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Universal relations among critical amplitude. Calculations up to order ϵ^2 for systems with continuous symmetry

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A general derivation of the two-scale-factor universality is given using the renormalized ϕ^4 theory. As a consequence ten universal relations among critical amplitudes are obtained, any other universal relations being a combination of them. The situation is shown to be the same as for the scaling laws for critical exponents. The calculations up to order ϵ^2 for systems with continuous symmetry are performed for some of these relations such as the specific-heat amplitudes ratio and R_ξ^+, which relates the amplitudes of the correlation length and of the specific heat, etc. Comparison with experiments and series results and, in particular, the discussion of the superfluid helium and $RbMnF_3$ cases, are repeated with our improved numerical results.

INTRODUCTION

The hypothesis of two-scale-factor universality,[1] which states that for a system near the critical point, the length scale is related to the thermodynamic scales, extends the concept of universality[2] related to the scaling hypothesis.[3] It has been shown recently,[4] through Wilson's renormalization-group approach,[5] that such a relation among the three fundamental scales does exist in the critical domain. In this paper we give a general derivation of the two-scale-factor universality in the language of the renormalized field theory developed by Brézin, Le Guillou, and Zinn-Justin (BLZ).[6]

As is shown here, these techniques are well adapted for the study of critical universality. We show that the correlation and thermodynamic functions are fully determined when two independent thermodynamic scales have been chosen. As a consequence, there exist ten universal relations among the twelve fundamental critical amplitudes, just as there exist ten relations among the twelve critical exponents. The universality of most of these relations has already been derived.[7] Some of them were only conjectured as universal[8]; we derive these explicitly, as well as a new one which relates the specific-heat and magnetic-suceptibility amplitudes on the critical isotherm to the amplitude of the correlation length on the critical isochore above T_c. We extend the order-ϵ^2 calculations of some of these universal relations to systems with continuous symmetry; below T_c this requires a treatment different from the Ising case, owing to the presence of Goldstone modes on the coexistence curve. For these calculations we use a technique developed by BLZ which never breaks the symmetry.[6] The ϵ expansions extrapolated to $d=3$ are compared to experiments and series results and lead, in general, to appreciably better results than the first order in ϵ. Finally, relying on these better numerical results we improve the discussion on physical systems proposed by Ferer[9] and completed by Hohenberg *et al.*[4]: Given a measure of the specific heat, and using the results of the ϵ expansion at $d=3$, we can predict values for the transverse correlation length.

In Sec. I, we present the general derivation of the two-scale-factor universality and the ten universal relations among critical amplitudes. Section II contains ϵ-expansion results up to order ϵ^2 of $A^+/A^-, R_\xi^+, \xi^+/\xi^T, R_\xi^T, \ldots$. Section III is devoted to a discussion of two physical systems: superfluid helium ($n=2$) and $RbMnF_3$ ($n=3$). The derivation of three of the universal relations, as well as some details of the calculations, are contained in the appendixes.

I. RENORMALIZED THEORY AND UNIVERSALITY: GENERAL DERIVATION OF THE TWO-SCALE-FACTOR UNIVERSALITY

In this section, we use, for the most part, the notations and results of BLZ.[6] First, we recall briefly the relations between the renormalized theory and the physical theory of critical phenomena without describing all the details and difficulties, in order to show clearly all the universal characters of the theories.

In the critical domain, we can replace the true Hamiltonian $\mathcal{H}$ by an effective Hamiltonian in terms of a local field $\vec{S}_0(x)$,[5] and we are led to a φ^4 field theory with an ultraviolet cutoff Λ:

$$\beta\mathcal{H} = \int d^d x\,\mathcal{H}(x)\ , \qquad (1.1)$$

$$\mathcal{H}(x) = \tfrac{1}{2}(\nabla\vec{S}_0)^2(x) + \tfrac{1}{2}r_0\vec{S}_0^2(x) + (u_0/4!)[\vec{S}_0^2(x)]^2\ ,$$

in which $\vec{S}_0(x)$ is an order parameter with n components and

$$\vec{S}_0^2(x)=\sum_{i=1}^{n} S_{0i}^2(x)\,,$$
$$(\nabla\vec{S}_0)^2(x)=\sum_{k=1}^{d}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_k}S_{0i}(x)\right)^2 .$$

The presence of an ultraviolet cutoff is the memory, in the critical theory, of the microscopic range a of the interaction ($\Lambda\sim 1/a$), and is reflected, in Eq. (1.1), through the Λ dependence of r_0 and u_0. All integrals in momentum space are cut off at infinity by Λ. In the critical domain, all dimensioned parameters are measured in terms of Λ: $r_0\sim\Lambda^2, u_0\sim\Lambda^{4-d}$. The critical domain is defined by the large cutoff limit since, with k for wave numbers and M for magnetization, it corresponds to

$$r_0(T)-r_{0c}\ll\Lambda^2\,,\quad k\ll\Lambda\,,\quad M\ll\Lambda^{d/2-1}. \tag{1.2}$$

The hypothesis of universality is generally stated as the independence of some quantities (such as critical indices or thermodynamic-amplitude ratios) of the microscopic description of the physical system, here, in particular, of $\Lambda\sim 1/a$ and $g_0=u_0\Lambda^{-\epsilon}$ ($\epsilon=4-d$). Thus universal quantities depend only on n and d.

Since we are interested in the large cutoff limit, it is natural to use the renormalized ϕ^4 theory, which gives us a finite theory in the limit of infinite Λ.

If we define a new field S, a new "mass" t, a new coupling constant g, related to S_0, r_0, and g_0, the Hamiltonian (1.1), in terms of these new variables, reads

$$\begin{aligned}\mathcal{H}(x)=&\tfrac{1}{2}Z(\Lambda)(\nabla\vec{S})^2(x)+\tfrac{1}{2}\delta m^2\vec{S}^2(x)\\&+Z^2(\Lambda)\tilde{Z}(\Lambda)(g/4!)\mu^{\epsilon}(\vec{S}^2)^2(x)\\&+\tfrac{1}{2}Z(\Lambda)\hat{Z}(\Lambda)t\vec{S}^2(x)\,,\end{aligned}\tag{1.3}$$

in which

$$\vec{S}_0(x)=Z^{1/2}\vec{S}(x)\,,\quad r_0-r_{0c}=\hat{Z}t\,,\quad g_0\Lambda^{\epsilon}=\tilde{Z}g\mu^{\epsilon}\,.$$

The functions Z, δm^2, $\tilde{Z}$, and $\hat{Z}$ are defined by conditions on the renormalized one-particle irreducible connected Green's functions $\Gamma^{(L,N)}(q_i;p_j;t,M,g,\mu,\Lambda)$ ($i=1,\ldots,L$; $j=1,\ldots,N$), in which N is the number of S fields and L is the number of S^2 fields appearing in the Green's function; μ is an arbitrary parameter, M is the magnetization. These conditions may be chosen on the critical theory in zero field as[6]

$$\Gamma^{(0,2)}(p,-p;0,0,g,\mu,\Lambda)|_{p^2=0}=0\,, \tag{1.4a}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(0,2)}(p,-p;0,0,g,\mu,\Lambda)\Big|_{p^2=e^l\mu^2}=1\,, \tag{1.4b}$$

$$\Gamma^{(0,4)}(p_i;0,0,g,\mu,\Lambda)|_{p_ip_j=(\mu^2/4)(4\delta_{ij}-1)}=g\mu^{\epsilon}\,, \tag{1.4c}$$

$$\Gamma^{(1,2)}(q;p_i;0,0,g,\mu,\Lambda)|_{q^2=\mu^2;p_1^2=p_2^2=3\mu^2/4}=1\,. \tag{1.4d}$$

With these renormalization conditions, involving an arbitrary parameter μ, the theory is well defined for all values of the temperature, including the critical point itself.

The relation between this renormalized theory and the physical one defined by Eq. (1.1) (the bare theory) is given by

$$\begin{aligned}&\Gamma^{(L,N)}_{\text{bare}}(q_j;p_i;r_0-r_{0c};M_0,g_0,\Lambda)\\&\quad=[Z(\Lambda)]^{-N/2}[\hat{Z}(\Lambda)]^{-L}\Gamma^{(L,N)}(q_j;p_i;t,M,g,\mu,\Lambda).\end{aligned}\tag{1.5}$$

If $N=0$ and $L=2$, there is an additive term in (1.5) which adds a simple constant to the specific heat.

The important result of renormalization theory is that, with the conditions (1.4), the $\Gamma^{(L,N)}(\Lambda)$ functions on the right-hand side of (1.5) have a finite limit when $\Lambda\to\infty$. This limit will be implied later for all renormalized quantities. Then in the large cutoff limit, we are interested in the leading dependence on Λ, which is explicitly factorized. We are thus in a good position to study universality.

When taking the large-Λ limit, one introduces two sources of nonuniversality: first this arbitrary parameter μ and, second, the arbitrariness of the renormalization scheme (1.4) [we choose, following BLZ, to characterize this freedom by the parameter l in Eq. (1.4b)].

In order to proceed further in the study of universality, we have to investigate the critical behavior of the theory. This is done, in a standard way, through the renormalization-group equation.

Following BLZ, the $\Gamma^{(L,N)}$ functions satisfy a differential equation,

$$\left[\mu\frac{\partial}{\partial\mu}+W(g)\frac{\partial}{\partial g}-\tfrac{1}{2}\eta(g)\left(N+M\frac{\partial}{\partial M}\right)-\left(\frac{1}{\nu(g)}-2\right)\left(L+t\frac{\partial}{\partial t}\right)\right]\Gamma^{(L,N)}(q_j;p_i;t,M,g,\mu)=\delta_{N0}\delta_{L2}\mu^{-\epsilon}B(g), \tag{1.6}$$

in which $W(g)$, $\eta(g)$, and $\nu(g)$ are calculable in power series of g; $B(g)$ comes from the additive renormalization constant that we have previously mentioned.

The solution of Eq. (1.6), can be expressed via an arbitrary parameter λ:

$$\Gamma^{(L,N)}(q_j;p_i;t,M,g,\mu)=\left[\exp\left(-\int_g^{g(\lambda)}\frac{\frac{1}{2}N\eta(g')+L(1/\nu(g')-2)}{W(g')}\,dg'\right)\right]\Gamma^{(L,N)}(q_j;p_i;t(\lambda),M(\lambda),g(\lambda),\lambda\mu),\quad(1.7)$$

in which $g(\lambda)$, $t(\lambda)$, and $M(\lambda)$ are defined by

$$\ln\lambda=\int_g^{g(\lambda)}\frac{dg'}{W(g')}\ ,\qquad(1.8a)$$

$$t(\lambda)=t\left\{\exp\left[-\int_g^{g(\lambda)}\left(\frac{1}{\nu(g')}-2\right)\frac{dg'}{W(g')}\right]\right\},\qquad(1.8b)$$

$$M(\lambda)=M\left\{\exp\left[-\frac{1}{2}\int_g^{g(\lambda)}\frac{\eta(g')}{W(g')}\,dg'\right]\right\},\quad g(1)=g\,.\qquad(1.8c)$$

A simple dimensional analysis and Eqs. (1.7) and (1.8) show that

$$\Gamma^{(L,N)}(p;t,M,g,\mu)=(\lambda\mu)^{d-2L-N(d-2)/2}\left(\frac{M(\lambda)}{M}\right)^N\left(\frac{t(\lambda)}{t}\right)^L\Gamma^{(L,N)}\left(\frac{p}{\lambda\mu};\frac{t(\lambda)}{\lambda^2\mu^2},\frac{M(\lambda)}{(\lambda\mu)^{d/2-1}},g(\lambda),1\right).\qquad(1.9)$$

In this equation we change the notation by writing (p) instead of $(q_j;p_i)$. Note that the μ dependence is given by canonical dimensions and we shall set μ equal to one, when no confusion may arise.

We can now fix the arbitrary parameter λ in such a way that a dimensionless ratio, such as $t(\lambda)/\lambda\mu^2$, is no longer critical and choose, for example,

$$t(\lambda)/\lambda^2\mu^2=1\,.\qquad(1.10)$$

Then Eq. (1.8b) shows that the limit $t\to 0$ (critical domain) corresponds to the limit $\lambda\to 0$, and if $W(g)$ has a nontrivial zero g^* with a positive derivative at g^* then $\lambda\to 0$ corresponds to $g\to g^*$.

When $\lambda\to 0$, Eq. (1.10) gives

$$\lambda=[X(g)t]^\nu=\bar{t}^{\,\nu}\,,\qquad(1.11a)$$

$$M(\lambda)/M=Y(g)\bar{t}^{\,-\nu\eta/2}\,,\qquad(1.11b)$$

$$t(\lambda)/t=X(g)\bar{t}^{\,2\nu-1}\,,\qquad(1.11c)$$

$$M(\lambda)/\lambda^{d/2-1}=Y(g)M\bar{t}^{\,-\nu(d-2+\eta)/2}=x\,.\qquad(1.11d)$$

$X(g)$ and $Y(g)$ are two nonuniversal constants (they depend on g and not just on g^*), $\nu\equiv\nu(g^*)$, and $\eta\equiv\eta(g^*)$.

Thus using Eq. (1.11), Eq. (1.9) becomes, in the critical domain,

$$\Gamma^{(L,N)}(p;t,M,g)\to Y^N X^L\bar{t}^{\,\nu[d-N(d-2+\eta)/2]-L}\times\Gamma^{(L,N)}(p\bar{t}^{\,-\nu};1,x,g^*,1)\,.\qquad(1.12)$$

This result shows that the microscopic (nonuniversal) information carried by g has not totally disappeared by going into the critical domain, since the two nonuniversal constants X and Y, which depend on g, are still present. But note also that all critical amplitudes will be expressed in terms of X and Y and in terms of the values of the $\Gamma^{(L,N)}$ at the fixed point g^*, so that any quantity independent of X and Y will be independent of g.

Furthermore, it will be independent of the renormalization scheme, since two renormalized theories, with two different renormalization schemes (say l_1 and l_2), are related to each other through

$$\begin{aligned}&\hat{Z}(l_1)t_1=\hat{Z}(l_2)t_2\,,\\&\tilde{Z}(l_1)g_1=\tilde{Z}(l_2)g_2\,,\\&Z(l_1)M_1^2=Z(l_2)M_2^2\,,\\&\Gamma^{(L,N)}(p;t_1,M_1,g_1,\mu)=(Z_{21})^{-N/2}(\hat{Z}_{21})^{-L}\\&\qquad\times\Gamma^{(L,N)}(p;t_2,M_2,g_2,\mu),\end{aligned}\qquad(1.13)$$

in which $Z_{12}=Z(l_1)/Z(l_2)=(Z_{21})^{-1}$, and similarly for $\hat{Z}_{12}$ and $\tilde{Z}_{12}$.

Equation (1.13) is easily obtained from (1.5), since two renormalized theories have to give back the same large cutoff limit of the bare theory. Comparing Eqs. (1.11) and (1.12), we see that a change of renormalization scheme is totally absorbed by a change of the two constants X and Y. Since Eq. (1.13) has the same form as Eq. (1.5), we can say that, in the large cutoff limit of the bare theory, any changes in the cutoff Λ or in the coupling constant g_0 are absorbed by changing only two scales: the scale of the temperature (related to X) and of the magnetization (related to Y). Furthermore, any dimensionless quantity independent of X and Y will be universal.

This ends the derivation of the two-scale-factor universality. As a consequence, just as there are twelve critical exponents and ten relations among them, we can define twelve fundamental critical amplitudes and ten relations among them. Four of these relations involving purely thermodynamic quantities have been recently reviewed in Ref. 7.

We are now in position to give all the relations among the critical amplitudes. These latter are generally defined as[8]

(i) $T>T_c$, $H=0$ (critical isochore):

$$\xi = \xi_0^+ t^{-\nu},$$

$$\chi = C^+ t^{-\gamma},$$

$$C_s = (A^+/\alpha) t^{-\alpha};$$

(ii) $T < T_c, H = 0$ (critical isochore):

$$\xi = \xi_0^-(-t)^{-\nu'},$$

$$\chi = C^-(-t)^{-\gamma'},$$

$$C_s = (A^-/\alpha')(-t)^{-\alpha'},$$

$$M = B(-t)^{\beta};$$

(iii) $T = T_c, H \neq 0$ (critical isotherm):

$$\xi = \xi_0^c |H|^{-\nu_c},$$

$$C_s = (A^c/\alpha^c)|H|^{-\alpha_c},$$

$$H = DM^{\delta},$$

$$\chi = C^c |H|^{-\gamma_c};$$

(iv) $T = T_c, H = 0$ (critical point):

$$\chi(p) = \hat{D} p^{\eta - 2};$$

in which ξ is the correlation length, χ is the susceptibility, C_s is the singular part of the specific heat, M is the magnetization. Note that the microscopic length a has been included in the definitions of ξ_0 and $\hat{D}$.

Since H and χ are related on the critical isotherm, we obtain the obvious universal relation

$$\delta C^c D^{1/\delta} = 1 .$$

The others are given in Table I.

Note that, for $n \geq 2$, C^- is not finite and therefore there remains eleven critical amplitudes with nine relations among them.

In Appendix A we prove universality for Q_2, R_D, and R_{A^c}. Apart from the latter, the universality of Q_2 and R_D (related to Q_3) was already conjectured in a previous paper.[8]

TABLE I. Universal relations among critical amplitudes.

Among thermodynamic amplitudes	Among correlation amplitudes	Mixing thermodynamic and correlation amplitudes
C^+/C^-	ξ_0^+/ξ_0^-	$R_\xi^+ = \xi_0^+ (A^+)^{1/d}$ [b]
A^+/A^-		$Q_2 = (C^+/C^c)(\xi_0^c/\xi_0^+)^{2-\eta}$ [c]
$R_\chi = C^+ D B^{\delta - 1}$ [a]		$R_D = \hat{D}^{(\delta+1)/2} D$
$R_C = A^+ C^+ / B^2$ [a]		$R_{A^c} = [(\xi_0^+)^2 (A^c)^{\nu}]^d (C^c)^{\alpha}$

[a] From Refs. 7 and 24.

[b] From Refs. 1 and 4.

[c] From Ref. 8. In this reference three quantities are proposed as universal, $Q_1 = C^c \delta (B^{\delta-1} C^+)^{-1/\delta}$ and $Q_3 = \hat{D} (\xi_0^+)^{2-\eta}/C^+$, which are of course related to the quantities defined in Table I; $R_\chi = (Q_1)^{-\delta}$; $Q_3^{(\delta+1)/2} = (R_D/R_\chi) \times [(R_\xi^+/R_C)^d]^{(\delta-1)/2}$. The third one, Q_2, is in this table.

II. ϵ EXPANSION: MAIN LINES OF CALCULATIONS AND RESULTS

In this section we present the results, up to order ϵ^2, for some universal quantities for a system with an n-component order parameter. The calculations are more complicated than for the Ising model. In the presence of a magnetic field or below T_c, we define two correlation functions (transverse and longitudinal). Moreover, below T_c on the coexistence curve, the Goldstone modes give additional singularities, and the correlations do not decay exponentially but according to a power law.[10,11] However, the steepest-descent method on functional integrals developed in Ref. 6 gives us a systematic method of calculation. We shall not write all the details of the calculations since it would be lengthy and tedious; we just give in Appendix B all the graphs that contribute to our calculations. We thus limit this section to the definitions and results with some comments.

With an n-component system, we can separate the correlation function into a longitudinal and a transverse part:

$$\Gamma_{ij}^{(0,2)} = \Gamma_L V_i V_j + \Gamma_T (\delta_{ij} - V_i V_j), \tag{2.1}$$

in which $\Gamma_{ij}^{(0,2)}(p) = [G(p)]_{ij}^{-1}$ is the inverse of the correlation function

$$G_{ij}(x, y) = \langle S_i(x) S_j(y) \rangle .$$

V_i is the unit vector in the direction of the magnetization:

$$V_i = M_i / (\vec{M}^2)^{1/2} .$$

We are going to define two correlation lengths, ξ_L and ξ_T, related to Γ_L and Γ_T. It is easy to check that Γ_L, Γ_T and the corresponding ξ_L, ξ_T are solutions of a renormalization-group equation which is given by Eq. (1.6) with $N = 2$, $L = 0$ for Γ_L, Γ_T and with $N = 0$, $L = 0$ without the right-hand side for ξ_L, ξ_T.

We can then derive the critical behavior in the same way for the longitudinal and the transverse parts. However, the procedure will be slightly different from the one followed in Sec. I.

First, the solutions for correlation lengths are, following the same arguments as in Sec. I,

$$\xi_{L,T}(t, M, g, \mu) = (\lambda\mu)^{-1} \xi_{L,T}\left(\frac{t}{\lambda^2 \mu^2}, \frac{M}{(\lambda\mu)^{d/2-1}}, g(\lambda), 1\right), \tag{2.2}$$

which in the critical domain leads to

$$\xi_{L,T}(t, M, g, \mu) \to \mu^{-1} \left(\frac{Xt}{\mu^2}\right)^{-\nu} \xi_{L,T}(1, x, g^*, 1). \tag{2.3}$$

This form [(2.3)] is obtained by fixing λ by Eq. (1.10), but we can choose, from Eq. (2.2), another condition, namely,

$$\xi_{L,T}\left(\frac{t}{\lambda}, \frac{M}{\lambda^{d/2-1}}, g(\lambda)\right) = 1 \qquad (2.4)$$

which corresponds to Wilson's prescription that we integrate the renormalization-group transformations until the correlation length becomes of order unity.

Then following the same arguments which gave us Eqs. (1.11) and (1.12) we obtain, in the critical domain $[\xi(t, M, g) \to \infty]$,

$$\Gamma_{L,T}(p; t, M, g) \to Y^2(\xi_{L,T})^{\eta-2} \times \Gamma_{L,T}(p\xi_{L,T}; \tilde{t}\,\xi_{L,T}^{1/\nu}; \tilde{M}\xi_{L,T}^{(d-2+\eta)/2}; g^*, 1), \qquad (2.5)$$

in which the constants X and Y are the same for the longitudinal part and the transverse part.

Thus, it is possible to define a correlation length as the second moment of the spin-spin correlation function:

$$\xi^2 = \frac{d}{dp^2}\,\Gamma^{(0,2)}\Big|_{p^2=0} \Big/ \Gamma^{(0,2)}(p^2=0) \qquad (2.6)$$

as long as $\Gamma^{(0,2)}(p^2=0)$ is not zero.

However, for the transverse modes, we know that for p small $\Gamma_T(p^2)$ behaves as[12]

$$\Gamma_T(p^2) \sim H/M + O(p^2). \qquad (2.7)$$

Consequently, if we want to define a correlation length which has a zero-field limit, we have to use another procedure.

From Eqs. (2.5)–(2.7) and dimensional analysis, it is easy to convince one's self that one can define ξ_T by the small-p limit of Γ_T,[13] such as

$$\Gamma_T(p)|_{H=0} \underset{p\to 0}{\sim} p^2(\xi_T)^{2-d}/M^2 + O(p^4). \qquad (2.8)$$

We could have chosen another definition for ξ_T, but the definition given by (2.8) is very useful since it is related (for $n=2$) to the superfluid density $\tilde{\rho}_S$ of ^{4}He,[11]

$$\xi_T^{2-d} = (\hbar^2/m_H^2 k_B T)\tilde{\rho}_S, \qquad (2.9)$$

in which m_H is the ^{4}He mass.

Of course, ξ_T, which satisfies the usual renormalization-group equation, combines with the specific heat below T_c to give the universal quantity

$$R_\xi^T = \xi_T(\alpha' t^2 C_S^-)^{1/d}. \qquad (2.10)$$

Below T_c the transverse part dominates the longitudinal part,[14] so that in Table I, we only replace ξ^- by ξ_T. Above T_c, we can define ξ^+ by (2.6), as usual.

Then with these definitions, and using the calculations briefly summarized in Appendix B, we have calculated R_ξ^+, ξ_0^+/ξ_0^T, A^+/A^-, and R_ξ^T up to order ϵ^2. The results are

$$(R_\xi^+)^d = \frac{nS}{4}\left[1 + \epsilon\,\frac{n-1}{n+1} + \epsilon^2\left(\frac{(n+2)(3n^2+50n+28)}{4(n+8)^3} + \frac{n^2+8n+48}{4(n+8)^2}\,\zeta(2) - \frac{3(n+2)}{(n+8)^2}\,I\right)\right] + O(\epsilon^3) \qquad (2.11)$$

$$\left(\frac{\xi_0^T}{\xi_0^+}\right)^{2-d} = 2^{(d-2)\nu-1}\,\frac{n+8}{\epsilon}\,S\left[1 - \epsilon\,\frac{17n+76}{2(n+8)^2} + \epsilon^2\left(\frac{\frac{21}{8}n^3 + \frac{65}{2}n^2 + 268n + 508}{(n+8)^4} + \frac{n^2+6n-14}{4(n+8)^2}\,\zeta(2) + \frac{12(5n+22)}{(n+8)^3}\,\zeta(3) - \frac{3(n+14)}{2(n+8)^2}\,I\right)\right] + O(\epsilon^2), \qquad (2.12)$$

$$\frac{A^+}{A^-} = 2^\alpha\,\frac{n}{4}\left[1 + \epsilon + \epsilon^2\left(\frac{3n^4+74n^3+708n^2+3264n+6400}{2(n+8)^4} + \frac{4-n}{2(n+8)}\,\zeta(2) - \frac{3(5n+22)}{(n+8)^2}\,\zeta(3) + \frac{9(4-n)}{4(n+8)^2}\,I\right)\right] + O(\epsilon^3), \qquad (2.13)$$

$$(R_\xi^T)^{d-2} = \frac{2^{\epsilon/d}\epsilon}{S^{2/d}(n+8)}\left[1 + \epsilon\,\frac{4n+2}{(n+8)^2} + \epsilon^2\left(\frac{-\frac{3}{8}n^4 - \frac{11}{2}n^3 + \frac{171}{8}n^2 + 138n + 256}{(n+8)^4} + \frac{n^2+4n+12}{8(n+8)^2}\,\zeta(2) + \frac{3n(5n+22)}{2(n+8)^3}\,\zeta(3) + \frac{9(n+12)}{8(n+8)^2}\,I\right)\right] + O(\epsilon^3), \qquad (2.14)$$

in which S is the surface of the d-dimensional sphere divided by $(2\pi)^d$:

$$S^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d), \qquad (2.15)$$

I is given by the integral

$$I = \int_0^1 dx\,\frac{\ln[x(1-x)]}{1-x(1-x)} = -2.349,$$

and $\zeta(2) = \frac{1}{6}\pi^2$, $\zeta(3) = 1.202\,06$.

The result (2.14) is obtained from the others [(2.11)–(2.13)] since

$$R_\xi^T = R_\xi^+\,\frac{\xi_0^T}{\xi_0^+}\left(\frac{A^-}{A^+}\right)^{1/d}. \qquad (2.16)$$

We have checked that our results [(2.11)–(2.13)] are in agreement with the large-n limits.[15]

Note that the powers of 2 in front of the right-hand side of Eqs. (2.12)–(2.14) differ from those in Eqs. (43) and (44) of Ref. 4. Although they lead to the same result at order ϵ, they are different at order ϵ^2; this will lead to somewhat different numerical values.

Extrapolated values of these quantities at $d=3$ are displayed in Table II. Comparison of ϵ series with experiment is ambiguous, for instance, it is very different to set $\epsilon=1$ in the ϵ expansion of A^+/A^- or in that of A^-/A^+. We choose the ϵ expansion which minimizes the oscillations between the first and second order. This can be achieved by calculating the quantity itself, its inverse, or its [1, 1] Padé approximant, according to the particular case.

The agreement with experiments or series result (Table II) is better than for the first-order result, except for A^+/A^- whose extrapolated values are far from experimental results. A similar situation has been observed for exponents[16]: When one computes graphs with three loops, the results become worse than at a lower order (two loops). This observation combined with the fact that the ϵ expansion of R_ξ^+ leads to good agreement with series and experimental results, in spite of the contributions of some three-loop graphs (A^+ up to order ϵ^2), shows that the more the three-loop graphs are calculated, the farther away the numerical results of the ϵ expansion are from the experimental value (three-loop-graph contributions are larger in A^- than in A^+).

For $n=1$, ξ_0^+/ξ_0^- has been calculated up to order ϵ^2: (Ref. 17)

$$\xi_0^+/\xi_0^- = 2^\nu\left[1+\tfrac{5}{24}\epsilon+\tfrac{1}{432}\epsilon^2\left(\tfrac{295}{24}+2I\right)\right]+O(\epsilon^3). \quad (2.17)$$

Then we can obtain R_ξ^- from Eq. (2.16) in which "T" is replaced by "$-$":

$$R_\xi^- = \tfrac{1}{4}S\left\{1-\tfrac{11}{6}\epsilon+\epsilon^2\left[\tfrac{2251}{1296}+\tfrac{5}{216}\zeta(2)+\zeta(3)-\tfrac{5}{27}I\right]\right\}. \quad (2.18)$$

We have also calculated R_D up to order ϵ:

$$R_D=\frac{1}{6}\left(\frac{g^*}{S}\right)^{(\delta-1)/2}\left[1+\frac{\epsilon}{2}\left(1-\ln 2-\frac{n-1}{n+8}\ln 3\right)\right]+O(\epsilon^3), \quad (2.19)$$

in which

$$g^*=\frac{6\epsilon}{n+8}\left[1+\epsilon\left(\frac{3(3n+14)}{(n+8)^2}-\frac{1}{2}\right)\right]+O(\epsilon^3)$$

and

$$\delta=3+\epsilon+O(\epsilon^2).$$

TABLE II. ϵ expansion for $\epsilon=1$ compared with series and experimental results for some universal quantities.

	$n=1$ Series	$n=1$ ϵ expansion	$n=2$ Series	$n=2$ ϵ expansion	$n=2$ e^{ts}	$n=3$ Series	$n=3$ ϵ expansion	$n=3$ e^{ts}
R_ξ^+	0.26 [a]	Order ϵ: 0.23 [a] Order ϵ^2: 0.27 [g]	0.36 [a]	ϵ: 0.30 [a] ϵ^2: 0.36 [g]		0.42 [a]	ϵ: 0.36 [a] ϵ^2: 0.42 [g]	0.45 [d]
A^+/A^-	0.51 [a]	ϵ: 0.55 [a] ϵ^2: 0.48 [g]	1.08 [a]	ϵ: 0.99 [a] ϵ^2: 0.88 [g]	1.07 [b]	1.52 [a]	ϵ: 1.36 [a] ϵ^2: 1.24 [g]	1.46 [f]
ξ_0^+/ξ_0^-	1.96 [a]	ϵ^2: 1.91 [a]						
R_ξ^-	0.17 [a]	ϵ: 0.09 [a] ϵ^2: 0.18 [g]						
ξ_0^+/ξ_0^T				ϵ: 0.27 [a] ϵ^2: 0.33 [g]	0.41 [c]		ϵ: 0.30 [a] ϵ^2: 0.38 [g]	0.5–0.7 [d]
R_ξ^T				ϵ: 0.95 [a] ϵ^2: 0.96 [g]	0.85 [b]		ϵ: 0.88 [a] ϵ^2: 0.90 [g]	1.8–1.2 [d]
R_D	14.18 [e]	ϵ: 11.02 [g] ϵ^2: ... [h]		ϵ: 8.22 [g]			ϵ: 6.69 [g]	

[a] See, in Ref. 4, Table I and references therein.
[b] From Refs. 18 and 19.
[c] From Refs. 18 and 19 and series values of R_ξ^+.
[d] See, in Ref. 4, Table II and references therein.
[e] From Ref. 8 and, in Ref. 7, Table III and series values of R_ξ^+.
[f] From Ref. 22.
[g] From this work.
[h] Q_1 has been calculated for $n=1$ up to order ϵ^2, but since R_C is not known at this order we cannot give the result for R_D.

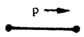
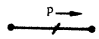
 $\equiv \left[p^2 + t + \frac{gM^2}{2}\right]^{-1}$  $\equiv \left[p^2 + t + \frac{gM^2}{6}\right]^{-1}$

FIG. 1. Longitudinal and transverse Feynman propagators.

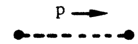
 $\equiv [p^2]^{-1}$

FIG. 2. Feynman propagator for counterterms.

III. APPLICATION TO PHYSICAL SYSTEMS

Relying on our better numerical results, we can improve the comparison with experiments discussed in Refs. 4 and 9. We can, through the numerical values of the quantities calculated in Sec. II, and with a measure of the specific heat, give a theoretical evaluation of ξ_T and compare with experiments. The experimental values and the reasoning are the same as those of Ref. 4.

A. Liquid helium

From a measurement of the superfluid density,[18] $\tilde{\rho}_S = 0.35(-t)^{0.67}$ cm^3, and the relation (2.9) with $T = 2.17$ K, we obtain an experimental estimate of the transverse correlation length:

$$\xi^T = 3.40(-t)^{-0.67} \text{ Å} . \tag{3.1}$$

From the ϵ expansion we have two ways to calculate ξ^T. The first uses R_ξ^T and a measurement of A^-, while the second uses R^+, ξ^+/ξ^-, and a measure of A^+/A^-. Using the experimental results at saturated vapor pressure,[19]

$$A = 1.65 \times 10^{22} \text{ cm}^{-3}, \quad A^+/A^- = 1.065, \quad \alpha = -0.0154 , \tag{3.2}$$

and the ϵ expansion given in Table II for $\epsilon = 1$, the first method gives

$$\xi^T = 3.9(-t)^{-0.67} \text{ Å} , \tag{3.3}$$

and the second method gives

$$\xi^T = 4.3(-t)^{-0.67} \text{ Å} . \tag{3.4}$$

These two results are closer to each other than those evaluated up to first order in ϵ (Ref. 4) (~10% instead of 30%), but they still differ appreciably from the experimental estimate [Eq. (3.1)]. (Note that in Ref. 4 there is an error in the experimental estimate of R_ξ^T: We find 0.85 for R_ξ^T instead of 0.90, and 0.41 for ξ_0^+/ξ_0^T instead of 0.39.) If the order ϵ^2 does not bring the theoretical estimate for ξ_T closer to its experimental value (3.1), we can assert that superfluidity is not due to the condensation of pairs of helium atoms,[20] since this would give an experimental value four times as big as in Eq. (3.1) and would be in a too large disagreement with the ϵ expansion

B. $RbMnF_3$

In the isotropic antiferromagnet it is possible to measure ξ^+ directly and ξ^T through the spin-wave velocity; following Ref. 4 we have, from experiments,[21]

$$\xi^+ = 2.1 t^{-0.71} \text{ Å}, \tag{3.5}$$

$$\xi^T = 6.2(-t)^{-0.54} \text{ Å} , \tag{3.6}$$

for which the scaling law $\nu = \nu'$ is violated.

Our estimates of ξ^+ and ξ^T from experimental values,[22]

$$A^+ = 9.87 \times 10^{21} \text{ cm}^{-3}, \quad A^+/A^- = 1.46, \quad \alpha = -0.135, \tag{3.7}$$

and ϵ expansion of R_ξ^+, R_ξ^T, and ξ^+/ξ^T for $n = 3$ and $\epsilon = 1$ listed in table are

$$\xi^+ = 1.96 t^{-0.71} \text{ Å} , \tag{3.8}$$

$$\xi^T = 4.76(-t)^{-0.71} \text{ Å} , \tag{3.9}$$

$$\xi^T = 5.4(-t)^{-0.71} \text{ Å} . \tag{3.10}$$

The values (3.9) and (3.10) correspond to the two possible ways to calculate ξ^T.

The results for ξ^+ are in good agreement with (3.5). It is interesting to note that the estimate for ξ^+ from series gives the same value[23] as (3.8).

For ξ^T, the same remarks as in the helium case can be made: The two values are closer to each other (compared to the first-order results[4]) and seem to converge towards a number that is still not very close to the experimental estimates (even if we take into account the strong violations of the scaling law, as was done in Ref. 4).

$$= J(p) = \int \frac{d^d q}{(2\pi)^d} \left[(p+q)^2 + t + \frac{gM^2}{2}\right]^{-1} \times \left[q^2 + t + \frac{gM^2}{6}\right]^{-1}$$

$$= J(0),$$

FIG. 3. Illustration of the Feynman rules for two graphs.

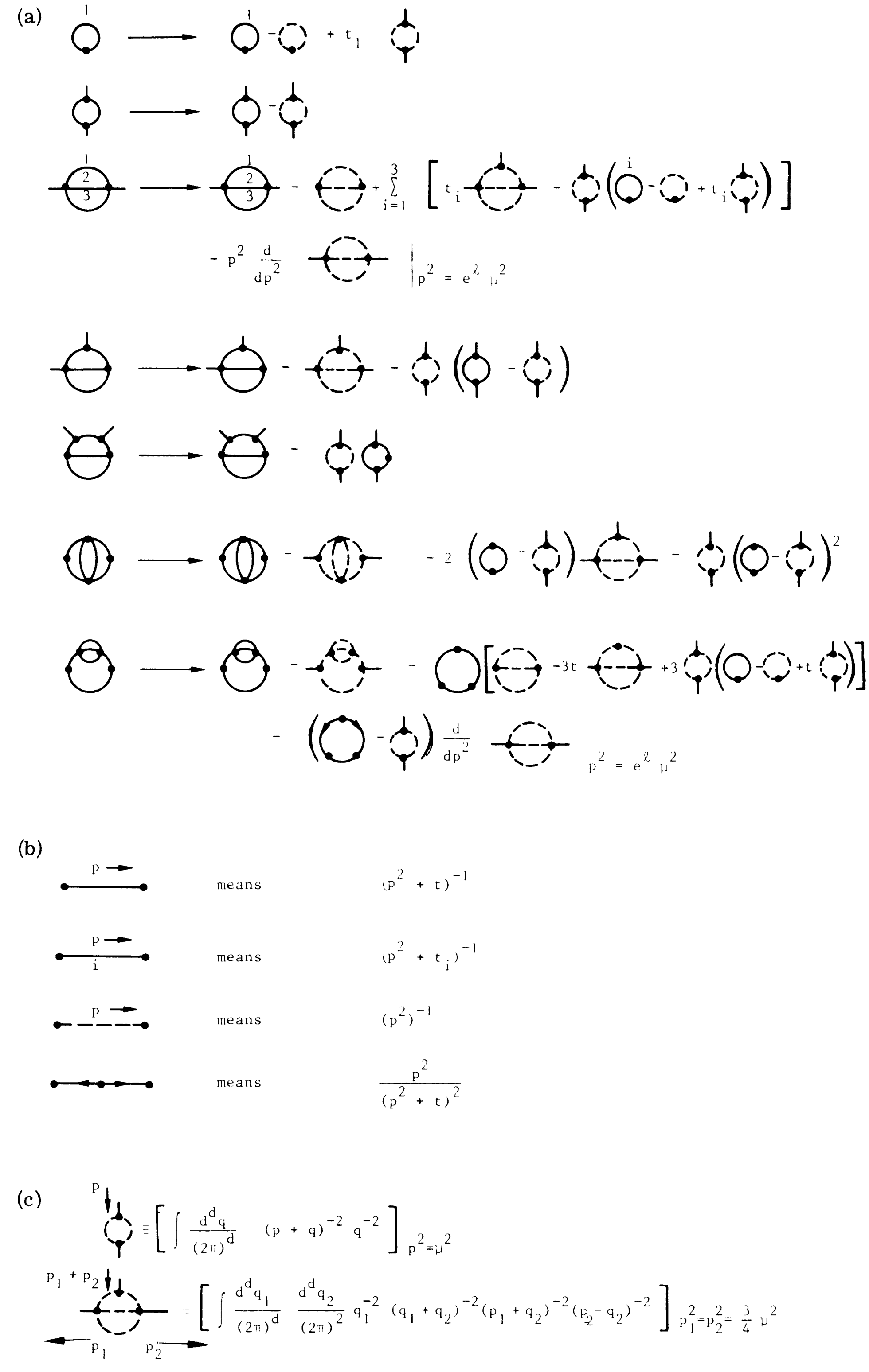


FIG. 4. (a) Finite combinations of graphs for $\epsilon = 0$, following renormalization conditions [Eq. (1.4)]. (b) Feynman propagators used in Fig. 4 (a). When there is no index associated with the lines of a graph, then each propagator has been the same "mass" t. (c) Two contributions to counterterms in which the subtraction point μ enters.

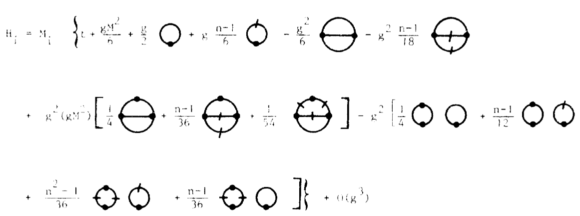


FIG. 5. Equation of state.

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APPENDIX A: UNIVERSALITY OF Q_2, R_D, AND R_{A^c}

In this appendix we do not prove the universality of all the quantities listed in Table I, since for seven of them this has already been done elsewhere. We only give the proof of universality of the quantities Q_2 (introduced by Tarko and Fisher[8]), R_D (related to Q_3[8]), and R_{A^c}, which has never been done. The proof for the seven others would follow the same arguments.

In our notations we have

$$\chi^{-1}(p)=\Gamma^{(0,2)}(p;t,M,g)\,, \tag{A1}$$

$$C_s=-\Gamma^{(2,0)}(0;t,M,g)\,, \tag{A2}$$

$$H=\Gamma^{(0,1)}(t,M,g)\,, \tag{A3}$$

$$\xi=\xi(t,M,g)\,. \tag{A4}$$

Then from Eqs. (1.12) and (2.3) it is easy to see that for $t>0$ (i.e., $T>T_c$) and $x=0(H=0)$,

$$(C^+)^{-1}=Y^2X^\gamma\Gamma^{(0,2)}(0;1,0,g^*)\,, \tag{A5}$$

$$\xi_0^+=X^{-\nu}\xi(1,0,g^*)\,; \tag{A6}$$

and for $t=0$ (i.e., $T=T_c$) and $H\neq 0$ we obtain, similarly,

$$H=Y^{\delta+1}\Gamma^{(0,1)}(0,1,g^*)M^\delta\,, \tag{A7}$$

$$\chi^{-1}=Y^{\delta+1}\Gamma^{(0,2)}(0;0,1,g^*)M^{\delta-1}\,, \tag{A8}$$

$$C_s=X^2Y^{\alpha(\delta+1)/\nu d}\Gamma^{(2,0)}(0;0,1,g^*)M^{\alpha(\delta+1)/d\nu}\,, \tag{A9}$$

$$\xi=Y^{-(\delta+1)/d}\xi(0,1,g^*)M^{-(\delta+1)/d}\,. \tag{A10}$$

Then it follows that

$$(C^c)^{-1}=Y^{(\delta+1)/\delta}\,\Gamma^{(0,2)}(0;0,1,g^*)\times[\Gamma^{(0,1)}(0;1,g^*)]^{(1-\delta)/\delta}\,, \tag{A11}$$

$$A^c=-X^2Y^{\alpha(\delta+1)/\delta\nu d}\Gamma^{(2,0)}(0;0,1,g^*)\times[\Gamma^{(0,1)}(0,1,g^*)]^{-\alpha(\delta+1)/d\delta}\,, \tag{A12}$$

$$\xi^c=Y^{(\delta+1)/d\delta}\xi(0,1,g^*)[\Gamma^{(0,1)}(0,1,g^*)]^{(\delta+1)/d\delta}\,, \tag{A13}$$

$$D=Y^{\delta+1}\Gamma^{(0,1)}(0,1,g^*)\,. \tag{A14}$$

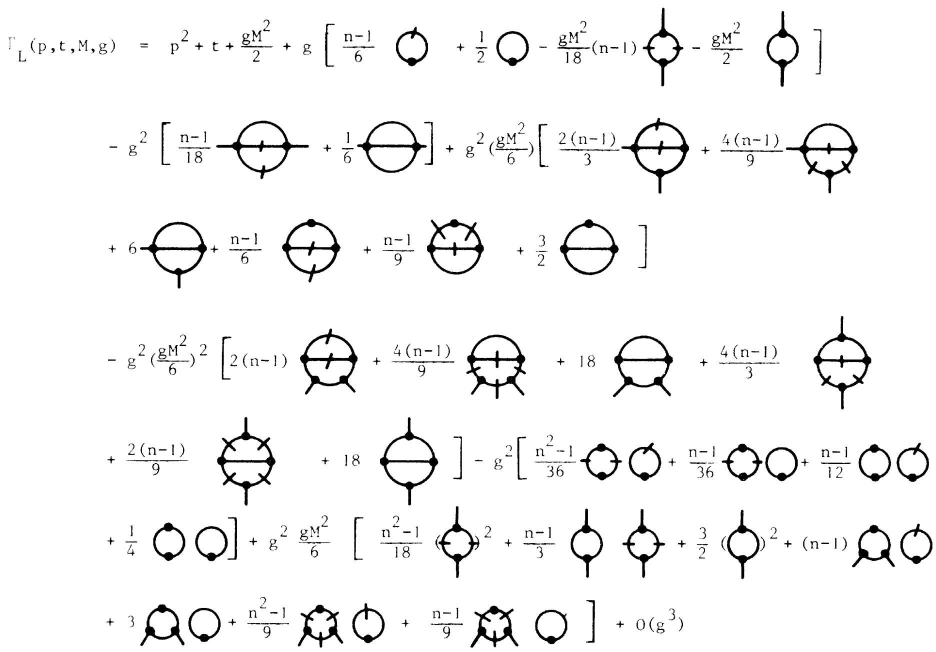

FIG. 6. Longitudinal part of the correlation function.

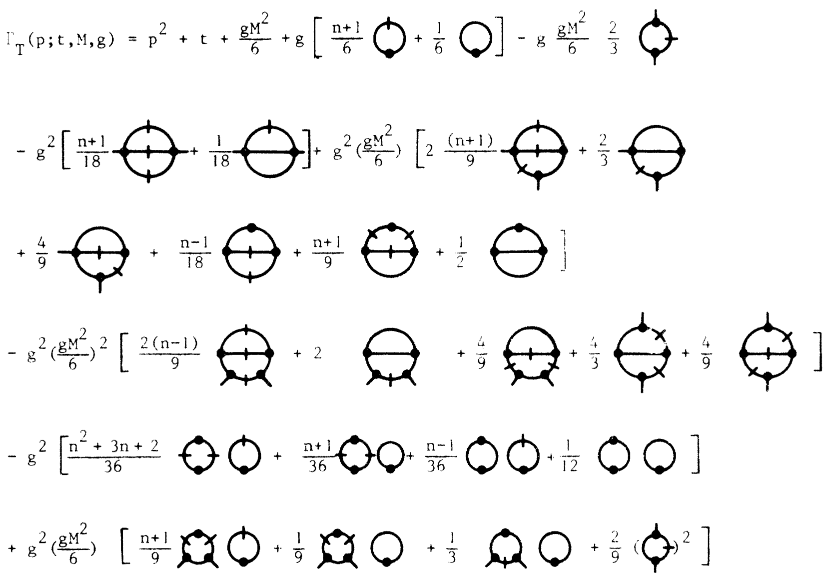

FIG. 7. Transverse part of the correlation function.

Finally at $t=0$ ($T=T_c$) and $H=0$ we have

$$\chi^{-1}(p)=Y^2p^{2-\eta}\Gamma^{(0,2)}(1;0,0,g^*)\,, \qquad (A15)$$

from which it follows that

$$\hat{D}^{-1}=Y^2\Gamma^{(0,2)}(1;0,0,g^*)\,. \qquad (A16)$$

With these definitions, it is now obvious to check that the combinations

$$Q_2=\frac{C^+}{C^c}\left(\frac{\xi_0^c}{\xi_0^+}\right)^{2-\eta},$$

$$R_{A^c}=[(\xi_0^+)^2(A^c)^\nu]^d(C^c)^\alpha\,,$$

and

$$R_D=\hat{D}^{(\delta+1)/2}D$$

are dimensionless (independent of μ, if we restore the μ dependence) and independent of X and Y, and thus, following the discussion in Sec. I, are universal.

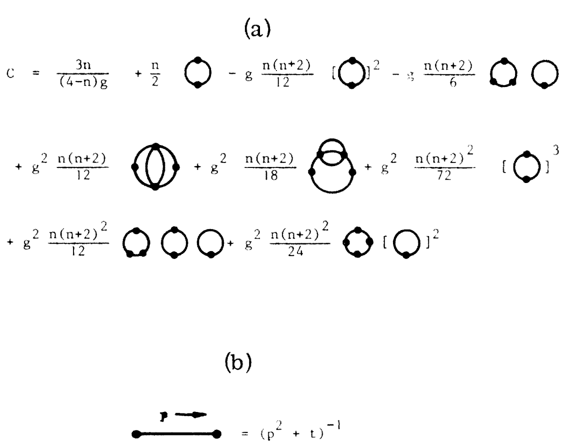

FIG. 8. (a) Specific heat above T_c. (b) Above T_c, for $H=0$, $M=0$, there is only one Feynman propagator: $(p^2+t)^{-1}$ (see Fig. 1).

APPENDIX B: ϵ EXPANSION ($\epsilon=4-d$)

We use for the ϵ calculations in the case of a system with O(n) symmetry, the steepest-descent method, developed in Ref. 6, applied to functional integrals. With this method, the O(n) symmetry is never broken by a translation on the mean value of the order parameter, and we do not need Ward identities to restore the symmetry.

We start with the Hamiltonian given by Eq. (1.3) and renormalization conditions given by Eq. (1.4). As was mentioned in Sec. II, there are two correlation functions (transverse and longitudinal), so there are two Feynman propagators (see Fig. 1). The counterterms (i.e., Z, $\hat{Z}$, δm^2, and so on) are calculated from renormalization conditions (1.4) at the critical point (Fig. 2).

For $d=4$, the integrals given in Fig. 3, for example, and counterterms, are built in such a way so as to subtract all the divergences of all the divergent graphs (integrals). Figure 4 exhibits all the divergent graphs (for $d=4$) we shall use, with their subtractions following the conditions (1.4). These subtractions will be now understood and the finite combinations of graphs will be represented by the divergent graphs themselves.

Using the expression for the equation of state (Fig. 5), the longitudinal (Fig. 6) and transverse (Fig. 7) parts of the correlation functions for $g=g^*$, and the ϵ expansion of g^*,[6] we have calculated, above T_c in zero field (i.e., $M=0$), the cor-

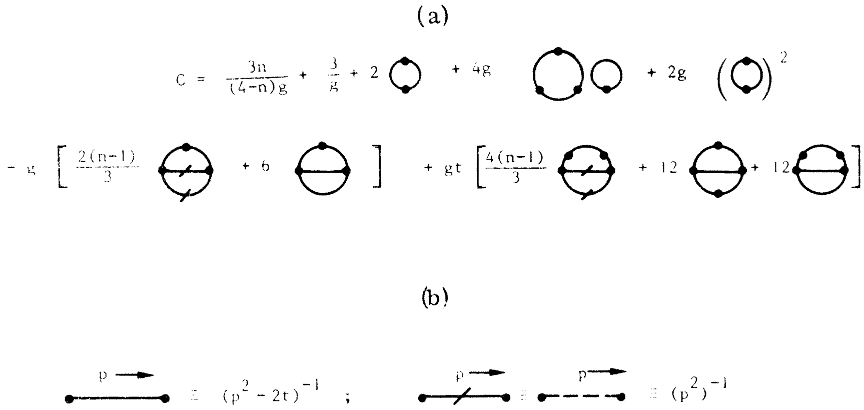

FIG. 9. (a) Specific heat below T_c ($t<0$). (b) Below T_c, in zero field, the Feynman propagators are those given in Fig. 1 with $\frac{1}{6}gM^2=-t$.

relation length defined by Eq. (2.6), and, below T_c in zero field (i.e., M is obtained from the equation of state for $H=0$), the transverse correlation length defined by Eq. (2.8).

We thus obtain ξ_0^+/ξ_0^T [Eq. (2.12)]. In order to obtain R_ξ^+ up to order ϵ^2, we need A^+ up to order ϵ^2.

The specific heat above T_c in zero field has the form

$$C=(A^+/\alpha)t^{-\alpha}+B\ . \tag{B1}$$

Figure 8 gives its expression in terms of graphs. Note that there are graphs with three loops. Fortunately, to obtain A^+ up to order ϵ^2 we have to know only the $1/\epsilon$ terms of these graphs at $t=0$. Since their counterterms have already been calculated[6] and by requiring a finite limit at $d=4$ of the combinations given in Fig. 4(a), we can extract the contribution of these graphs to the second-order in ϵ for A^+.

The specific heat is defined up to a constant; following BLZ, we fit this constant in such a way that C satisfies exactly the renormalization-group equation by introducing the term $-3n/(4-n)g^*$.[6]

Having A^+ and ξ_0^+ we obtain R_ξ^+ given by (2.11). Below T_c in zero field, the specific heat has the following form:

$$C=(A^-/\alpha')(-t)^{-\alpha'}+B\ , \tag{B2}$$

where B is the regular part of C and is the same constant as in Eq. (B1). Since $\alpha=\alpha'$, we calculate B using the specific heat above T_c (Fig. 8), A^+ up to order ϵ^2, and α up to order ϵ^3, which has been already calculated.[16] Having B, we calculate A^- up to order ϵ^2 through only two-loop graphs. Figure 9 gives the expression in terms of graphs of the specific heat below T_c in zero field.

Then we obtain A^+/A^- [Eq. (2.13)]. And finally, combining R_ξ^+, ξ_0^+/ξ_0^T, and A^+/A^-, we obtain R_ξ^T [Eq. (2.14)].

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