sigma_0\vec{\nabla}T, \tag{2.5d}$$

where

$$\overline{\sigma}_0=\sigma_0-c_0\frac{\partial\sigma}{\partial c}\bigg|_{T,p}, \tag{2.6}$$

and $\vec{v}_s$ is the superfluid velocity. The eigenmodes of (2.5) which correspond to first and second sound are readily constructed by expanding ρ, μ, and T in the deviations from equilibrium $p(x,t)$, $q(x,t)$, and $c(x,t)$ and looking for excitations which vary as $e^{i\vec{k}\cdot(\vec{x}-\vec{u}t)}$. When $u=u_2$, the amplitude of p is a factor ρ_s smaller than either q or c. To lowest order, this slow mode only involves q, c, and $\vec{v}_s$. When $u=u_1$, the amplitudes of q and c are now a factor of ρ_s smaller than p, although there is a component of the fast mode proportional to $(C_p/C_V-1)v_s$. The crucial point is that the pressure is decoupled from second sound to lowest order, and relaxes instead at the first sound frequency.

As a result of the rapid relaxation of the pressure to its equilibrium value, we can neglect Eq. (2.5a) and work at constant pressure. The remaining equations in (2.5) can be written more symmetrically in terms of a potential,

$$W_2\equiv W_1+\frac{1}{2}\int d^dx\,\rho_s|v_s|^2. \tag{2.7}$$

Equations (2.5b)–(2.5d) now take the form

$$\frac{\partial q}{\partial t}=\overline{\sigma}_0\vec{\nabla}\cdot\frac{\delta W_2}{\delta\vec{v}_s}, \tag{2.8a}$$

$$\frac{\partial c}{\partial t}=\frac{c_0}{\rho_0}\vec{\nabla}\cdot\frac{\delta W_2}{\delta\vec{v}_s}, \tag{2.8b}$$

$$\frac{\partial\vec{v}_s}{\partial t}=\overline{\sigma}_0\vec{\nabla}\frac{\delta W_2}{\delta q}+\frac{c_0}{\rho_0}\vec{\nabla}\frac{\delta W_2}{\delta c}, \tag{2.8c}$$

where (2.4) was used to eliminate μ and T. The couplings which appear in (2.8) will determine the form of the nondissipative couplings in our phenomenological model. The second sound velocity which results from (2.8) agrees with Khalatnikov's expression[28] to $O(\rho_s/\rho_0)$.

We now consider the diffusive hydrodynamics of q and c above T_c. Three independent transport coefficients appear, as is the case quite generally for binary mixtures.[30] Keeping with the notation of Ref. 30, the hydrodynamic equations read

$$\frac{\partial c}{\partial t}=D\frac{\partial c}{\partial\mu}\bigg|_{T,p}\rho_0^{-1}\nabla^2\frac{\delta W_1}{\delta c}+k_T D\nabla^2\frac{\delta W_1}{\delta q}, \tag{2.9a}$$

$$\frac{\partial q}{\partial t}=\left(\kappa_{\mathrm{Tr}}+\rho_0 D k_T^2\frac{\partial\mu}{\partial c}\bigg|_{T,p}\right)\nabla^2\frac{\delta W_1}{\delta q}+k_T D\nabla^2\frac{\delta W_1}{\delta c}, \tag{2.9b}$$

where the thermal conductivity κ_{Tr} was denoted by κ in Refs. 27 and 30. (We wish to reserve κ to designate the inverse correlation length.) Again, the form of Eq. (2.9) will guide our construction of a phenomenological model. We note that the eigenfrequencies of (2.9) agree with Eq. (15) of the paper by Griffin.[31]

A complete phenomenological model of ^{3}He-^{4}He mixtures near the λ transition must include the correct free-energy functional,[21] all dissipative terms permitted by symmetry, and the nondissipative couplings implied by the "Poisson-bracket" relations.[25,26] One must then verify that no relevant operators have been overlooked. The model we propose is

$$W=\int d^d x\,\left(\tfrac{1}{2}r_0|\psi|^2+\tfrac{1}{2}|\vec{\nabla}\psi|^2+\tilde{u}_0|\psi|^4+\tfrac{1}{2}\chi_0^{-1}c^2+\gamma_0 c|\psi|^2+\tfrac{1}{2}C_0^{-1}q^2\right), \tag{2.10a}$$

$$\frac{\partial\psi}{\partial t}=-2\Gamma_0\frac{\delta W}{\delta\psi^*}-ig_{1,0}\psi\frac{\delta W}{\delta q}-ig_{2,0}\psi\frac{\delta W}{\delta c}+\zeta, \tag{2.10b}$$

$$\frac{\partial c}{\partial t} = \lambda_0 \nabla^2 \frac{\delta W}{\delta c} + L_0 \nabla^2 \frac{\delta W}{\delta c} + 2 g_{2,0} \mathrm{Im}\left(\psi^* \frac{\delta W}{\delta \psi^*}\right) + \theta , \tag{2.10c}$$

$$\frac{\partial q}{\partial t} = K_0 \nabla^2 \frac{\delta W}{\delta q} + L_0 \nabla^2 \frac{\delta W}{\delta c} + 2 g_{1,0} \mathrm{Im}\left(\psi^* \frac{\delta W}{\delta \psi^*}\right) + \varphi . \tag{2.10d}$$

The noise sources $\zeta(x,t)$, $\theta(x,t)$, and $\varphi(x,t)$ satisfy

$$\langle \zeta(1) \zeta^*(1') \rangle = 4 \mathrm{Re}\Gamma_0 \delta(1-1') , \tag{2.11a}$$

$$\langle \theta(1) \theta(1') \rangle = -2\lambda_0 \nabla^2 \delta(1-1') , \tag{2.11b}$$

$$\langle \varphi(1) \theta(1') \rangle = -2L_0 \nabla^2 \delta(1-1') , \tag{2.11c}$$

$$\langle \varphi(1) \varphi(1') \rangle = -2K_0 \nabla^2 \delta(1-1') , \tag{2.11d}$$

while all other noise correlations vanish. The superfluid transition is described by a complex order parameter ψ, and we will make the usual identification[32] of the gradient of the phase of ψ with the superfluid velocity $\vec{v}_s$.[33] A term proportional to $|\psi|^6$ should in principle be added to (2.10a),[1] but this is not needed to describe the tricritical behavior for $u_0 > 0$.[11] The nonlinear coupling γ_0 has been added to make the fluctuations in c diverge near four dimensions. If the parameter q in (2.10a) is identified with (2.2), thermodynamic considerations imply that its fluctuations are finite except at isolated points, e.g., $c_0 = 0$.[34] By integrating out c and q it is easy to verify that (2.10a) reproduces the correct static critical behavior of ψ with the bare Landau-Ginzburg-Wilson vertex[21];

$$u_0 \equiv \tilde{u}_0 - \tfrac{1}{2} \chi_0 \gamma_0^2 . \tag{2.12}$$

If only the quadratic terms were retained in (2.10a), we believe that (2.10) would be equivalent to the model of Refs. 16 and 17.

By applying a renormalization-group transformation to (2.10), one can integrate the equations of motion out of the critical regime to obtain the physical or renormalized susceptibilities and transport coefficients (see Secs. III and IV). Renormalized quantities can always be written as a "bare" quantity, i.e., one that explicitly appears in (2.10) after l iterations, plus all diagrams. The diagrams decrease in importance relative to the "bare" quantities, $r(l)$, $\lambda(l)$, etc., as one iterates the recursion relations. The renormalized quantities are the only ones that are measurable experimentally and by definition do not change with iteration except for factors which compensate for the length, time, and field rescalings. Such quantities will be denoted by a subscript R to emphasize that they are functions of parameters or coupling constants that need not be themselves experimentally accessible. In Refs. 26 and 27 physical susceptibilities and transport coefficients analogous to what we term "renormalized" were denoted without a subscript R but were typically functions of only temperature.

At this point one can identify the renormalized susceptibilities by comparing with (2.3):

$$\chi_R = \rho_0^{-1} \left.\frac{\partial c}{\partial \mu}\right|_{T,p} , \tag{2.13}$$

$$C_R = C_0 = \rho_0 \left.\frac{\partial \sigma}{\partial T}\right|_{c,p} . \tag{2.14}$$

Comparison with (2.9) gives the diffusion constants:

$$D = \lambda_R / \chi_R , \tag{2.15}$$

$$k_T = L_R / D , \tag{2.16}$$

$$\kappa_{\mathrm{Tr}} = K_R - L_R^2 / \lambda_R . \tag{2.17}$$

The renormalized kinetic coefficient Γ_R can be complex, but we require $\mathrm{Re}\Gamma_R > 0$. No additional kinetic coefficients can appear in the equation for ψ above T_c by symmetry. We identify the couplings $g_{1,0}$ and $g_{2,0}$ by comparison with (2.8) below T_c by allowing only fluctuations in the phase of $\psi = \psi_0 e^{i\varphi}$. Near $d = 4$, $\rho_s = k_B T_c m_4^2 \psi_0^2 / \hbar^2$, where m_4 is the mass of a ^{4}He atom, and

$$\mathrm{Im}\left(\psi^* \frac{\delta F}{\delta \psi^*}\right) = -\rho_s \left(\frac{\hbar^2}{m_4^2}\right) \nabla^2 \varphi \equiv -\rho_s \left(\frac{\hbar}{m_4}\right) \vec{\nabla} \cdot \vec{v}_s .$$

Comparing with (2.8) we obtain

$$g_{1,0} = k_B T_c \bar{\sigma} / \hbar , \tag{2.18}$$

$$g_{2,0} = k_B T_c c_0 / \rho_0 \hbar , \tag{2.19}$$

where $g_{1,0}$ and $g_{2,0}$ are unrenormalized coupling constants. A more complete derivation of the nondissipative couplings is given in Ref. 26.

III. STATIC RECURSION RELATIONS AND SCALING FUNCTIONS

A. Recursion relations

The static recursion formulas relevant to our model of ^{3}He-^{4}He mixtures were determined in Ref. 21. The variable $q(x,t)$ was not considered, but it does not affect the previous analysis because it is decoupled from the static order parameter and concentration fluctuations.

It is convenient to cast the recursion relations derived by Halperin *et al.*[21] into differential form. With this in mind, we consider the usual renormalization-group transformation and eliminate degrees of freedom with momenta in the range $e^{-l} \leqslant |\vec{q}| \leqslant 1$. We have fixed the cut off Λ at unity for convenience. In addition, we make a scale transformation on the basic variables:

$$x \to x' = e^{-l} x , \tag{3.1a}$$

$$\omega \to \omega' = \exp\left(\int_0^l z(l')\,dl'\right)\omega\,, \tag{3.1b}$$

$$\psi \to \psi' = \exp\left(\int_0^l a(l')\,dl'\right)\psi\,, \tag{3.1c}$$

$$c \to c' = \exp\left(\int_0^l c(l')\,dl'\right)c\,, \tag{3.1d}$$

$$q \to q' = \exp\left(\int_0^l e(l')\,dl'\right)q\,. \tag{3.1e}$$

The scale factor $c(l)$ should not be confused with the fluctuating concentration $c(x,t)$. If $z(l)$, $a(l)$, $c(l)$, and $e(l)$ are constants independent of l, (3.1) is just the transformation of Ref. 21, with $b = e^l$. However, there are advantages in allowing for this more general possibility.

With this rescaling, differential recursion relations follow immediately from the results of Ref. 21 on setting $b \approx 1 + \delta$ and expanding in δ:

$$dr(l)/dl = [d - 2a(l)]r(l) + 4(n+2)K_4 u(l)/[1 + r(l)]\,, \tag{3.2a}$$

$$du(l)/dl = [d - 4a(l)]u(l) - 4(n+8)K_4 u^2(l)\,, \tag{3.2b}$$

$$d\chi(l)/dl = \chi(l)[2c(l) - d + 2nK_4\gamma^2(l)\chi(l)]\,, \tag{3.2c}$$

$$d\gamma(l)/dl = \gamma(l)[d - 2a(l) - c(l) - 4(n+2)K_4 u(l) - 2nK_4\gamma^2(l)\chi(l)]\,, \tag{3.2d}$$

$$dC(l)/dl = C(l)[2e(l) - d]\,. \tag{3.2e}$$

The equations quoted here apply to a general n-component order parameter. We have included an equation describing the trivial renormalization of the $q(x,t)$ field, and have defined a constant geometrical factor,

$$K_4 = 1/8\pi^2\,. \tag{3.3}$$

The recursion formula for the coefficient of the gradient coupling in (2.10a) has not been shown, but it will remain fixed at unity provided we take

$$a(l) = \tfrac{1}{2}[d - 2 + \eta(l)]\,. \tag{3.4}$$

The function $\eta(l)$ is zero to $O(\epsilon)$, and reduces to the critical exponent η at a fixed point. To further simplify the analysis, we choose $c(l)$ and $e(l)$ to keep $\chi(l)$ and $C(l)$ fixed at their initial values:

$$c(l) = \tfrac{1}{2}d - nK_4\gamma^2(l)\chi(l)\,, \tag{3.5}$$

$$e(l) = \tfrac{1}{2}d\,. \tag{3.6}$$

The frequency rescaling factor $z(l)$ does not appear in the static recursion relations. Only three equations now remain, namely those for $r(l)$, $u(l)$, and

$$v(l) \equiv K_4\gamma^2(l)\chi(l)\,. \tag{3.7}$$

They are

$$dr(l)/dl = 2r(l) + 4(n+2)K_4 u(l)/[1 + r(l)]\,, \tag{3.8a}$$

$$du(l)/dl = \epsilon u(l) - 4(n+8)K_4 u^2(l)\,, \tag{3.8b}$$

$$dv(l)/dl = \epsilon v(l) - 8(n+2)K_4 v(l)u(l) - 2nv^2(l)\,. \tag{3.8c}$$

With $n = 2$, these equations control both critical and tricritical behavior in ^{3}He-^{4}He mixtures. Upon setting $n = 1$, they will be important in our discussion of magnetic tricritical behavior.

To calculate static and dynamic scaling functions, it is necessary to solve the system (3.8). The solution of (3.8b) is immediate:

$$u(l) = u_0 e^{\epsilon l}/Q(l)\,, \tag{3.9a}$$

$$Q(l) = 1 + 4(n+8)K_4 u_0(e^{\epsilon l} - 1)/\epsilon\,. \tag{3.9b}$$

Equation (3.8a) has been discussed at length previously,[11,35] but its full solution will not be needed here. The solution of (3.8c) is obtained in Appendix A, with the result

$$v(l) = \frac{4K_4(4-n)v_0 u_0 e^{\epsilon l}}{2nv_0 Q(l) + [4K_4(4-n)u_0 - 2nv_0]Q(l)^{(2n+4)/(n+8)}}\,. \tag{3.10}$$

Nelson and Fisher[36] have also considered the system (3.8) for $n = 1$, which arose in an analysis of metamagnetic tricritical behavior. Flows similar to theirs are found when we plot trajectories obtained from (3.9) and (3.10) in the u-v subspace. These are shown in Fig. 1 for $n = 2$.[37] Fixed points B and C lead to $n = 2$ static critical

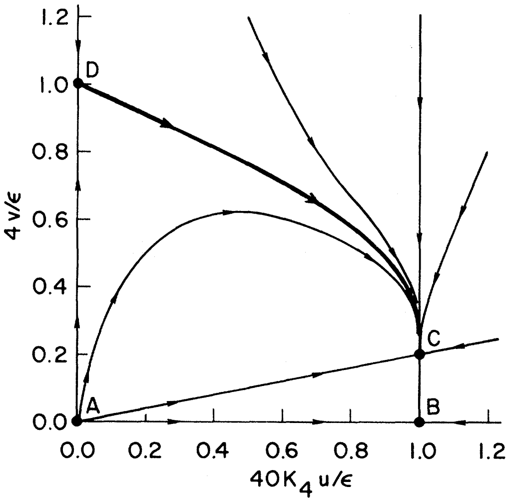

FIG. 1. Hamiltonian flows in the u-v subspace generated by (3.8b) and (3.8c) with $n = 2$. Fixed points A and D are Gaussian, while B and C lead to XY critical exponents. Crossovers from tricritical to λ-line behavior are dominated by the bold trajectory joining D to C.

exponents, while A and D lead to Gaussian critical behavior.

According to the hypothesis of Riedel and Wegner,[2] the change over from tricritical to λ-line properties is accompanied by crossover Hamiltonian flow from a Gaussian to the critical fixed point. It is clear from Fig. 1 that the *stable* tricritical and critical fixed points are D and C, respectively. The asymptotic trajectory of physical interest is thus the bold line joining D to C.

As emphasized by Halperin *et al.*,[21] the eigenvalue controlling the growth of perturbations about fixed point B in the direction of C is expected quite generally to be α/ν, where α and ν are the usual λ-line critical exponents. To first order in ϵ, $\alpha \to 0$ as $n \to 4$, and the fixed point C merges with B at $n = 4$, becoming unphysical for $n > 4$. Since α is known to be slightly negative for $n = 2$ in three dimensions,[38] we expect that only fixed points A, B, and D will appear in this case. Nevertheless, the relevant trajectory controlling crossover from critical to λ-line behavior would be a curve joining D to B, and *not* a curve from A to B.

B. A static crossover scaling function

Experimentally measured decay rates are determined by the ratio of a transport or Onsager kinetic coefficient to a static susceptibility.[22] Thus the mass diffusion constant D in ^{3}He-^{4}He mixtures is

$$D = \lambda_R / \chi_R \,, \tag{3.11}$$

where λ_R is a renormalized transport coefficient, and χ_R is the concentration susceptibility,

$$\chi_R \equiv \int d\vec{x} \langle c(\vec{x}, 0) c(\vec{0}, 0) \rangle \,. \tag{3.12}$$

Here, $\langle\ \rangle$ denotes the usual static thermodynamic ensemble average.

Although the thermodynamic functions relevant to tricritical phenomena have been calculated to $O(\epsilon)$ elsewhere,[11,35] we will sketch an illustrative calculation of the concentration susceptibility. This will be the only static quantity needed in the present work, and will serve as a model for the dynamic scaling function calculations in Sec. IV.

Thermodynamic functions can be calculated by integrating recursion relations until the resulting Hamiltonian is noncritical. Renormalization theory can be used to relate the quantity of interest to that same quantity calculated with the noncritical Hamiltonian. If we integrate until the "renormalized mass" $r(l^*)$ is of order unity, a Landau theory with only small fluctuation corrections should be sufficient. As demonstrated in detail in Ref. 35, this condition will be satisfied, provided we choose $l = l^*$ such that

$$t(l^*) \equiv t_0 e^{2l^*} / Q(l^*)^{(n+2)/(n+8)} = 1 \,. \tag{3.13}$$

The parameter t_0 is given by

$$t_0 = r_0 + 2K_4(n+2)u_0 \,, \tag{3.14}$$

and is zero on the λ line and at the tricritical point.[11,35] To a leading approximation, the solution of (3.13) is simply

$$e^{l^*} = t_0^{-1/2} \,. \tag{3.15}$$

The concentration susceptibilities calculated with renormalized and unrenormalized Hamiltonians are related by a scale transformation. On transforming (3.12) as indicated by (3.1), a homogeneity law is readily derived, namely,

$$\begin{aligned} \chi_R(t_0, u_0, v_0) &= \exp\left(dl - 2\int_0^l c(l')\,dl' \right) \\ &\quad \times \chi_R(t(l), u(l), v(l)) \\ &= \exp\left(2n \int_0^l v(l')\,dl' \right) \\ &\quad \times \chi_R(t(l), u(l), v(l)) \,. \end{aligned} \tag{3.16}$$

Evaluating the right-hand side at $l = l^*$, $\chi_R(t(l^*), u(l^*), v(l^*))$ can be replaced by its bare value, namely $\chi(l^*) \equiv \chi_0$ to first order in ϵ. Evaluating the prefactor analytically (see Appendix A), we obtain finally

$$\begin{aligned} \chi_R(t_0, u_0, v_0) = \chi_0 [& 2nv_0 Q(l^*)^{(4-n)/(n+8)} \\ & + 4K_4(4-n)u_0 - 2nv_0] / 4K_4(4-n)u_0 \,, \end{aligned} \tag{3.17}$$

where

$$Q(l^*) = 1 + 4K_4(n+8)u_0(t_0^{-\epsilon/2} - 1)/\epsilon \,. \tag{3.18}$$

A universal scaling function is obtained by taking the limit $t_0 \ll 1$, $u_0 \ll \epsilon$, at fixed v_0 of order ϵ:

$$\begin{aligned} \chi_R(t_0, u_0, v_0) \approx & \frac{n\chi_0}{2K_4(4-n)u_0} \\ & \times \{ [1 + 4K_4(n+8)u_0 t_0^{-\epsilon/2}/\epsilon]^{(4-n)/(n+8)} - 1 \} \,. \end{aligned} \tag{3.19}$$

This can be written in scaling form,

$$\chi_R \approx t_0^{-\epsilon/2} \psi(u_0 t_0^{-\epsilon/2}) \,, \tag{3.20}$$

where

$$\psi(x) = (4-n)^{-1} x^{-1} [(1+x)^{(4-n)/(n+8)} - 1] \,. \tag{3.21}$$

We have suppressed certain nonuniversal prefactors, and have chosen a convenient rescaling of the argument of the scaling function,

$$u_0 t_0^{-\epsilon/2} \to \epsilon u_0 t_0^{-\epsilon/2} / 4K_4(n+8) \,. \tag{3.22}$$

It follows from (3.20) and (3.21) that χ_R diverges as $t_0^{-\epsilon/2}$ on approaching the tricritical point with $u_0=0$, but eventually behaves as $t_0^{-\alpha}$ for $u_0\neq 0$. Here, α is the λ-line specific-heat index to first order in ϵ,

$$\alpha=\frac{1}{2}\,\frac{(4-n)}{n+8}\,\epsilon\,. \tag{3.23}$$

Similar crossover behavior will be displayed by the renormalized kinetic coefficients.

IV. DYNAMIC RECURSION RELATIONS AND SCALING FUNCTIONS

A. Recursion relations

It remains to determine recursion relations for the *dynamic* parameters Γ, λ, K, L, g_1, and g_2 appearing in our model by "thinning" the degrees of freedom in (2.10).[21] This may be accomplished by a number of diagrammatic formalisms.[39,40] The relevant Feynman diagrams are summarized in Appendix B. Here, we will pass immediately to the results.

It is convenient to display the recursion relations in differential form, keeping in mind the rescalings (3.1). In terms of the reduced parameters

$$f_1(l)=K_4g_1^2(l)/K(l)\mathrm{Re}\Gamma(l)\,, \tag{4.1a}$$

$$f_2(l)=K_4g_2^2(l)/\lambda(l)\mathrm{Re}\Gamma(l)\,, \tag{4.1b}$$

$$w_1(l)\equiv w_1'(l)+iw_1''(l)=\Gamma(l)C(l)/K(l)\,, \tag{4.1c}$$

$$w_2(l)\equiv w_2'(l)+iw_2''(l)=\Gamma(l)\chi(l)/\lambda(l)\,, \tag{4.1d}$$

$$w_3(l)=L(l)/\lambda^{1/2}(l)K^{1/2}(l)\,, \tag{4.1e}$$

the dynamic recursion formulas are

$$d\Gamma(l)/dl=\Gamma(l)[z(l)-2+A(l)/B(l)]\,, \tag{4.2a}$$

$$d\lambda(l)/dl=\lambda(l)[z(l)-2-4v(l)+\tfrac{1}{2}f_2(l)]\,, \tag{4.2b}$$

$$dK(l)/dl=K(l)[z(l)-2+\tfrac{1}{2}f_1(l)]\,, \tag{4.2c}$$

$$\begin{aligned}dL(l)/dl=L(l)[z(l)&-2-2v(l)\\&+f_1^{1/2}(l)f_2^{1/2}(l)/2w_3(l)]\,,\end{aligned} \tag{4.2d}$$

$$dg_1(l)/dl=g_1(l)[z(l)-\tfrac{1}{2}d]\,, \tag{4.2e}$$

$$dg_2(l)/dl=g_2(l)[z(l)-\tfrac{1}{2}d-2v(l)]\,. \tag{4.2f}$$

Two auxiliary functions appear in (4.2a), namely,

$$\begin{aligned}A(l)=[w_1'(l)/w_1(l)]\{&f_1(l)[1+w_2(l)]+f_2(l)[1+w_1(l)]-2f_1^{1/2}(l)f_2^{1/2}(l)w_3(l)\}\\&-4v(l)w_2(l)[1+w_1(l)]-4i[f_2(l)v(l)w_2'(l)]^{1/2}[1+w_1(l)]+4i[f_1(l)v(l)w_2'(l)]^{1/2}w_3(l)\,,\end{aligned} \tag{4.3a}$$

and

$$B(l)=[1+w_1(l)][1+w_2(l)]-w_3^2(l)\,. \tag{4.3b}$$

The kinetic coefficient $\Gamma(l)$, and hence $w_i(l)$, will in general be complex with real and imaginary parts, $w_i'(l)$ and $w_i''(l)$, respectively. The rescalings (3.4)–(3.6) chosen in Sec. III have been used, with, of course, $n=2$. The scale function $z(l)$ is still unspecified, but could be used to fix either $\mathrm{Re}\Gamma(l)$, $\lambda(l)$, $K(l)$, or $L(l)$ at its initial value. The simplicity of the equations for $g_1(l)$ and $g_2(l)$ is a consequence of gauge invariance, and holds to all orders in ϵ.[26]

The recursion relations (4.2) can be used to construct equations for the $f_i(l)$ and $w_i(l)$:

$$df_1(l)/dl=f_1(l)\{\epsilon-\tfrac{1}{2}f_1(l)-\mathrm{Re}[w_1(l)A(l)/B(l)w_1'(l)]\}\,, \tag{4.4a}$$

$$df_2(l)/dl=f_2(l)\{\epsilon-\tfrac{1}{2}f_2(l)-\mathrm{Re}[w_1(l)A(l)/B(l)w_1'(l)]\}\,, \tag{4.4b}$$

$$dw_1(l)/dl=w_1(l)[A(l)/B(l)-\tfrac{1}{2}f_1(l)]\,, \tag{4.4c}$$

$$dw_2(l)/dl=w_2(l)[A(l)/B(l)-\tfrac{1}{2}f_2(l)+4v(l)]\,, \tag{4.4d}$$

$$dw_3(l)/dl=\tfrac{1}{2}[f_1(l)f_2(l)]^{1/2}-\tfrac{1}{4}w_3(l)[f_1(l)+f_2(l)]\,. \tag{4.4e}$$

Fortunately, the rather formidable complexity of these five coupled differential equations [with complex $w_1(l)$ and $w_2(l)$] simplifies considerably in the limit of large l where one can work within a subspace characterized by

$$f_1(l)=f_2(l)\equiv f(l)\,, \tag{4.5}$$

$$w_2(l)=\infty\,, \tag{4.6}$$

$$w_3(l)=1\,, \tag{4.7}$$

with $w_1(l)$ real. The first relation is established by taking the ratio of (4.4a) and (4.4b). Equations (4.4c) and (4.4d) imply that w_2 grows more rapidly than w_1 and one can show that (4.6) is the only stable possibility. Equation (4.7) is a consequence of (4.5). These conclusions have been verified by numerically integrating (4.4) from arbitrarily chosen initial conditions. Exponents and crossover functions can be analyzed within the subspace of (4.5)–(4.7) but the corrections to scaling require (4.4). Relatively simple differential equations describe the evolution of the remaining parameters $f(l)$ and

$$w(l)\equiv w_1'(l)\,, \tag{4.8}$$

namely,

$$df(l)/dl=\epsilon f(l)+4v(l)f(l)-\tfrac{1}{2}f^2(l)-f^2(l)/[1+w(l)]\,, \tag{4.9}$$

$$dw(l)/dl=-\tfrac{1}{2}f(l)w(l)-4v(l)w(l)+f(l)w(l)/[1+w(l)]\,. \tag{4.10}$$

B. Homogeneity relations and critical exponents

The critical exponents and scaling functions which will follow from (4.9) and (4.10) can be summarized in terms of homogeneity relations. These can be derived for the physical transport coefficients by integrating the recursion relations out of the critical region, and then comparing the resulting "dressed" equations of motion with linearized hydrodynamics.[41] Denoting the physical transport coefficients at zero momentum and frequency by Γ_R, λ_R, K_R, L_R, we readily deduce the relations

$$\Gamma_R(t_0,u_0,v_0)=\exp\left(2l-\int_0^l z(l')\,dl'\right)\times\Gamma_R(t(l),u(l),v(l))\,, \tag{4.11a}$$

$$\lambda_R(t_0,u_0,v_0)=\exp\left(2l-\int_0^l z(l')\,dl'+4\int_0^l v(l')\,dl'\right)\times\lambda_R(t(l),u(l),v(l))\,, \tag{4.11b}$$

$$K_R(t_0,u_0,v_0)=\exp\left(2l-\int_0^l z(l')\,dl'\right)\times K_R(t(l),u(l),v(l))\,, \tag{4.11c}$$

$$L_R(t_0,u_0,v_0)=\exp\left(2l-\int_0^l z(l')\,dl'+2\int_0^l v(l')\,dl'\right)\times L_R(t(l),u(l),v(l))\,. \tag{4.11d}$$

To make further progress, we will evaluate the right-hand side at $l=l^*$ such that the correlation length $\xi(l^*)$ is of order unity. As discussed in Sec. III for the statics, the appropriate choice is $l^*\approx-\frac{1}{2}\ln t_0$, which will allow us to replace quantities like $\Gamma_R(t(l^*),u(l^*),v(l^*))$ by their "bare" values, in this case simply $\Gamma(l^*)$. The fluctuation corrections to this approximation are small and nonsingular and can be neglected to first order in ϵ. The "bare" quantities $\Gamma(l^*)$, $\lambda(l^*)$, $K(l^*)$, and $L(l^*)$ are related to Γ_0, λ_0, K_0, and L_0 by the recursion relations. Within the subspace determined by the restriction (4.5)–(4.7), these are

$$d\Gamma(l)/dl=\Gamma(l)\{z(l)-2-4v(l)+f(l)/[1+w(l)]\}\,, \tag{4.12a}$$

$$d\lambda(l)/dl=\lambda(l)[z(l)-2-4v(l)+\tfrac{1}{2}f(l)]\,, \tag{4.12b}$$

$$dK(l)/dl=K(l)[z(l)-2+\tfrac{1}{2}f(l)]\,, \tag{4.12c}$$

$$dL(l)/dl=L(l)[z(l)-2-2v(l)+\tfrac{1}{2}f(l)]\,. \tag{4.12d}$$

Equation (4.12) is readily integrated, and on substitution in (4.11) gives finally

$$\Gamma_R(t_0,u_0,v_0)=\Gamma_0\exp\left(-4\int_0^{l^*}v(l)\,dl+\int_0^{l^*}\frac{f(l)\,dl}{1+w(l)}\right), \tag{4.13a}$$

$$\lambda_R(t_0,u_0,v_0)=\lambda_0\exp\left(\frac{1}{2}\int_0^{l^*}f(l)\,dl\right), \tag{4.13b}$$

$$K_R(t_0,u_0,v_0)=K_0\exp\left(\frac{1}{2}\int_0^{l^*}f(l)\,dl\right), \tag{4.13c}$$

$$L_R(t_0,u_0,v_0)=L_0\exp\left(\frac{1}{2}\int_0^{l^*}f(l)\,dl\right). \tag{4.13d}$$

The equations (4.13) are fundamental to our discussion of exponents and scaling functions: The integrals in the exponentials will be dominated by fixed points of the recursion relations for $f(l)$ and $w(l)$. The stable fixed point of (4.9) and (4.10) when $v(l^*)$ is at its critical value (fixed point C of Fig. 1, $u_0\neq 0$) is

$$f^*=\epsilon,\quad w^*=\tfrac{3}{7};\quad v^*=\epsilon/20\,. \tag{4.14}$$

Consequently, as $t\to 0$ for fixed $u_0\neq 0$

$$\Gamma_R\approx\Gamma_0 e^{\epsilon l^*/2}\sim\kappa^{-\epsilon/2}\,, \tag{4.15}$$

$$\lambda_R\approx\lambda_0 e^{\epsilon l^*/2}\sim\kappa^{-\epsilon/2} \tag{4.16}$$

with equations for K_R and L_R similar to (4.16). Here κ is the inverse correlation length,

$$\kappa\equiv\xi^{-1}\sim t_0^{\nu}\,, \tag{4.17}$$

and we have made use of the requirement

$$\xi(l^*)=e^{l^*}\xi\approx 1\,, \tag{4.18}$$

which is equivalent to (3.13). Equation (4.16) appears to be in accord with mode coupling predictions for the λ line in ^{3}He-^{4}He mixtures.[16-18] Equations (4.15) and (4.16) hold to all orders in ϵ as for pure ^{4}He but note that there are no factors of $\kappa^{\alpha/2\nu}$ even near four dimensions in accordance with dynamic scaling.[26]

The stable fixed point of (4.9) and (4.10) when $v(l)$ is at its *tricritical* value (fixed point D in Fig. 1, $u_0=0$) is

$$f^*=4\epsilon/3,\quad w^*=0;\quad v^*=\epsilon/4\,. \tag{4.19}$$

The corresponding predictions for the renormalized kinetic and transport coefficients are

$$\Gamma_R\sim\kappa^{-\epsilon/3}\,, \tag{4.20}$$

$$\lambda_R\sim K_R\sim L_R\sim\kappa^{-2\epsilon/3}\,. \tag{4.21}$$

If we were able to assert that our recursion relations were nonsingular to all orders in ϵ it would follow that $\Gamma_R\lambda_R\sim\Gamma_R K_R\sim\Gamma_R L_R\sim\kappa^{-\epsilon}$ exactly. Equation (4.21) does *not* agree with previous work,[16-18] which predicts, e.g., $\lambda_R\sim\kappa^{-\epsilon/2}$ at the tricritical point. Results in agreement with the mode-coupling analysis are obtained if we (incorrectly) set $v(l)$ to zero from the start. Then (4.9) and (4.10) drive $f(l)$ and $w(l)$ to a fixed point with $f^* = \epsilon$, $w^* = 1$, and we obtain

$$\Gamma_R \sim \lambda_R \sim K_R \sim L_R \sim \kappa^{-\epsilon/2} , \tag{4.22}$$

both at the tricritical point and on the λ line. We emphasize that the *tricritical* behavior implied by (4.22) does not correspond to a stable fixed point of the recursion relations.

It is worth stressing that the inverse correlation length which enters (4.15), (4.16), (4.20), (4.21), and (4.22) behaves very differently on the λ-line and at the tricritical point. Indeed, the correlation length $\xi = \xi(t_0, u_0)$ exhibits complicated crossover behavior in its own right. Specifically, we expect[1,2] that $\xi \sim t_0^{-\nu}$, where

$$\nu = \tfrac{1}{2} + \tfrac{1}{10}\epsilon + \cdots \approx \tfrac{2}{3}, \quad u_0 \neq 0 \ (\lambda \text{ line}) , \tag{4.23a}$$

$$\nu = \tfrac{1}{2}, u_0 = 0 \ (\text{tricritical point}) . \tag{4.23b}$$

The scaling function which describes $\xi(t_0, u_0)$ in the vicinity of the tricritical point is defined by the relation

$$\xi(t_0, u_0) \approx t_0^{-1/2} \varphi(u_0 / t_0^{\epsilon/2}) , \tag{4.24}$$

where the function $\varphi(x)$ is[42]

$$\varphi(x) = (1 + x)^{1/5} \tag{4.25}$$

to first order in ϵ. We have normalized $\varphi(x)$ to be unity at $x = 0$, and have rescaled its argument according to (3.22).

C. Crossover scaling functions and "intermediate" λ-line behavior

The homogeneity or "matching" relations (4.13) can be used to obtain the physical transport coefficients themselves, in addition to the dynamic critical exponents. One need only determine quantities like $\exp(\frac{1}{2}\int_0^{l^*} f(l)\,dl)$, which can be calculated numerically, if necessary.

A plot of the function $f(l)$, obtained by integrating (4.9) and (4.10), is shown in Fig. 2. The initial conditions were chosen to be close to the stable dynamic tricritical fixed point. Although $f(l)$ remains initially at its tricritical value $f^* = \frac{4}{3}\epsilon$, and ultimately crosses over to its critical value $f^* = \epsilon$, there is a large flat intermediate region where $f(l)$ remains fixed at $f(l) \approx \frac{4}{5}\epsilon$! Since the transport coefficients λ_R, K_R, and L_R are determined by the integral of $f(l)$ from $l = 0$ to $l = l^* = -\frac{1}{2}\ln t_0$, it appears that they will display *three* distinct varieties of critical behavior. Interposed between the asymptotic tricritical and critical regimes is critical behavior controlled by a third fixed point:

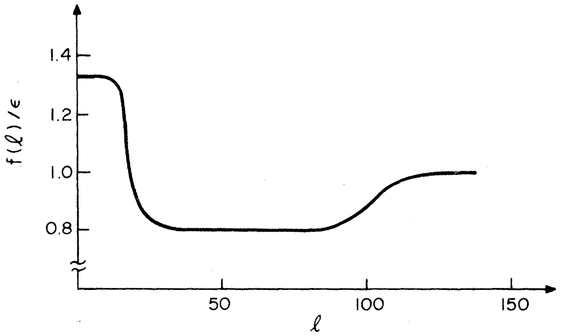

FIG. 2. Plot of the dynamic parameter $f(l)$ vs l for initial conditions close to the stable tricritical fixed point: $f_0 = 1.3333$, $w_0 = 10^{-5}$, $v_0 = 0.2499$, and $u_0 = 10^{-7}$. Although $f(l)$ remains initially close to its tricritical value $4\epsilon/3$, and eventually locks into its λ-line value ϵ at large l, there is a large intermediate region where $f(l) \approx 4\epsilon/5$. The region between $l \approx 30$ and $l \approx 90$ is dominated by a third fixed point, which controls "intermediate" λ-line behavior.

$$f^* = \tfrac{4}{5}\epsilon, \quad w^* = 0; \quad v^* = \epsilon/20 . \tag{4.26}$$

Although this fixed point is unstable, trajectories connecting the tricritical and critical fixed points approach it arbitrarily closely as $u_0 \to 0$. In the regime dominated by this fixed point we find effective exponents;

$$\Gamma_R \sim \kappa^{-3\epsilon/5} , \tag{4.27}$$

$$\lambda_R \sim K_R \sim L_R \sim \kappa^{-2\epsilon/5} . \tag{4.28}$$

Our predictions for the dynamic critical exponents near the tricritical point in ^{3}He-^{4}He mixtures are summarized in Fig. 3. This shows the basic tricritical phase diagram in the (r_0, u_0) plane. The λ line is given by the condition $t_0 \equiv r_0 + 4K_4(n+2)u_0 = 0$ to first order in ϵ. The dashed lines separate regions of different dynamical behavior: In region I, the transport coefficients behave as in Eq. (4.21), while in region II they appear to diverge according to (4.28). As $t_0 \to 0$ for fixed u_0, one ultimately enters region III, where the behavior of the transport coefficients is given by (4.16). The boundary between regions I and II is given asymptotically by

$$u_0 / t_0^{\epsilon/2} = \text{const} . \tag{4.29}$$

The boundary between regions II and III is given by a relation of the form

$$u_0 / t_0^{\tilde{\varphi}} = \text{const} , \tag{4.30}$$

where it appears that $\tilde{\varphi} < \frac{1}{2}\epsilon$.

Static quantities, such as the concentration susceptibility and the correlation length, exhibit tricritical exponents in region I, and critical line exponents in regions II and III.[1,2] There is an intermediate regime only for dynamic quantities.

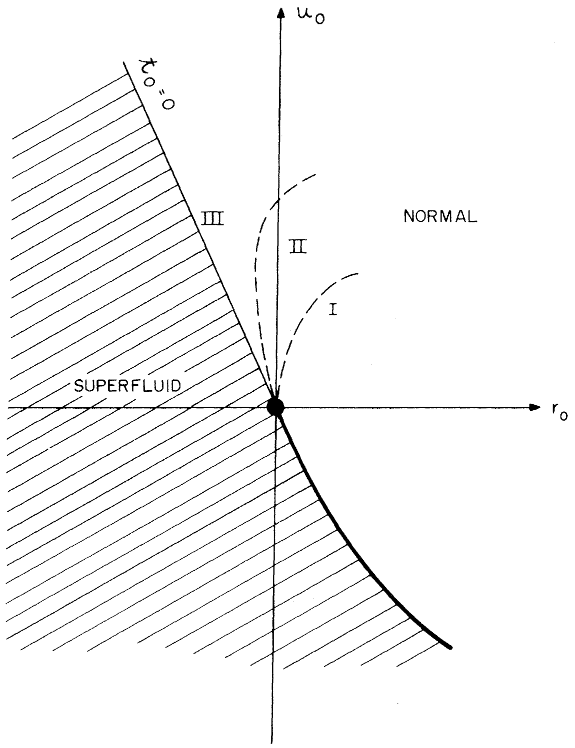

FIG. 3. Phase diagram for ^{3}He-^{4}He mixtures in the r_0-u_0 plane. To first order in ϵ, the λ line is given by the straight line $t_0=0$, and terminates at the tricritical point $(r_0,u_0)=(0,0)$. A six point coupling $|\psi|^6$ is necessary to describe the bold first-order line in the region $u_0<0$, which is not treated here. According to the theory to $O(\epsilon)$, three distinct varieties of dynamic behavior should be encountered in regions I, II, and III in the normal phase. Interposed between the tricritical region (I) and the λ-line region (III) is a new variety of dynamic critical behavior (region II).

The prediction $\tilde{\varphi} < \frac{1}{2}\epsilon$ was obtained by studying plots of $f(l)$ with different initial conditions. As $u_0 \rightarrow 0$, the point at which $f(l)$ crosses over from $\frac{4}{3}\epsilon$ to $\frac{4}{5}\epsilon$ moves to the right, but the width of the flat region where $f(l) \approx 0.8\epsilon$ grows more rapidly. If $\tilde{\varphi}$ were equal to $\frac{1}{2}\epsilon$, the flat region would be of fixed width. It thus appears that the dynamic scaling functions will cross over first from tricritical to "intermediate" λ-line behavior and only asymptotically attain their λ-line limit. The true λ-line properties will not appear in the universal scaling functions, i.e., in the limit $u_0, t_0 \rightarrow 0$; $u_0 t_0^{-\epsilon/2}$ finite.

The scaling functions for the first crossover can be determined analytically, because $w(l) \approx 0$ throughout the changeover from tricritical to "intermediate" λ-line behavior. Equation (4.9) for $f(l)$ now simplifies:

$$df(l)/dl \approx \epsilon f(l) + 4v(l)f(l) - \tfrac{3}{2}f^2(l) . \qquad (4.31)$$

The solution of this equation, as well as the quantity $\exp(\frac{1}{2}\int_0^{l^*} f(l)\,dl^*)$ is determined analytically in Appendix A. Analysis, similar to that shown explicitly in Sec. III B for the concentration susceptibility, then leads straightforwardly to a prediction for the transport coefficients in the limit $t_0, u_0 \rightarrow 0$:

$$\lambda_R \sim K_R \sim L_R \approx t_0^{-\epsilon/3}\,\Phi(u_0/t_0^{\epsilon/2}) , \qquad (4.32)$$

where

$$\Phi(x) = x^{-2/3}\{-x + \tfrac{5}{6}[(1+x)^{6/5} - 1]\}^{1/3} . \qquad (4.33)$$

The analogous equations for Γ_R are

$$\Gamma_R \approx t_0^{-\epsilon/6}\Psi(u_0/t^{\epsilon/2}) , \qquad (4.34)$$

and

$$\Psi(x) = \{-x + \tfrac{5}{6}[(1+x)^{6/5} - 1]\}^{2/3} / \{x^{1/3}[-1 + (1+x)^{1/5}]\} . \qquad (4.35)$$

Equations (4.33) and (4.35) are valid for all x, $0 \leq x < \infty$, to first order in ϵ. As usual, overall prefactors in the scaling functions have been suppressed, and the arguments of $\Phi(x)$ and $\Psi(x)$ have been rescaled according to (3.22).

V. MAGNETIC TRICRITICAL POINTS

Fortunately, the dynamics of Ising antiferromagnets in a magnetic field are, in principle, simpler that the dynamics of ^{3}He-^{4}He mixtures. Propagating modes are not expected to appear in the ordered phases of these systems.

A free-energy functional appropriate to these systems, which can exhibit tricritical points in strong enough magnetic fields,[1] has been derived by Nelson and Fisher.[36] This functional is identical to that used by Halperin *et al.*[21] for their models *C* and *D*, and also resembles the functional *W* used here. The field ψ now represents a one-component staggered magnetization, and *c* corresponds to a uniform magnetization. The coupling γ_0 was found to be proportional to the magnetic field, which was assumed to drive the effective four point coupling $u_0 = \tilde{u}_0 - \frac{1}{2}\chi_0\gamma_0^2$ to zero at the tricritical point.[36] An additional energy field can be easily incorporated into the analysis of Ref. 36. The resulting static recursion relations are just those discussed in Sec. III, evaluated with $n=1$.

Dynamics at these magnetic tricritical points of course depends crucially on the conservation laws. If the conservation of magnetization and energy is strongly broken by a large spin-lattice interaction, model A of Ref. 21 should be an appropriate description. Additional nonconserved modes have no effect on the order parameter relaxation.[21] Near the critical line, the renormalized kinetic coefficient Γ_R is expected to go slowly to zero as

$T \to T_c$.[21] To first order in ϵ, Γ_R is actually finite.[21] Repeating the analysis of Ref. 21 with $u_0 = 0$ one finds immediately that Γ should be finite and nonzero at the tricritical point to all orders in ϵ. Thus ψ should obey the conventional theory[22] of critical slowing down at the tricritical point, in agreement with recent Monte Carlo studies on a kinetic Ising model by Müller-Krumbhaar and Landau.[43]

If either magnetization or energy is conserved, it can be shown that the remaining fast mode has no effect on the critical dynamics.[21,26] The appropriate dynamic model is then model C with $n = 1$. We remind the reader that difficulties encountered in our model of ^{3}He-^{4}He mixtures only occur in model C for $2 < n < 4$.

Huber[23] has constructed a hydrodynamic theory of $FeCl_2$ assuming that both energy and magnetization are conserved. Our model of ^{3}He-^{4}He mixtures should be an appropriate description of this possibility, provided we take $g_1 = g_2 = 0$ and set $n = 1$. The corresponding recursion relations are just given by (4.2), where the factors $-4v(l)$ and $-2v(l)$ in Eqs. (4.2b) and (4.2d) must be replaced by $-2v(l)$ and $-v(l)$, respectively. Of course, we also set $f_1(l) = f_2(l) = 0$ and evaluate the function $v(l)$ from (3.10) with $n = 1$. The analysis of the resulting recursion relations is straightforward. Although the discussion is complicated by the existence of a redundant marginal operator, this operator has no effect on the dynamics. The critical and tricritical behavior is just that appropriate to model C.

Apparently, the dynamics of most Ising antiferromagnets in a magnetic field can be described by either model A or model C. As discussed in Ref. 21, it is difficult to distinguish the critical dynamics of model A and model C experimentally. In the former case, the exponent z which describes the slowing down of the order parameter is $z = 2 + O(\epsilon^2)$, while in the latter, $z = 2 + \alpha/\nu$. Because α/ν is rather small, it is difficult to distinguish these predictions.

Fortunately, the situation at the *tricritical* point of these models is rather different because of the large tricritical exponent $\alpha = \frac{1}{2}\epsilon$. As we have seen, model A is rather uninteresting at the tricritical point. To lowest order, the crossover scaling function which describes the renormalization of Γ is just a constant. Tricritical crossover in model C can be studied with the aid of a scaling relation for the renormalized kinetic coefficient Γ_R, similar to those derived here for ^{3}He-^{4}He mixtures:

$$\Gamma_R(t_0, u_0, v_0) = \Gamma_0 \exp - 2\int_0^{l^*} v(l)\, dl \, . \tag{5.1}$$

As usual, the upper limit of the integral is to be evaluated with $l^* \approx -\frac{1}{2}\ln t_0$. Equation (5.1) follows immediately, upon recasting the results of Ref. 21 in differential form. With the aid of Appendix A, (5.1) is readily evaluated:

$$\Gamma_R(t_0, u_0, v_0) = \frac{6K_4 u_0}{6K_4 u_0 - v_0 + v_0 Q(l^*)^{1/3}}\, \Gamma_0 \, . \tag{5.2}$$

We are led to a scaling prediction in the limit of small t_0 and u_0 with fixed v_0 of order ϵ:

$$\Gamma_R(t_0, u_0, v_0) \approx t_0^{\epsilon/2}\Upsilon(u_0/t_0^{\epsilon/2}) \, , \tag{5.3}$$

where

$$\Upsilon(x) = x/[(1+x)^{1/3} - 1] \, . \tag{5.4}$$

An overall (nonuniversal) v_0-dependent prefactor has been surpressed in (5.3), and the argument of $\Upsilon(x)$ has been rescaled as in (3.22).

According to (5.3) and (5.4), Γ_R goes to zero rather rapidly, $\Gamma_R \sim t_0^{\alpha_t}$, at the tricritical point, but varies more slowly near the critical line—the controlling exponent here is the Ising *critical* specific-heat index,[21] $\alpha = \frac{1}{6}\epsilon + O(\epsilon^2)$. One might hope that the rather large effect at the tricritical point could be observed experimentally. At the very least, the large differences in the predicted tricritical dynamics of models A and C, as summarized in Eq. (1.1), should allow these possibilities to be distinguished experimentally.

VI. ^{3}He-^{4}He MIXTURES IN THREE DIMENSIONS

A. Relation to model C

The experimental relevance of our results for magnetic tricritical phenomena is adequately described in the Introduction and Sec. V so we will concentrate here on the helium mixtures. Unfortunately, the implications of our $O(\epsilon)$ analysis in Sec. IV may be seriously modified by effects appearing at $O(\epsilon^2)$. These difficulties were first noted by Halperin, Hohenberg, and Ma in an analysis of model C in which a conserved energy mode is coupled to a nonconserved order parmeter.[21] For $2 < n < 4$ they found that on a scale corresponding to the nominal characteristic frequency of the order parameter, the energy relaxed more slowly yet the two modes, remained coupled. This suggests a violation of dynamic scaling in which the order parameter correlations do not scale with a single characteristic frequency. Formally, the existence of a mode that is slower than and coupled to the order parameter is manifested by nonanalytic recursion relations and a Feynman graph expansion that fails to exponentiate. These difficulties have not been fully resolved.

Our model is more complicated than model C in that there are two conserved densities and propagating modes below T_c, but the same difficulties are expected to appear. The results of Sec. IV can be used to calculate the eigenfrequencies of the response matrix of c and q. The transport coefficients are the sum of a regular and singular part:

$$\begin{pmatrix} \lambda_R, & L_R \\ L_R, & K_R \end{pmatrix} = \underline{A} + \kappa^{-y}\,\underline{B}\,, \tag{6.1}$$

where $\mathrm{Det}(\underline{B}) = 0$ because $w_3^* = 1$. The eigenfrequencies of

$$\underline{\Omega} = \begin{pmatrix} \lambda_R \chi_R^{-1}, & L_R C_R^{-1} \\ L_R \chi_R^{-1}, & K_R C_R^{-1} \end{pmatrix} k^2 \tag{6.2}$$

scale for $k = \kappa$ as $\kappa^{2+\alpha/\nu}$ and κ^{2-y}, while the order parameter relaxes as $\kappa^{2-\epsilon+y}$ with $y = \epsilon/2$ on the λ line and $y = 2\epsilon/3$ at the tricritical point. In the latter region, the order parameter is a factor $\kappa^{4\epsilon/3}$ faster than the slowest diffusion mode. This is a large effect in contrast to model C at an ordinary critical point where the corresponding frequencies differ by at most $\kappa^{\alpha/\nu}$ or κ^{η}. If there turn out to be experimental manifestations of the breakdown of dynamic scaling in model C like systems $(2 < n < 4)$, they should be most evident at the tricritical point. For analogous reasons, our $O(\epsilon)$ results are more likely to be incorrect at the tricritical point than elsewhere. Along the λ line the order parameter is again faster than the slowest diffusion mode but in three dimensions the two are decoupled because $\alpha < 0$ implies $v^* = 0$. Our results may prove more reliable along the λ line but one must artificially set $v(l)$ to zero ignoring the fact that α is positive to first order in ϵ.

B. Experimental consequences

With these reservations, we will present the implications of our $O(\epsilon)$ calculations for the three experimentally accessible transport coefficients above T_c, D, k_T, and κ_{Tr} in the hope they will remain valid when the difficulties with model C are resolved. This may mean that one considers only transport coefficients measured at zero frequency or alternately only those defined by a moment of a correlation function with no obvious equivalence between the two definitions. Correlation functions may differ significantly from a Lorentzian.

Our predictions for the diffusion coefficient take the form, $D \sim \kappa^x$, where

$$x = \tfrac{1}{3}\epsilon + O(\epsilon^2) \quad \text{tricritical (region I)}\,, \tag{6.3a}$$

$$x = -\epsilon/5 + O(\epsilon^2) \quad \text{intermediate (region II)}\,, \tag{6.3b}$$

$$x \equiv -\epsilon/2 + \tilde{\alpha}/\nu$$
$$= -\tfrac{3}{10}\epsilon + O(\epsilon^2) \quad \lambda \text{ line (region III)}\,, \tag{6.3c}$$

and

$$\tilde{\alpha} = \max(0, \alpha)\,.$$

The three regions are pictured in Fig. 3 and the boundary lines are given by (4.29) and (4.30). Static quantities do not vary differently in regions II and III.

Equation (6.3a) should be contrasted with the predictions of mode-coupling theory,[16-18] $x = \epsilon/2$, that we believe to be incorrect due to the assumption that the fluctuations of c and ψ are Gaussian and independent. The exponent α measures this coupling and is clearly important at the tricritical point. For similar reasons, model F is required to describe the transition in pure He near four dimensions when $\alpha > 0$.[26] The first equality in (6.3c) holds to all orders in ϵ.

It should also be emphasized that the crossover of D from (6.3a) to (6.3c) can in no sense be expressed in terms of static correlation functions. Equations (4.9) and (4.10) possess their own characteristic flows and in particular exhibit an unstable intermediate fixed point, even though they are driven by a static parameter v from one asymptotic regime to another. The complete crossover from the tricritical regime to the λ line was computed from (4.9), (4.10) and (4.13) numerically to see the importance of the intermediate regime. In Fig. 4 we have plotted an effective exponent for $\lambda_R(t_0, u_0) = \chi_R(t_0, u_0) D$ defined as

$$y_{\mathrm{eff}} = \partial(\ln \lambda_R)/\partial(\ln t_0) \tag{6.4}$$

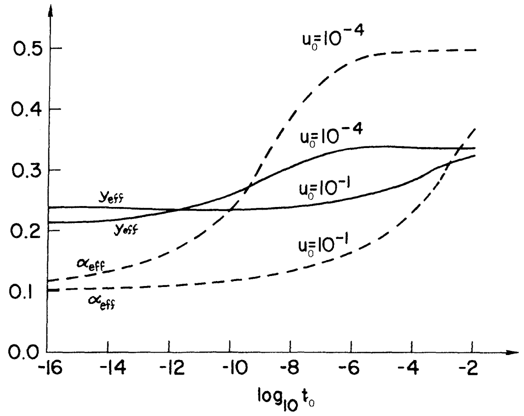

FIG. 4. Effective critical exponents for the static concentration susceptibility χ_R (dashed curves) and the renormalized transport coefficient λ_R (solid curves). The plots are shown for two distinct values of the parameter u_0. For $u_0 = 10^{-1}$, the plot of the effective exponent for λ_R dips slightly towards its "intermediate" λ-line value 0.2 before settling in at its true λ-line value 0.25. For $u_0 = 10^{-4}$, this dip is amplified into a genuine crossover from tricritical to "intermediate" behavior. The λ-line behavior does not appear. Because t_0 is raised to an $O(\epsilon)$ power, ν may be replaced by $\frac{1}{2}$.

for two values of u_0. The concentration susceptibility χ_R is known experimentally. The intermediate region (II) appears only as a slight dip for $u_0 = 10^{-1}$ but dominates the small t_0 regime for $u_0 = 10^{-4}$. For $u_0 < 10^{-2}$, the analytic expressions in (4.32) and (4.33) should be adequate for the experimentally accessible range of t_0. Unfortunately, the difference between the λ line and intermediate exponents is only $\epsilon/10$ in (6.3), (0.25 vs 0.20 in Fig. 4). The effective exponent of the concentration susceptibility from (3.20),

$$\alpha_{\rm eff} = \partial(\ln \chi_R)/\partial(\ln t_0)\,, \tag{6.5}$$

has been plotted in Fig. 4 to first order in ϵ with $\epsilon = 1$ for comparison.

In Fig. 5 we have plotted D vs t_0 for small values of u_0 as obtained from (4.32) and (4.33). The diffusion coefficient initially decreases with a slope of $\frac{1}{6}$ [Eq. (6.3a) with $\epsilon = 1$] and then diverges in the intermediate region with an exponent $\frac{1}{10}$ (6.3b). [Since the exponents in (6.3) are already of $O(\epsilon)$, we have replaced ν by $\frac{1}{2}$]. If our graphs had extended to smaller t_0 or included larger u_0, the slope would steepen a second time corresponding to the λ-line exponent of $\frac{3}{20}$. There will of course be significant correction terms to D along the λ line coming from the slow approach of v to its fixed point value 0, corresponding to $\alpha < 0$ in three dimensions.[26]

There are no universal amplitudes along the λ line that involve only D, k_T, $\kappa_{\rm Tr}$, and static quantities. The analog of R_λ in Ref. 26 does not exist because $w_2 \to \infty$ (4.6).

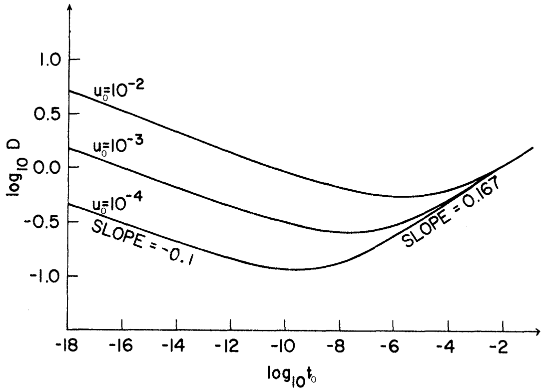

FIG. 5. Plot of the mass-diffusion constant D vs $\log_{10} t_0$ for a variety of different values of u_0 from (3.20) and (4.32). Although D initially tends to zero, it eventually diverges as the λ line is approached. The asymptotic slopes represent results to first order in ϵ, where the slope 0.1 actually corresponds to "intermediate" λ-line behavior. These plots correspond roughly to an experiment in which the chemical potential difference $\mu = \mu_3 - \mu_4$ is varied at fixed temperature. In *real* experiments, however, the temperature is usually varied at fixed concentration. The eighteen decades in t_0 displayed here would correspond to "only" nine decades in temperature in such a realistic experiment.

Roughly speaking, the parameter t_0 which appears in Figs. 3, 4, and 5 represents the deviation of the chemical potential difference $\mu = \mu_3 - \mu_4$ from its tricritical value, while u_0 is essentially a temperature variable. Most experiments are done by varying the temperature at fixed concentration. One consequence is that the eighteen decades in the variable t_0 shown in Fig. 5 would correspond to "merely" nine decades of temperature in a real experiment!

Ahlers and Pobell have measured D along the λ line at several concentrations.[44] They found an effective exponent similar to Eq. (6.3c) but with some concentration dependence that may be due to correction terms. The diffusion constant D has recently been determined in the tricritical region by ultrasonic attenuation and appears to agree with the order ϵ result in (6.3a).[45]

The thermal diffusion ratio k_T is a measure of the coupling between the concentration and diffusion modes. From (2.16) and (4.13) we find that

$$k_T \sim \kappa^{-\bar{\alpha}/\nu}$$

in agreement with the mode-coupling calculations,[16-18] and in qualitative accord with recent experiments.[45]

The thermal conductivity is given by (2.17) or,

$$\kappa_{\rm Tr} = K_R(1 - w_{3,R}^2)\,. \tag{6.6}$$

Although K_R diverges, (4.13c), $w_{3,R}^2 = L_R^2/\lambda_R K_R$ tends to 1 and (6.6) is indeterminate. The correct answer is obtained by retaining the corrections to the leading singularities of K and w_3 implied by (4.4). To $O(\epsilon)$ one finds

$$\begin{aligned} &\kappa_{\rm Tr} = c_1 + c_2 \kappa^x\,, \\ &x = 2\epsilon/3 \;\; \text{tricritical}\,, \\ &x = \epsilon/2 \;\; \text{lambda line}\,. \end{aligned} \tag{6.7}$$

Note that c_1 and c_2 are constants independent of the singular thermodynamic susceptibility, $(\partial c/\partial\mu)_{T,p}$ in contrast to pure He, where $\kappa_{\rm Tr}$ involves C_p and correction terms $\sim \partial \ln C_p/\partial \ln t_0$. The constants diverge as the concentration of ^{3}He tends to zero and the model represented by (2.10) is no longer adequate because $(\partial\sigma/\partial T)|_{c,p}$ was assumed to be constant. A cusp in $\kappa_{\rm Tr}$ is clearly visible in the data of Ref. 19. The original mode-coupling calculations of ^{3}He-^{4}He mixtures also predicted that $\kappa_{\rm Tr}$ remains finite at the transition.[16] The details of the derivation of (6.7) are given at the end of Appendix A.

In conclusion, we remind the reader that our results for ^{3}He-^{4}He mixtures must be viewed with some suspicion near the tricritical point because of the resemblance of our model to model C of Ref. 21. It is quite intriguing that these difficulties appear in such an extensively investigated system. Dynamic tricritical phenomena apparently still contains some mysteries although tricritical statistics is believed to be well understood.

ACKNOWLEDGMENT

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APPENDIX A: SOLUTIONS OF DIFFERENTIAL EQUATIONS

We first solve Eq. (3.8c), which we write in the form

$$\frac{dv(l)}{dl}=\epsilon v(l)-2Av(l)u(l)-2nv^2(l)\,,$$

$$A=4K_4(n+2)\,, \tag{A1}$$

where the function $u(l)$ is known,

$$u(l)=\frac{u_0e^{\epsilon l}}{1+Bu_0(e^{\epsilon l}-1)/\epsilon}\equiv u_0e^{\epsilon l}/Q(l)\,,$$

$$B=4K_4(n+8)\,. \tag{A2}$$

The terms linear in $v(l)$ can be combined with $dv(l)/dl$ to give

$$\frac{d}{dl}\left[\exp\left(-\epsilon l+2A\int_0^l u(l')dl'\right)v(l)\right]$$

$$=-2n\exp\left(-\epsilon l+2A\int_0^l u(l')dl'\right)v^2(l)\,. \tag{A3}$$

On defining,

$$\overline{v}(l)\equiv\exp\left(-\epsilon l+2A\int_0^l u(l')dl'\right)v(l)$$

$$=e^{-\epsilon l}Q(l)^{2A/B}v(l)\,, \tag{A4}$$

and,

$$dx\equiv e^{\epsilon l}Q(l)^{-2A/B}dl\,, \tag{A5}$$

Eq. (A3) becomes

$$d\overline{v}/dx=-2n\overline{v}^2\,, \tag{A6}$$

which can be integrated immediately:

$$\overline{v}(x)=\overline{v}(0)/[1+2n\overline{v}(0)x]\,. \tag{A7}$$

Equation (A5) can also be integrated,

$$x(l)=[Q(l)^{1-(2A/B)}-1]/u_0(B-2A)\,. \tag{A8}$$

On combining (A7), (A4), and (A8) we obtain,

$$v(l)=\frac{(B-2A)u_0v_0e^{\epsilon l}}{2nv_0Q(l)+[u_0(B-2A)-2nv_0]Q(l)^{2A/B}}\,, \tag{A9}$$

which is (3.10).

To calculate quantities like $\exp(\int_0^l v(l')\,dl')$, one can integrate (A9) directly, or instead simply rewrite (A1) in the form

$$\frac{d}{dl}\left[e^{-\epsilon l}Q(l)^{2A/B}\exp\left(2n\int_0^l v(l')\,dl'\right)v(l)\right]=0\,. \tag{A10}$$

It follows that

$$\exp\left(2n\int_0^l v(l')\,dl'\right)=v_0e^{\epsilon l}Q(l)^{-2A/B}/v(l)\,. \tag{A11}$$

Equation (4.31),

$$\frac{df(l)}{dl}=\epsilon f(l)+4v(l)f(l)-\tfrac{3}{2}f^2(l)\,, \tag{A12}$$

can be solved in much the same way. Here $v(l)$ is given by (A9) with $n=2$. On rewriting (A12),

$$\frac{d}{dl}\left[\exp\left(-\epsilon l-4\int_0^l v(l')\,dl'\right)f(l)\right]=-\tfrac{3}{2}f^2(l)\,, \tag{A13}$$

we are led to define,

$$\overline{f}(l)\equiv\exp\left(-\epsilon l-4\int_0^l v(l')\,dl'\,f(l)\right)$$

$$=e^{-2\epsilon l}Q(l)^{4/5}[v(l)/v_0]f(l)\,, \tag{A14}$$

and,

$$dy\equiv\exp\left(\epsilon l+4\int_0^l v(l')\,dl'\right)dl$$

$$=e^{2\epsilon l}Q(l)^{-4/5}[v_0/v(l)]\,dl\,, \tag{A15}$$

where we have made use of (A11). Equation (A13) becomes

$$\frac{d\overline{f}}{dy}=-\tfrac{3}{2}\overline{f}^2\,, \tag{A16}$$

and thus,

$$\overline{f}(l)=\overline{f}(0)/[1+\tfrac{3}{2}\overline{f}(0)y(l)]\,. \tag{A17}$$

The function $y(l)$ follows from (A15) and (A9):

$$y(l)=(1-v_0/2K_4u_0)(e^{\epsilon l}-1)/\epsilon$$

$$+(v_0/96K_4^2u_0^2)[Q(l)^{6/5}-1]\,. \tag{A18}$$

Equations (A17), (A18), and (A14) then determine our final result,

$$f(l)=\frac{f_0e^{\epsilon l}[(1-v_0/2K_4u_0)+(v_0/2K_4u_0)Q(l)^{1/5}]}{1+\tfrac{3}{2}f_0\{(1-v_0/2K_4u_0)(e^{\epsilon l}-1)/\epsilon+(v_0/96K_4^2u_0^2)[Q(l)^{6/5}-1]\}}\,. \tag{A19}$$

To determine $\exp(\int_0^l f(l)\,dl)$, we rewrite (A12),

$$\frac{d}{dl}\left[\exp\left(-\epsilon l-4\int_0^l v(l')\,dl'+\frac{3}{2}\int_0^l f(l')\,dl'\right)f(l)\right]=0 \tag{A20}$$

and obtain finally

$$\exp\left(\frac{3}{2}\int_0^l f(l')\,dl'\right)=f_0\exp\left(\epsilon l+4\int_0^l v(l')\,dl'\right)\Big/f(l)\,. \tag{A21}$$

Equation (6.7) for κ_{Tr} follows once the corrections to (4.7) are known. They are computed from the relaxation of the five parameters $\varphi_i(l)$ of (4.4) to their fixed-point values φ_i^* when $|\varphi_{i,0}-\varphi_i^*|\ll\varphi_i^*$. The relevant equations become

$$\begin{aligned} d[f_1(l)-f_2(l)]/dl&=-\tfrac{1}{2}f^*[f_1(l)-f_2(l)]\\ dw_3(l)/dl&=-[f_1(l)-f_2(l)]^2/16f^*\\ &\quad-\left(\frac{f_1(l)+f_2(l)}{4}\right)[w_3(l)-1]\,. \end{aligned} \tag{A22}$$

Their solutions are

$$f_1(l)-f_2(l)=e^{-f^*l/2}(f_{1,0}-f_{2,0})\,, \tag{A23}$$

and

$$w_3(l)-1=\left((w_{3,0}-1)-(1-e^{-f^*l/2})\frac{(f_{1,0}-f_{2,0})^2}{8f^{*2}}\right)\exp-\frac{1}{4}\int_0^l(f_1(l')+f_2(l'))\,dl'\,. \tag{A24}$$

The integration of (4.20) is immediate [see (4.13c)],

$$\kappa_{\mathrm{Tr}}\sim K(l^*)[1-w_3^2(l^*)]\sim\exp\left[\frac{1}{4}\int^{l^*}(f_1-f_2)\right][2+e^{-f^*l^*/2}(w_{3,0}-1)]\,. \tag{A25}$$

The inhomogeneous term in (A24) is negligible in comparison with the integral in (A25) since the latter has a term $\propto(f_{1,0}-f_{2,0})$. Equation (6.7) follows with $c_2\propto(w_{3,0}-1)$ or $(f_{1,0}-f_{2,0})$.

APPENDIX B: PERTURBATION THEORY AND DYNAMIC RECURSION RELATIONS

The dynamic recursion relation method has been explained in detail elsewhere.[21, 26, 27, 39] The calculations for ^{3}He-^{4}He mixtures, although rather involved, are very similar to those described in Ref. 26 for pure ^{4}He (model F). The salient features will be sketched in this Appendix.

We consider the Fourier transformed equations of motion (2.10), neglecting for the moment the four point coupling $\tilde{u}_0$. On defining the bare propagators,

$$G_0^{-1}(k,\omega)=-i\omega+\Gamma_0(r_0+k^2)\equiv-i\omega+\Gamma_0(k) \tag{B1}$$

and

$$\underline{D}_0^{-1}(k,\omega)=\begin{pmatrix}-i\omega+K_0C_0^{-1}k^2 & L_0\chi_0^{-1}k^2\\ L_0C_0^{-1}k^2 & -i\omega+\lambda_0\chi_0^{-1}k^2\end{pmatrix}, \tag{B2}$$

these can be written as coupled integral equations,

$$\begin{aligned}\psi(k,\omega)={}&G_0(k,\omega)\zeta(k,\omega)-ig_{1,0}C_0^{-1}G_0(k,\omega)\int_{p\Omega}\psi(p,\Omega)q(k-p,\omega-\Omega)-ig_{2,0}\chi_0^{-1}G_0(k,\omega)\int_{p\Omega}\psi(p,\Omega)c(k-p,\omega-\Omega)\\ &-ig_{2,0}\gamma_0G_0(k,\omega)\int_{p_1,\Omega_1}\int_{p_2,\Omega_2}\psi(p_1,\Omega_1)\psi^*(p_2,\Omega_2)\psi(k-p_1-p_2,\omega-\Omega_1-\Omega_2)\\ &-2\gamma_0\Gamma_0G_0(k,\omega)\int_{p\Omega}\psi(p,\Omega)c(k-p,\omega-\Omega)\,,\end{aligned} \tag{B3}$$

$$\begin{aligned}\begin{pmatrix}q(k,\omega)\\ c(k,\omega)\end{pmatrix}={}&\underline{D}_0(k,\omega)\begin{pmatrix}\theta(k,\omega)\\ \varphi(k,\omega)\end{pmatrix}+\frac{1}{2}i\,\underline{D}_0(k,\omega)\begin{pmatrix}g_1\\ g_2\end{pmatrix}\int_{p\Omega}[p^2-(k-p)^2]\psi^*(p,\Omega)\psi(k-p,\omega-\Omega)\\ &-\lambda_0\gamma_0k^2\underline{D}_0(k,\omega)\begin{pmatrix}0\\ 1\end{pmatrix}\int_{p\Omega}\psi^*(p,\Omega)\psi(k-p,\omega-\Omega)\,,\end{aligned} \tag{B4}$$

where the equations for $q(k,\omega)$ and $c(k,\omega)$ have been arranged in a convenient vector form. Quantities such as $\psi(k,\omega)$ and $c(k,\omega)$ are Fourier transforms of the same objects in real space and time, and $\int_{p\Omega}$ means

$$\int_{|\vec{p}|<\Lambda}\frac{d^dp}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\,.$$

Henceforth, we take the cutoff Λ to be unity.

Denoting the matrix diffusion propagator $\underline{D}_0(k,\omega)$

by a wavy line, and the order-parameter propagator $G_0(k, \omega)$ by a straight line, one readily constructs a graphical series which formally solves these equations. The graphs which appear are in one-to-one correspondence with those in Appendix B of Ref. 26, and can be derived either by iterating the equations of motion and averaging over the noise sources, or by using the formalism of Martin, Siggia, and Rose.[40]

The four point coupling $\tilde{u}_0$ is of course necessary in (B3) and (B4) to reproduce the correct static recursion relations. Otherwise, however, this coupling has no effect on the dynamic recursion relations to lowest order.

To obtain recursion relations, we need the lowest-order self-energies from (B3) and (B4). Writing the renormalized order parameter propagator in the form

$$G_R^{-1}(k, \omega) = G_0^{-1}(k, \omega) - \Sigma_{\hat{\psi}\psi}(k, \omega)\,, \tag{B5}$$

surpressing subscripts on quantities such as $\Gamma_0(q) = \Gamma_0(r_0 + q^2)$, λ_0, L_0, etc., and defining

$$\begin{aligned}\Delta(k, q, \omega) \equiv {} & [-i\omega + \Gamma(q) + KC^{-1}(\vec{k} - \vec{q})^2] \\ & \times [-i\omega + \Gamma(q) + \lambda\chi^{-1}(\vec{k} - \vec{q})^2] \\ & - L^2\chi^{-1}C^{-1}(\vec{k} - \vec{q})^4\,,\end{aligned} \tag{B6}$$

we find

$$\begin{aligned}\Sigma_{\hat{\psi}\psi}(k, \omega) = {} & (-i\omega)2ig_1C^{-1}L\gamma \int_q \frac{(\vec{k} - \vec{q})^2}{(r + q^2)\Delta(k, q, \omega)} \\ & + 2ig_2\gamma \int_q \frac{[\Gamma(q) + \lambda\chi^{-1}(\vec{k} - \vec{q})^2][-i\omega + \Gamma(q) + KC^{-1}(k - q)^2] - L^2\chi^{-1}C^{-1}(\vec{k} - \vec{q})^4}{(r + q^2)\Delta(k, q, \omega)} \\ & + 4\Gamma\chi\gamma^2 \int_q \frac{[\Gamma(q) + \lambda\chi^{-1}(\vec{k} - \vec{q})^2][-i\omega + \Gamma(q) + KC^{-1}(\vec{k} - \vec{q})^2] - L^2\chi^{-1}C^{-1}(\vec{k} - \vec{q})^4}{(r + q^2)\Delta(k, q, \omega)} \\ & - g_1^2C^{-1}(r + k^2) \int_q \frac{-i\omega + \Gamma(q) + \lambda\chi^{-1}(\vec{k} - \vec{q})^2}{(r + q^2)\Delta(k, q, \omega)} - g_2^2\chi^{-1}(r + k^2) \int_q \frac{-i\omega + \Gamma(q) + KC^{-1}(\vec{k} - \vec{q})^2}{(r + q^2)\Delta(k, q, \omega)} \\ & + 2g_1g_2L\chi^{-1}C^{-1}(r + k^2) \int_q \frac{(\vec{k} - \vec{q})^2}{(r + q^2)\Delta(k, q, \omega)} - 2i\Gamma\gamma g_1C^{-1}L(r + k^2) \int_q \frac{(\vec{k} - \vec{q})^2}{(r + q^2)\Delta(k, q, \omega)} \\ & + 2i\Gamma\gamma g_2(r + k^2) \int_q \frac{-i\omega + \Gamma(q) + KC^{-1}(\vec{k} - \vec{q})^2}{(r + q^2)\Delta(k, q, \omega)} - 2ig_2\gamma \int_q \frac{1}{(r + q^2)}\end{aligned} \tag{B7}$$

where

$$\int_q \equiv \int \frac{d^4q}{(2\pi)^4}\,.$$

The last "Hartree" graph serves to cancel a portion of the second term in (B7), and we have surpressed a Hartree term proportional to $\tilde{u}_0$ which contributes only to the statics.

Defining self-energies for the diffusion propagator,

$$D_R^{-1}(k, \omega) = \begin{pmatrix} -i\omega + K_0C_0^{-1}k^2 - \Pi_{\hat{q}q}(k, \omega) & L_0\chi_0^{-1}k^2 - \Pi_{\hat{q}c}(k, \omega) \\ L_0C_0^{-1}k^2 - \Pi_{\hat{c}q}(k, \omega) & -i\omega + \lambda_0\chi_0^{-1}k^2 - \Pi_{\hat{c}c}(k, \omega) \end{pmatrix}, \tag{B8}$$

and again surpressing subscripts, we find

$$\Pi_{\hat{q}q}(k, \omega) = -g_1^2C^{-1} \int_q \left(\frac{[q^2 - (q + k)^2]^2}{(r + q^2)[r + (q + k)^2][-i\omega + \Gamma(q + k) + \Gamma^*(q)]}\right), \tag{B9}$$

$$\begin{aligned}\Pi_{\hat{c}c}(k, \omega) = {} & -g_2^2\chi^{-1} \int_q \left(\frac{[q^2 - (q + k)^2]^2}{(r + q^2)[r + (q + k)^2][-i\omega + \Gamma(q + k) + \Gamma^*(q)]}\right) \\ & + 2k^2\lambda\gamma^2 \int_q \left(\frac{1}{r + q^2} + \frac{1}{r + (q + k)^2}\right)\left(\frac{\Gamma + \Gamma^*}{-i\omega + \Gamma(q + k) + \Gamma^*(q)}\right),\end{aligned} \tag{B10}$$

$$\Pi_{\hat{c}q} = -g_1g_2C^{-1} \int_q \left(\frac{[q^2 - (q + k)]^2}{(r + q^2)[r + (q + k)^2][-i\omega + \Gamma(q + k) + \Gamma^*(q)]}\right), \tag{B11}$$

$$\Pi_{\hat{q}c} = C\chi^{-1}\Pi_{\hat{c}q} + 2k^2L\gamma^2 \int_q \left(\frac{1}{r + q^2} + \frac{1}{r + (k + q)^2}\right)\left(\frac{\Gamma + \Gamma^*}{-i\omega + \Gamma(q + k) + \Gamma^*(q)}\right). \tag{B12}$$

Both the notation used, and the precise meaning of expressions like $\Sigma_{\hat{\psi}\psi}(k, \omega)$ and $\Pi_{\hat{q}c}(k, \omega)$ should be clear from the discussion in Appendix B of Ref. 26.

Dynamic recursion relations can now be constructed in the usual way,[21,26,27,39] by taking (B7) and (B9)–(B12) and restricting the momentum integrals to a shell $e^{-l} \leqslant |\vec{q}| \leqslant 1$. The parameters entering the self-energies must now be interpreted as $\Gamma(q; l) \equiv \Gamma(l)(r(l) + q^2)$, $g_1(l)$, $\lambda(l)$, etc., and one must also take account of the rescalings (3.1). These self-energies (defined as integrals over a shell) can be combined with the bare propagators to give new equations of motion. "Partially dressed" parameters such as $K(l)$, $\Gamma(l)$, etc., replace the bare parameters. On taking the limit $e^l = b \to 1$, we obtain the differential recursion relations displayed in Eq. (4.2). Appendix B of Ref. 26 should be consulted for details. The rescaling parameter A has been omitted here since it does not affect the lowest-order recursion relations.

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