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Electrostatic edge modes along a parabolic wedge

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We have calculated the electrostatic edge modes of a dielectric wedge whose boundary is a parabolic cylinder. The spectrum of edge modes is found to be discrete, and to depend on the one-dimensional wave vector q which characterizes the propagation of these modes along the wedge. This is in agreement with the results of recent, similar calculations by Davis for a wedge whose boundary is a hyperbolic cylinder. The dispersion curves for the first four mdoes are plotted for the case of a parabolic wedge characterized by a free-electron metal dielectric constant. The relation of the present results to those of the original electrostatic edge mode calculation by Dobrzynski and Maradudin and to those of Davis is discussed.

Several years ago Dobrzynski and one of the authors of this paper obtained the dispersion relation for the electrostatic edge modes of a dielectric wedge.[1] These are solutions of Laplace's equation which propagate in a wavelike fashion along the apex of a dielectric wedge formed by the intersection of two semi-infinite planes, but whose amplitudes decay exponentially with increasing distance from the apex both into the wedge and into the vacuum outside it. The frequencies of these modes were found to depend on a continuously varying parameter (the separation constant in the solution of Laplace's equation by separation of variables), and to be independent of the one-dimensional wave vector q characterizing their advance along the apex of the wedge.

Recently, Davis[2] has shown that rounding the edge of the wedge by taking the boundary to be a hyperbolic cylinder renders the spectrum of edge modes discrete rather than continuous, and causes it to be a function of q.

In the present paper we carry out a calculation, similar to that of Davis, of the electrostatic edge modes of a dielectric wedge whose boundary is a parabolic cylinder. Unlike the case treated by Davis, the present calculation can be carried out analytically, and yields results in terms of tabulated functions. The spectrum of edge modes is found to be discrete and to depend on the wave vector q.

We introduce parabolic cylinder coordinates[3]

$$x = \xi\eta, \quad -\infty \le \xi \le \infty, \quad 0 \le \eta \le \infty, \tag{1a}$$

$$y = \tfrac{1}{2}(\eta^2 - \xi^2), \tag{1b}$$

$$z = z. \tag{1c}$$

The dielectric occupies the region $0 \le \eta \le \eta_0$, $-\infty \le \xi \le \infty$ (see Fig. 1), and is characterized by an isotropic dielectric constant $\epsilon(\omega)$. Outside the wedge ($\eta \ge \eta_0$, $-\infty \le \xi \le \infty$) there is vacuum ($\epsilon = 1$).

As in Ref. 1 we ignore nonlocal effects, and the magnetic permeability is assumed to be unity everywhere. Furthermore, we work in the electrostatic limit, in which effects due to the finiteness of the speed of light are ignored. Then, with $\Phi(\xi,\eta,z;t) = \Phi(\xi,\eta,z)e^{-i\omega t}$ being the scalar potential of the electromagnetic field, we seek solutions to Laplace's equation $\nabla^2\Phi = 0$ that decrease exponentially with η as $\eta \to \infty$, increase exponentially with η as $\eta \to \eta_0$ from below, and also decrease exponentially as $|\xi| \to \infty$.

Because the system under consideration posses-ses infinitesimal translational invariance along the z axis, we choose for Φ an expression of the form $\Phi(\xi,\eta,z) = \varphi(\xi,\eta)e^{iqz}$. Writing the Laplacian operator in parabolic cylinder coordinates and making use of this expression for Φ, we obtain a partial differential equation for $\varphi(\xi,\eta)$ which can be solved by separation of variables. Setting $\varphi(\xi,\eta) = F(\xi)G(\eta)$, the equations for $F(\xi)$ and $G(\eta)$ are, respectively,

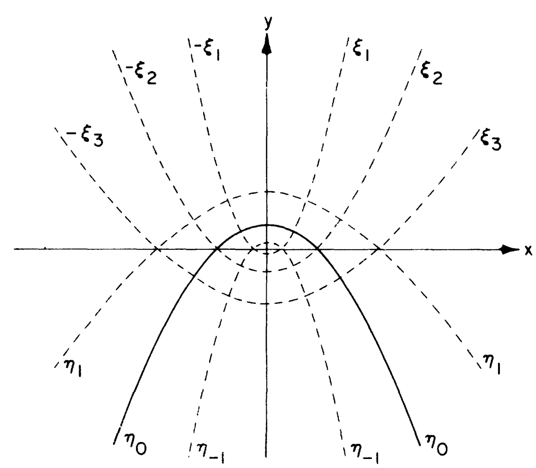

FIG. 1. Parabolic cylinder coordinates. The figure shows the cross section of the surfaces of constant ξ and η. The z axis is perpendicular to the drawing. The boundary of the dielectric medium is the surface $\eta = \eta_0$. The medium occupies the region $\eta \le \eta_0$.

$$\frac{d^2F(\xi)}{d\xi^2}+(E-q^2\xi^2)F(\xi)=0 \quad (2)$$

and

$$\frac{d^2G(\eta)}{d\eta^2}-(E+q^2\eta^2)G(\eta)=0. \quad (3)$$

In Eqs. (2) and (3), E is the separation constant.

We consider first the solution to Eq. (2). The similarity between this equation and the harmonic-oscillator problem of quantum mechanics is apparent. We can therefore immediately write the solutions of Eq. (4) which tend to zero as $|\xi|\to\infty$,

$$F_n(\xi)=C_{nq}e^{-q\xi^2/2}H_n(q^{1/2}\xi), \quad (4)$$

corresponding to the following discrete values of the separation constant E:

$$E_n=(2n+1)q, \quad n=0,1,2,3,\ldots . \quad (5)$$

In Eq. (4), $H_n(x)$ is the nth Hermite polynomial, while C_{nq} is a normalization constant.

We have thus shown that the solutions of Eq. (2), and consequently of Eq. (3), can be labeled by the discrete quantum number n. As a consequence of this the potential function $\Phi(\xi,\eta,z)$ will be written $\Phi_{nq}(\xi,\eta,z)$ to indicate explicitly its dependence on this quantum number, as well as on the continuously varying quantum number q. We emphasize that in the case of the edge modes studied by Dobrzynski and Maradudin,[1] there is no restriction on the separation constant, which can thus vary continuously. Furthermore, as they must, the solutions of Eq. (1) have been shown to have a definite parity under reflection in the plane defined by the line $\xi=0$ and the z axis (the yz plane); that is, if n is even then $\Phi_{nq}(\xi,\eta,z)=\Phi_{nq}(-\xi,\eta,z)$ and if n is odd, then $\Phi_{nq}(\xi,\eta,z)=-\Phi_{nq}(-\xi,\eta,z)$.

We proceed now to consider the solution to Eq. (3). Making the change of variables $x=(2q)^{1/2}\eta$, defining $a_n\equiv n+\frac{1}{2}$ and $g_n(x)\equiv G_n(\eta)$, we obtain the equation

$$\frac{d^2}{dx^2}g_n(x)-\left(a_n+\frac{x^2}{4}\right)g_n(x)=0. \quad (6)$$

The solutions to Eq. (6) are the parabolic cylinder functions.[4] Inside the wedge, $\eta\le\eta_0$ $[x\le x_0=\eta_0(2q)^{1/2}]$, we keep the solution that increases exponentially with x. This solution is usually denoted by $V(a_n,x)$. We shall need below its asymptotic behavior[4] as $x\to\infty$ (physically, this is the $q\to\infty$ limit):

$$g_n(x)=V(n+\tfrac{1}{2},x)$$

$$\xrightarrow[x\to\infty]{}\left(\frac{2}{\pi}\right)^{1/2}e^{x^2/4}x^n\left(1+\frac{n(n-1)}{x^2}+\cdots\right). \quad (7)$$

Outside the wedge, $\eta\ge\eta_0$ the solution to Eq. (6) that decreases exponentially as $x\to\infty$ is denoted by $U(a_n,x)$. Its asymptotic behavior is given by

$$g_n(x)=U(n+\tfrac{1}{2},x)$$

$$\xrightarrow[x\to\infty]{}\frac{e^{-x^2/4}}{x^{n+1}}\left(1-\frac{(n+1)(n+2)}{2x^2}+\cdots\right). \quad (8)$$

In summary, the solutions to Laplace's equation which are localized at the apex of our parabolic wedge are

$$\Phi_{nq}(\xi,\eta,z)=\begin{cases}A_{nq}F_n(\xi)V[n+\frac{1}{2},(2q)^{1/2}\eta]e^{iqz}, & \eta\le\eta_0,\quad -\infty\le\xi\le\infty\ ;\\ B_{nq}F_n(\xi)U[n+\frac{1}{2},(2q)^{1/2}\eta]e^{iqz}, & \eta\ge\eta_0,\quad -\infty\le\xi\le\infty\ .\end{cases} \quad (9)$$

The two boundary conditions of the problem are the continuity of $\Phi(\xi,\eta,z)$ and of $\epsilon\partial\Phi(\xi,\eta,z)/\partial\eta$ at $\eta=\eta_0$. Imposing them on the solutions (9) we obtain, from the requirement that the coefficients A_{nq} and B_{nq} be nonzero, the dispersion relation for the electrostatic modes localized at the apex of a parabolic wedge

$$\epsilon(\omega)=\frac{V[n+\frac{1}{2},(2q)^{1/2}\eta_0]}{V'[n+\frac{1}{2},(2q)^{1/2}\eta_0]}\,\frac{U'[n+\frac{1}{2},(2q)^{1/2}\eta_0]}{U[n+\frac{1}{2},(2q)^{1/2}\eta_0]}\ . \quad (10)$$

In Eq. (10) the prime indicates differentiation with respect to the argument.

In contrast to Ref. 2, the $q\to\infty$ limit of the dispersion relation can be studied analytically. Using Eqs. (7) and (8) one can show that when x_0 $[=(2q)^{1/2}\eta_0]$ tends to infinity Eq. (10) reduces to

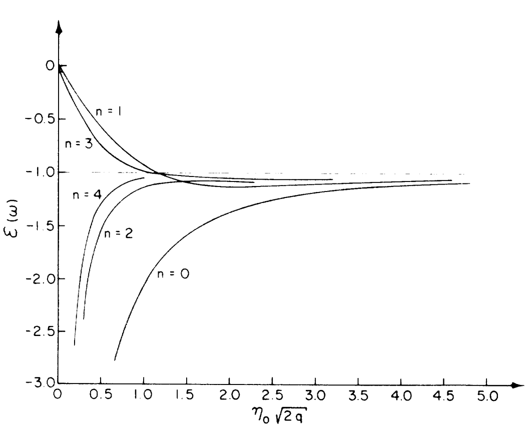

FIG. 2. First five allowed values of $\epsilon(\omega)$ as functions of $(2q)^{1/2}\eta_0$. Even values of n correspond to even modes, and odd values to odd modes.

$$\epsilon(\omega) = -(1 + 2/x_0^2 + \cdots) \ . \tag{11}$$

We emphasize that Eq. (11) holds for all n. Thus, in the $q = \infty$ limit, the dispersion relation of the edge modes coincides with that ($\epsilon = -1$) for surface plasmon modes bound to the plane interface between a dielectric medium and vacuum. This was to be expected, since a fluctuation of vanishingly small wavelength cannot probe the curvature of the interface.

If we assume that the material composing the wedge is a free-electron metal, then $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$, where ω_p is the bulk plasma frequency of the metal. Substituting this form for $\epsilon(\omega)$ into Eq. (11) we can solve explicitly for the frequency of the edge modes in the $q \to \infty$ limit:

$$\omega \to (\omega_p/\sqrt{2})x_0/(1 + x_0^2)^{1/2} \quad (x_0 \to \infty). \tag{12}$$

In Fig. 2 we present the dispersion relation, Eq. (10), for the first five values of n. In obtaining these curves use was made of the tabulated[4] values of the parabolic cylinder functions and of the recurrence relations satisfied by them. We remark that for the even modes $\epsilon(\omega) \to -\infty$ as $q \to 0$, whereas for the odd modes $\epsilon(\omega) \to 0$ as $q \to 0$. These limits follow from the expressions for $V(a_n, 0)$ and $V'(a_n, 0)$ given in Ref. 4.

In Fig. 3 we give the curves for ω/ω_p as a function of $x_0 = (2q)^{1/2}\eta_0$ obtained from Eq. (10) and the use of the free-electron gas dielectric constant.

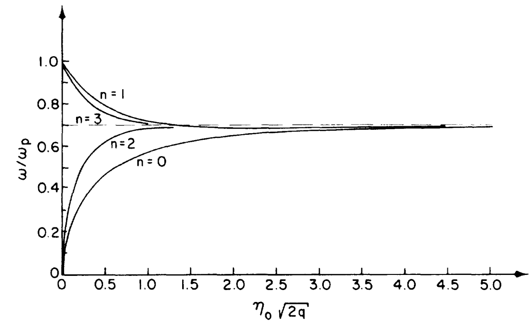

FIG. 3. Frequencies of electrostatic edge modes along a parabolic wedge characterized by a free-electron dielectric constant $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$. Only the first four are shown. Even values of n correspond to even modes, and odd values to odd modes.

The results of the present calculation confirm the result obtained by Davis for a hyperbolic wedge that rounding the tip of a dielectric wedge gives rise to a discrete set of electrostatic edge modes whose frequencies are functions of the (one-dimensional) wave vector characterizing their propagation along the apex of the wedge.

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