# Strong-coupling correction to the low-frequency electrical conductivity of superconductors and Josephson junctions

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(Received 4 December 1975)

Values of the strong-coupling correction to the low-frequency local-limit conductivity of a number of superconductors are given. These values also describe the strong-coupling correction to the critical current of a Josephson tunnel junction. The correction is given for indium, amorphous gallium, $\beta$-phase gallium, amorphous bismuth, $Pb_{0.8}Bi_{0.2}$, $Pb_{0.7}Bi_{0.3}$, amorphous $Pb_{0.45}Bi_{0.55}$, $Tl_{0.9}Bi_{0.1}$, $Pb_{0.8}Tl_{0.2}$, and previously published values are listed for tin, lead, mercury, thallium, and indium-thallium alloys. It is shown that there is a simple empirical relation between these values and the effective electron-electron coupling parameter $(\lambda - \mu^*)$.

The influence of strong electron-phonon coupling on the electromagnetic properties of superconductors has been of interest for many years, and our understanding of the essential features of this problem is probably approaching completion.[1-5] Strong-coupling affects the critical current of Josephson tunnel junctions in essentially the same way as it does the conductivity of bulk superconductors[6-8]; carefully prepared junctions exhibit critical currents approaching the theoretical values.

We report here some previously unpublished theoretical values of the low-frequency electrical conductivities of some strong-coupling superconductors. We show that these values and ones which have been previously published can be related directly to McMillan's[9] effective electron-electron coupling parameter $(\lambda - \mu^*)$. The value of the conductivity can be used to calculate the penetration of electromagnetic waves through thin films[3,10] and the amplitude of the supercurrent in a Josephson tunnel junction as well, so the conductivity has considerable technological importance as well as intrinsic interest.

Although the effects of strong coupling on superconductors and on Josephson junctions are similar, the two seem conceptually rather different. We discuss them separately. We assume that the electron mean free path $l$ is very much less than the electromagnetic coherence length $\xi_0 = \hbar v/\pi\Delta_0$, where $v$ is the Fermi velocity and $\Delta_0$ is the energy-gap parameter. This condition will be satisfied if the samples are thin films.

For superconductors, a consequence of the small mean free path is that the current density $J$ is an essentially local response to the electromagnetic field, and can therefore be described in terms of a superconducting-state-to-normal-state conductivity ratio $\sigma_s/\sigma_n$ which depends on the angular frequency $\omega$, but not on the electromagnetic wave number.[11] We are concerned here with angular frequencies which are greater than zero, but less than $2\Delta_0/\hbar$. In this frequency range there is no energy dissipation (assuming zero temperature and that there are no pair-breaking interactions[12]), so the real part $\sigma_1(\omega)$ of $\sigma_s$ vanishes, and $\sigma_s$ is a purely imaginary function of $\omega$, which is called $-i\sigma_2(\omega)$. In the same frequency region, $\sigma_n$ is a real constant if $l \ll \xi_0$. Hence, $\sigma_s/\sigma_n$ is an imaginary function of $\omega$ for our purposes. We are going to focus particularly on very low frequencies, $\omega \ll 2\Delta_0/\hbar$, for which $\sigma_2(\omega)$ is inversely proportional to $\omega$:

$$\sigma_2/\sigma_n = 2A/\pi\omega, \tag{1}$$

where $A$ is a constant. (For a typical value $\Delta_0 \simeq 1$ meV, $2\Delta_0/h \simeq 5\times10^{11}$ Hz.)

Mattis and Bardeen calculated the electromagnetic response of a weak-coupling superconductor, and obtained a value $A_{MB}$ for $A$, given by[13]

$$A_{MB} = (\tfrac{1}{4}\pi^2)2\Delta_0/\hbar. \tag{2}$$

The calculation of $\sigma_s/\sigma_n$ for a strong-coupling superconductor can be carried out according to the theory of Nam,[4] as modified by Swihart and Shaw,[5] provided electron tunneling data are available. The strong coupling alters $A$ by an amount $\delta A$, so that

$$A = A_{MB} + \delta A. \tag{3}$$

The resulting change $\sigma_2/\sigma_n$ according to Eq. (1) is $2\delta A/\pi\omega$. In fact, it is known that this is, within about 1%,[14,15] the entire change in $\sigma_s/\sigma_n$ caused by strong coupling for angular frequencies up to $2\Delta_0/\hbar$, even though the validity of Eq. (1) breaks down as $\omega$ approaches $2\Delta_0/\hbar$.

A close connection between the change in the conductivity due to strong coupling and the change in the critical current $I_c$ of a Josephson tunnel junction arising from strong coupling was noted by Fulton and McCumber.[6] Recently Harris[7,8] has shown that the connection arises through the coherence effects dominating the response in the two cases. The connection is simple:

$$I_c = \frac{\hbar}{2eR_N} \lim_{\omega \to 0+} \frac{\omega\sigma_2(\omega)}{\sigma_n} = A\left(\frac{\hbar}{\pi e R_N}\right) , \qquad (4)$$

where $R_N$ is the normal-state resistance of the Josephson junction. Usually the theoretical value of $I_c$ for a junction made of strong-coupling materials is expressed as a fraction of its value $I_{cw}$ for a junction made of material in the weak-coupling limit. Thus

$$I_c/I_{cw} = (A_{MB} + \delta A)/A_{MB} . \qquad (5)$$

It is therefore explicitly clear that the fractional change in $I_c$ due to strong coupling is identical to the fractional change in $A$ due to strong coupling.

Calculated values of $\delta A$ for a variety of materials have already been published. Unpublished values of $\delta A$, calculated from tunneling data for amorphous bismuth,[16] amorphous gallium,[16,17] and $\beta$-phase gallium[17] have been used to find the transmission of far-infrared electromagnetic waves through thin films, as has the published value for lead.[5] The agreement between experiment and theory was very good for all four materials.[1,2]

We report in Table I the values of $\delta A$ normalized to $2\Delta_0$ for all these materials.[18] We also give in Table I a number of previously unpublished values of $\delta A$ for other materials: $\delta A$ was calculated by numerically evaluating $\sigma_2/\sigma_n$ in the limit of small

TABLE I. Values of parameters related to strong coupling for a variety of materials.

| Material | References Data | References Calc. | $\hbar\delta A/2\Delta_0$ [a] | $I_c/I_{cw}$ [a] | $\lambda$ | $\mu^*(\omega_{ph})$ |
|---|---|---|---|---|---|---|
| Sn [b] | c | d | −0.209 | 0.915 | 0.72 | 0.092 |
| Tl | c | e | −0.234 | 0.905 | 0.795 | 0.111 |
| In [b,f] | c | g | −0.247 | 0.900 | 0.805 | 0.097 |
| Pb [b] | c | d | −0.502 | 0.797 | 1.55 | 0.105 |
| Hg [b] | h | i | −0.518 | 0.790 | 1.6 | 0.098 |
| $\beta$ Ga [j] | c | g | −0.328 | 0.867 | 0.97 | 0.092 |
| Amorphous Ga [b,j] | c | g | −0.492 | 0.801 | 1.62 | 0.095 |
| Amorphous Bi [b,j] | c | g | −0.589 | 0.761 | 2.05 | 0.11 |
| $In_{0.9}Tl_{0.1}$ | c | e | −0.261 | 0.894 | 0.85 | 0.103 |
| $In_{0.73}Tl_{0.27}$ | c | e | −0.299 | 0.879 | 0.93 | 0.110 |
| $In_{0.67}Tl_{0.33}$ | c | e | −0.281 | 0.886 | 0.90 | 0.110 |
| $In_{0.5}Tl_{0.5}$ | c | e | −0.249 | 0.899 | 0.83 | 0.110 |
| $Tl_{0.9}Bi_{0.1}$ | c | g | −0.268 | 0.891 | 0.78 | 0.099 |
| $Pb_{0.8}Tl_{0.2}$ | c | g | −0.513 | 0.792 | 1.53 | 0.101 |
| $Pb_{0.8}Bi_{0.2}$ | c | g | −0.595 | 0.759 | 1.88 | 0.093 |
| $Pb_{0.7}Bi_{0.3}$ | c | g | −0.622 | 0.748 | 2.01 | 0.092 |
| Amorphous $Pb_{0.45}Bi_{0.55}$ | c, k | g | −0.648 | 0.737 | 2.59 | 0.116 |

[a] The accuracy of these quantities is not known well. However, the uncertainty introduced by the present calculations is small. Larger uncertainties arise from limitations in the accuracy of the original tunneling data and from the spacing of the points for which the complex gap parameter was obtained when the tunneling data were inverted.

[b] Values of $I_c/I_{cw}$ are also given for Sn, In, Pb, Hg, amorphous Bi, and amorphous Ga by J. P. Carbotte and P. Vashishta [Can. J. Phys. 49, 1493 (1971)]. Their values do not differ significantly from those listed here, except as noted in Ref. 18.

[c] J. M. Rowell, W. L. McMillan, and R. C. Dynes, J. Phys. Chem. Ref. Data (to be published).

[d] Reference 14.

[e] D. W. Taylor and P. Vashishta, Phys. Rev. B 5, 4410 (1972).

[f] Reference e also gives $I_c/I_{cw}$ for In; it is nearly the same as ours.

[g] This work.

[h] W. N. Hubin and D. M. Ginsberg, Phys. Rev. 188, 716 (1969).

[i] Reference 15.

[j] Reference 18.

[k] Reference 21.

frequency[15] as indicated by Eq. (1). For completeness previously published values of $\delta A$ are listed for other materials as well. For each material we have also given in Table I the fractional reduction in the critical current of a Josephson tunnel junction according to Eq. (5).

It is seen that $\delta A$ is not negligible compared with $A_{MB}$, and that $\delta A$ is negative in all cases. The sign of $\delta A$ follows from the electromagnetic sum rule,[19,20] the Kramers-Kronig relations[11] between $\sigma_1$ and $\sigma_2$, and the fact that strong coupling adds oscillator strength to $\sigma_1$ in the frequency range $\omega > 2\Delta_0/\hbar$.[1,2,5]

It is useful to determine empirically whether the dimensionless ratio $\hbar\,\delta A/2\Delta_0$ is related directly to McMillan's[9] electron-phonon coupling parameter $\lambda$ and the Coulomb pseudopotential $\mu^*$,[21,22] since large values of $(\lambda-\mu^*)$ should correspond to strong coupling.[9] The parameters $\lambda$ and $\mu^*$ are found by electron tunneling[23]; we use the best available values. References are indicated in Table I. The relation between $\hbar\delta A/2\Delta_0$ and $(\lambda-\mu^*)$ is shown in Fig. 1. Apparently there is an essentially monotonic relation between these two variables. The curve in the figure, although it is simply drawn by eye to fit the points, can hopefully be used to find $\delta A$ and $I_c/I_{cw}$ for materials for which those parameters have not been directly calculated or measured.

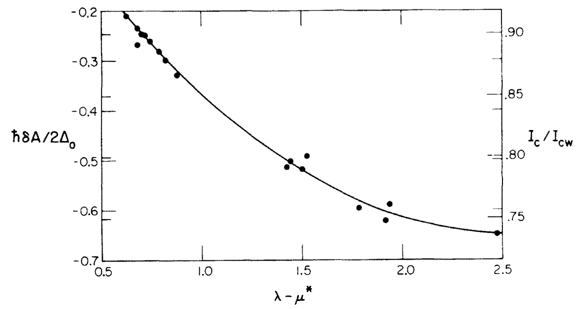


FIG. 1. Strong-coupling correction to $\hbar A/2\Delta_0$, defined in Eq. (1), as a function of McMillan's effective electron-electron coupling parameter $(\lambda-\mu^*)$. The solid line is simply drawn to fit the points, and is not based on theory.

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*Research supported in part by the National Science Foundation under Grant Nos. NSF-DMR 7307581 and NSF-DMR 7203026.

†Research supported in part by the Office of Naval Research.


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[22]The historical measure of strong coupling is $2\Delta_0/kT_c$. However, we have determined empirically that when $\delta A/2\Delta_0$ is plotted against $\lambda-\mu^*$, there is much less scatter of the data from a smooth curve than is the case when $\delta A/2\Delta_0$ is plotted against $2\Delta_0/kT_c$.


[23]W. L. McMillan and J. M. Rowell, in Ref. 11, Chap. 11.