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Transport entropy of vortices in superconductors with paramagnetic impurities*

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The entropy S_D transported by a unit segment of a moving vortex line, in a type-II superconductor in the flux-flow regime, is calculated microscopically for gapless superconductors containing arbitrary amounts of paramagnetic and nonmagnetic impurities, assuming low average magnetic induction $B \simeq 0$, and large Ginzburg-Landau parameter $\kappa \gg 1$. The calculation is based on a new prescription for calculating certain heat-current-related transport properties in magnetic conductors, recently derived by the author to resolve a contradiction with either the third law of thermodynamics or Onsager's principle. For high concentrations of magnetic impurities, when the gaplessness condition is satisfied at absolute zero temperature, we successfully show for the first time in the low-field limit that a requirement from the third law, viz., $S_D \to 0$ as $T \to 0$, is exactly satisfied, thereby giving strong support to the underlying new method for calculating S_D and other related transport properties. Combining the present results with our earlier results on S_D in the high-field limit $B \simeq H_{c2}$, we predict S_D to first rise linearly as B is lowered from H_{c2}, and then to bend upward to approach a finite limit as B is further lowered to approach zero, for practically all concentrations of magnetic and nonmagnetic impurities. The exact amount of upward bending depends on the concentrations of the two types of impurities, but is generally larger for dirtier systems. As a side product of this work, we also give a plausible identification of the physical meaning of an anomalous quantity u_1 in the set of dynamic equations first derived by Éliashberg as being proportional to the local temperature deviation from equilibrium.

I. INTRODUCTION

Recently, the author has revised the existing procedure for calculating certain heat-current or temperature-gradient related off-diagonal transport coefficients of a *magnetic* conductor such as a type-II superconductor, in order to resolve a conflict with either the third law of thermodynamics or the Onsager's reciprocity principle. In a previous paper (henceforth referred to as I),[1] the author has presented a general argument in irreversible thermodynamics which justifies this revision. To further support the new prescription obtained in I for calculating heat-current responses in magnetic conductors, the author has applied it to calculate the transport entropy of vortices in gapless type-II superconductors containing an arbitrary amount of magnetic and nonmagnetic impurities, and for both the high-field ($B \simeq H_{c2}$) and the low-field ($B \simeq 0$) limits. The high-field calculation is relatively simple, with results already presented in I. The low-field calculation, however, is much more involved and is only very briefly mentioned in that paper, in order not to obscure the more fundamental points made there. The purpose of the present paper is then to report the details of the low-field calculation, and also to present some overall analysis of our results on S_D obtained in both field limits.

For a type-II superconductor in the Abrikosov state[2] with an external magnetic field in the z direction, the transverse thermal and electric transport properties are strongly coupled, so that in general one must write

$$(\vec{j}^h)_{av} = \overleftrightarrow{K}\cdot(-\vec{\nabla}T)_{av} + \overleftrightarrow{\alpha}\cdot(\vec{\mathcal{E}})_{av}, \qquad (\vec{j})_{av} = \overleftrightarrow{\beta}\cdot(\vec{\nabla}T)_{av} + \overleftrightarrow{\sigma}\cdot(\vec{\mathcal{E}})_{av}, \tag{1}$$

where all vectors and tensors have indices running through x and y only. In Eqs. (1), $\vec{j}^h$ and $\vec{j}$ are the local heat- and charge-current densities, T and μ are the local temperature and chemical potential (with T_0 and μ_0 denoting their corresponding equilibrium values), $\vec{\mathcal{E}} \equiv \vec{E} - \vec{\nabla}\mu/e$ is the effective local electric field (with $\vec{E}$ being the true local electric field and e the electronic charge), and we have used $(\cdots)_{av}$ to denote a space average, reserving $\langle\cdots\rangle$ for a statistical ensemble average. The Onsager theorem requires that $\alpha_{ij}(\vec{H}_{ext}) = -T_0\beta_{ji}(-\vec{H}_{ext})$,[3] which for isotropic systems reduces to $\alpha_{ij}(\vec{H}_{ext}) = -T_0\beta_{ij}(\vec{H}_{ext})$. If one further ignores small contributions related to the Hall effect,[4] then one has the following simple structures for the four tensorial transport coefficients (which are invariant with respect to rotation about the z axis):

$$\overleftrightarrow{K} = K_s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \overleftrightarrow{\sigma} = \sigma_s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$\overleftrightarrow{\alpha} = \alpha_s\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \overleftrightarrow{\beta} = \beta_s\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

so that without loss of generality one may rewrite Eqs. (1) simply as

$$(j_x^h)_{av} = K_s(-\nabla_x T)_{av} + \alpha_s(\mathcal{E}_y)_{av},$$
$$(j_y)_{av} = \beta_s(\nabla_x T)_{av} + \sigma_s(\mathcal{E}_y)_{av}. \quad (2)$$

The Onsager relation then reduces simply to $\alpha_s = \beta_s T_0$. The four transport coefficients in Eqs. (2) will be called, respectively, thermal conductivity (K_s), flux-flow conductivity (σ_s), Ettingshausen coefficient (α_s), and Nernst coefficient (β_s), although experimentally one often measures certain combinations of these coefficients.[3]

As is now well known, for a type-II superconductor in the presence of an electric field $(\vec{\mathcal{E}})_{av}$ perpendicular to the average magnetic induction $\vec{B} \equiv (\vec{b})_{av}$, the vortex lines must move (in the absence of pinning forces) with a uniform velocity $\vec{v} \perp (\vec{\mathcal{E}})_{av}$ and $\vec{B}$, such that $(\vec{\mathcal{E}})_{av} = -\vec{v}\times\vec{B}$.[5] (Throughout this paper we shall put $\hbar = c = k_B = 1$ where k_B is Boltzmann's constant.) Since vortex cores are expected to carry more entropy than their superconducting surroundings, such a flux flow is expected to always result in a entropy transport which explains why $\alpha_s \neq 0$. The "transport entropy" S_D is defined as the amount of *deliverable* entropy carried by a unit segment of every moving vortex line. This definition is equivalent to the mathematical statement that $(\vec{j}^h)_{av} = (B/\phi_0)T_0 S_D \vec{v}$, where the B/ϕ_0 factor gives the number of vortex lines per unit area and ϕ_0 is the flux quantum. It then follows from $v = (\mathcal{E})_{av}/B$ that

$$S_D = \alpha_s \phi_0 / T_0. \quad (3)$$

The third law of thermodynamics implies that $S_D \to 0$ as $T_0 \to 0$, which is true whether one would question the physical interpretation of S_D, since the third law may be directly applied to the entropy current $\vec{j}^h/T_0$, with Eq. (3) merely treated as a mathematical definition for S_D.

To calculate α_s and therefore S_D microscopically, one needs a microscopic prescription for evaluating a heat-current linear response. The original prescription existing in the literature is to evaluate the statistical average (over a linearly perturbed ensemble) of a "heat-current-density" operator $\hat{j}_0^h \equiv \hat{j}^E - (\mu_0/e)\hat{j}$, where $\hat{j}^E$ and $\hat{j}$ are the energy- and charge-current-density (vector) operators, respectively.[6,7] As was reviewed in I, this old prescription violates the third law of thermodynamics, since it gives a transport entropy S_D^0 which diverges like T_0^{-1} at low temperatures. The new prescription deduced in I reads

$$\vec{j}^h = \vec{j}_0^h + \vec{\mathcal{E}} \times \vec{m}, \quad (4)$$

where $\vec{j}_0^h \equiv \langle \hat{j}_0^h \rangle$; $m(\vec{r})$ is the local magnetization density at equilibrium, and is given by $\vec{\nabla}\times\vec{m} = \langle \hat{j} \rangle_{eq}$, plus the boundary condition that $\vec{m} \equiv 0$ outside the sample.[8] The second term in Eq. (4) is new. It implies also a new magnetic contribution to the Ettingshausen coefficient

$$\alpha_s = \alpha_s^0 + |(\vec{\mathcal{E}} \times \vec{m})_{av}| / |(\vec{\mathcal{E}})_{av}|, \quad (5)$$

where α_s^0 denotes the value calculated with the operator $\hat{j}_0^h$. As we shall further demonstrate in this paper, the transport entropy S_D calculated with this new prescription does satisfy the third law of thermodynamics under a variety of conditions. If the effective electric field $\vec{\mathcal{E}}$ is uniform in space, as is the case when α_s is evaluated only to lowest order in $(H_{c2} - B)/H_{c2}$ near the second-order phase-transition line, then Eq. (5) reduces simply to

$$\alpha_s = \alpha_s^0 + M, \quad (6)$$

where $\vec{M} \equiv (\vec{m})_{av}$. Equation (6) has already been proposed by Maki,[7] although he did not realize that the validity of this equation is only restricted to the case $H_{c2} - B \ll H_{c2}$, and that the argument he used to justify Eq. (6) is actually logically inconsistent (see I). We mention here in passing that the argument presented in I also implies a correction term for the Nernst coefficient β_s:

$$\beta_s = \beta_s^0 + |(\vec{m} \times \vec{\nabla} T/T_0)_{av}| / |(\vec{\nabla} T)_{av}|, \quad (7)$$

which reduces to $\beta_s = \beta_s^0 + M/T_0$ only near H_{c2}. Thus near H_{c2} one has $\alpha_s = \beta_s T_0$ if only $\alpha_s^0 = \beta_s^0 T_0$, as we have pointed out in I. But below H_{c2}, one most likely has $\alpha_s^0 \neq \beta_s^0 T_0$ in order to uphold the Onsager relation $\alpha_s = \beta_s T_0$. We have not yet tested Eq. (7) to explicitly verify this last statement.

Returning to Eq. (4), we note a subtle point concerning the definition of $\hat{j}^E$ which appears in the definition of $\hat{j}_0^h$. Equation (4) is derived in I from the local thermodynamic relation $T\delta s = \delta\epsilon - \mu\delta n + \vec{m}\cdot\delta\vec{b}$, where for an equilibrium system, ϵ is simply the ensemble average of the Hamiltonian density: $\epsilon \equiv \langle \hat{h} \rangle$. The relation between $\hat{j}^E$ and $\hat{h}$ is then the equation of continuity $\partial\hat{h}/\partial t + \vec{\nabla}\cdot\hat{j}^E = 0$. We have already pointed out in I that in a transport situation when an electric field is applied to the sample, the Hamiltonian density changes to $\hat{h}_T$, which includes a scalar potential term $\hat{\rho}\phi$ in a general gauge. The definition of ϵ must then be generalized to $\epsilon \equiv \langle \hat{\epsilon} \rangle$ with $\hat{\epsilon} \equiv \hat{h}_T - \hat{\rho}\phi$, in order to maintain the gauge invariance of ϵ and $\vec{j}^E$. Note that $\hat{\epsilon} = \hat{h}$ only if one works in a pure scalar gauge, otherwise $\hat{\epsilon}$ contains correction terms due to a time-dependent vector potential $\delta A(\vec{r}, t)$ which must be added to the original static vector potential $A_0(\vec{r})$, in order to account for the local electric field $\vec{E} = -\vec{\nabla}\phi - \partial\delta\vec{A}/\partial t$. The equation of continuity defining $\hat{j}^E$ then changes to $\partial\hat{\epsilon}/\partial t + \vec{\nabla}\cdot\hat{j}^E = \hat{j}\cdot\vec{E}$, which contains a *gauge-invariant* source term. For a weak-coupling superconductor, the Hamiltonian density may be taken as (after introducing second quantization)

$$\hat{h}_T = (2m_e)^{-1}(\vec{\Pi}\hat{\psi}_\alpha)^\dagger \cdot \vec{\Pi}\hat{\psi}_\alpha^\dagger + \hat{\psi}_\alpha^\dagger V^{\text{imp}}{}_{\alpha\beta}\hat{\psi}_\beta + g\hat{\psi}_\alpha^\dagger\hat{\psi}_\beta^\dagger\hat{\psi}_\beta\hat{\psi}_\alpha + \hat{\psi}^\dagger e\phi\hat{\psi}, \tag{8}$$

where $\vec{\Pi} \equiv \vec{p} - e\vec{A}$ with $\vec{p} \equiv i\vec{\nabla}$ and $\vec{A} \equiv \vec{A}_0 + \delta\vec{A}$. Using Heisenberg's equation of motion to evaluate $\partial\hat{\epsilon}/\partial t$, one may verify the above continuity equation for $\hat{\epsilon}$, and identify

$$\hat{j}^E = (2m_e)^{-1}\left[(\vec{\Pi}\hat{\psi}_{H\alpha})^\dagger\left(\frac{i\partial}{\partial t} - e\phi\right) + \text{H.c.}\right] + \frac{\mu_0}{e}\hat{j} \equiv \hat{j}_0^h + \frac{\mu_0}{e}\hat{j}, \tag{9}$$

where, in anticipation of our later need to employ the thermal Green's-function techniques, we have introduced the (modified) Heisenberg-picture operators $\hat{\psi}_H$ and $\hat{\psi}_H^\dagger$ using $\hat{K} \equiv \hat{H}_T - \mu\hat{N}$ instead of the total Hamiltonian operator $\hat{H}_T$ itself in defining the evolution operator.[9] This accounts for the $(\mu_0/e)\hat{j}$ term in Eq. (9), allowing the first term of $\hat{j}^E$ alone to be identified as the *nonmagnetic part* of the heat-current operator $\hat{j}_0^h$. This expression for $\hat{j}_0^h$ has already been given by Caroli and Maki[6] (but mistaken as the complete $\hat{j}^h$), although to our knowledge it has not been used so far for evaluating heat-current responses in any gauge in which $\phi \neq 0$. The calculation to be presented in this paper, however, will be completely gauge invariant. We shall then see the necessity of having the explicitly ϕ-dependent terms in $\hat{j}_0^h$ in order to ensure the gauge-independent nature of a heat current. We stress that the origin of these ϕ-dependent terms is our identification that $\hat{\epsilon} \equiv \hat{h}_T - \hat{\rho}\phi$.[10]

Equations (4) and (9) are our starting point for calculating the heat-current response $(j_x^h)_{\text{av}}$ due to an applied field $(\mathcal{E}_y)_{\text{av}}$ in a *gapless* type-II superconductor containing an arbitrary amount of magnetic and nonmagnetic impurities, in the low-field limit $B \simeq 0$, where B is assumed to be in the z direction. A theory of superconducting alloys containing paramagnetic impurities was first worked out by Abrikosov and Gor'kov[11] who showed that if one denotes the spin-flip lifetime of the electrons as τ_s, and the equilibrium value of the order parameter as $\Delta_0(T)$, then the energy gap of the excitations in this system diminished and tends to zero as $\tau_s\Delta_0 \to 1$ from above, and the system becomes "gapless" when $\tau_s\Delta_0 \leq 1$. Gor'kov and Éliashberg[12,13] (GE) then showed that in the extreme gapless limit $\tau_s\Delta_0 \ll 1$, the dynamics of the system may be described by a complete set of "time-dependent Ginzburg-Landau (TDGL) equations." This set of TDGL equations were derived under the condition of very high or very low concentrations of magnetic impurities together with the "dirty-limit" assumption $\tau_1 \ll \tau_s$, where τ_1 is the total scattering lifetime of the electrons.[14] Without altering the condition $\tau_s\Delta_0 \ll 1$, the set of TDGL equations has been previously extended by Hu and Thompson[15] to allow for arbitrary amounts of magnetic and nonmagnetic scatterings, for the purpose of studying the dependence of flux-flow resistivity $\rho_s = (\sigma_s)^{-1}$ on the two parameters τ_s and τ_1, which are now limited by the natural restriction $\tau_s \geq \tau_1$ only, due to the way they are defined. The calculation to be presented in this paper on the transport entropy S_D is for the same type of systems and under the same general condition on τ_s and τ_1. As was already pointed out in Ref. 15, the condition $\tau_s\Delta_0 \ll 1$ is roughly equivalent to $T_C(T_C - T) \ll (3.5\tau_s)^{-2} \simeq \frac{1}{12}(T_{c0} - T_c)^2$, when T_{c0} and T_c denote the transition temperatures before and after magnetic impurities are introduced into the system, respectively. This requires one to work either in a very narrow temperature range near T_c, or with a system containing a high concentration of magnetic impurities, so that its T_c is much suppressed from T_{c0} (by at least a factor of $\sim\frac{1}{6}$). While this stringent condition may make the direct verification of our results difficult, the present work does serve another important purpose, that is the establishment of a correct procedure for calculating α_s and S_D, which can then be applied to other systems, such as to the more complex case of a type-II superconductor with a finite energy gap. To this goal we shall show that both terms of Eq. (4) approach constant limits as $T_0 \to 0$, thus individually they give T_0^{-1}-divergent contributions to the entropy current $\vec{j}^h/T_0$ and the transport entropy S_D. Then we shall explicitly demonstrate that these two divergent contributions exactly cancel each other, in the limit $T_0 \to 0$, leaving a T_0^2 dependence for $\vec{j}^h$, and linear temperature dependences for the entropy current and S_D. Owing to the lack of an exact solution even for an isolated vortex at equilibrium, our calculation of S_D is actually only performed numerically under a certain approximation. However, it cannot be overemphasized that the approximation is introduced only *after* we have effected the *exact* cancellation of the divergent contributions, or else the calculated results for S_D would be totally unphysical in the low temperature limit.

This paper is organized as follows. In Sec. II, we briefly review the real-time Feynman-diagram method as was developed by GE, for studying the dynamics of superconducting systems and its application to the derivation of a complete set of TDGL equations for gapless superconductors containing arbitrary amounts of magnetic and nonmagnetic impurities. We also establish in Sec. II a systematic order analysis for applying the GE method to calculate any physical quantity such as $\vec{j}_0^h$ for such systems. In Sec. III we use the GE method and the order analysis of Sec. II to derive an expression for $\vec{j}_0^h$ in terms of the dynamic variables in the complete set of TDGL equations. We then show in Sec. IV that at least for $B \simeq 0$ there is an *exact* cancellation in S_D of the T_0^{-1}-divergences due to the two terms in Eq. (4), leaving a linear temperature dependence for S_D at low temperatures, in consistency with the third law of thermodynamics. This result near $B\simeq 0$ has not been achieved by anybody previous to us. Section IV, therefore, establishes on solid ground that Eq. (4) is the correct heat-current expression for magnetic conductors. Section V is devoted to an explicit evaluation of the transport entropy S_D in the low-B limit for gapless superconductors with arbitrary concentrations of magnetic and nonmagnetic impurities, and then to present an overall analysis of our results on S_D for both $B\simeq 0$ and $B \simeq H_{c2}$ (calculated previously[1]) in order to reveal the B dependence of S_D for various values of τ_s and τ_1. Finally, a short conclusion is given in Sec. VI, which also contains a discussion of the physical meaning of an anomalous term u_1 in the TDGL equations first discovered by Éliashberg.[13] As we shall see, our heat-current study strongly suggests that apart from a proportionality constant, u_1 should be identified as the local temperature deviation from equilibrium δT, at least in an effective sense, and the word "effective" may be dropped when the magnetic-impurity concentration is high. If this physical interpretation of u_1 is further confirmed, then the TDGL equations of GE would have already incorporated the Clem mechanism[16] (i.e., local heat-flow) for extra dissipation in a flux-flow state, and the recent attempt by Cohen and Rickayzen[17] to put this effect artificially into the TDGL equations of GE would become redundant.

Two appendices conclude this paper. In Appendix A we illustrate the real-time Feynman-diagram method of GE for studying the dynamic properties of weak coupling superconductors, by presenting the details of our evaluation of two particular diagrams that contribute to the calculation of $\vec{j}_0^h$. In Appendix B we discuss a subtle point concerning the anomalous quantity u_1, whose gradient appears as a contribution to $\vec{j}_0^h$. It will be shown that the solution for u_1 as provided by Gor'kov and Kopnin for an isolated vortex line[18] requires a subtle modification, otherwise it would give a spurious (and unphysical) contribution to $(\vec{j}_0^h)_{av}$. This modification is necessitated by the long-range nature of u_1 (i.e., $u_1 \propto r^{-1}$ even for $r \gg \lambda$),[18] which implies that an "isolated-vortex limit" really does not exist for this quantity. This property of u_1, when compared with Clem's solution of δT near an isolated vortex line,[16] further supports our identification that $u_1 \propto \delta T$. It should be stressed that the modification of u_1 (for $r \gg \lambda$ only) removes the spurious contribution to $(\vec{j}_0^h)_{av}$ *without* affecting the earlier results on flux-flow resistivity.[15,18]

II. GOR'KOV-ÉLIASHBERG METHOD, ITS APPLICATION TO SUPERCONDUCTING ALLOYS WITH PARAMAGNETIC IMPURITIES, AND AN ORDER ANALYSIS

To study dynamic properties of a many-body system at finite temperatures, the general procedure is to first calculate certain imaginary-time-ordered thermal correlation functions, and then to perform analytic continuation from the positive imaginary frequency axis to the real frequency axis, with respect to every external frequency, in order to obtain the corresponding physical, causal correlation functions. For weak-coupling superconducting alloys containing paramagnetic impurities, it was shown by Gor'kov and Éliashberg[12,13] that the analytic continuation step may be performed formally on each Feynman diagram, thereby establishing a direct real-time Feynman-diagram method for studying nonstationary problems at finite temperatures. In establishing such a method, GE take advantage of the fact that for the system concerned, every Feynman diagram involved is composed of only one fermion loop, which may in general thread through n frequency-bearing vertices (which include one response vertex usually taking the leftmost position, and $n-1$ cause vertices). After the formal analytical continuation, such a diagram becomes $n+1$ diagrams: one retarded regular diagram, one advanced regular diagram, and $n-1$ anomalous diagrams. To obtain the retarded regular diagram from the corresponding thermal Feynman diagram, one replaces every normal-electron thermal propagator: $G^{\pm}(\epsilon - \sum_{j=1}^{i} \omega_j) \equiv (\epsilon - \sum \omega_j \pm \xi)^{-1}$, where $\epsilon = i(2n+1)\pi T$, $\omega_j = i2n_j\pi T$, by the corresponding retarded propagator: $G^{\pm R}(\epsilon - \sum \omega_j) \equiv (\epsilon - \sum \omega_j \pm \xi + i\delta)^{-1}$, where ϵ and all ω_j are now real frequencies. The summation over the fermion frequency, $T\sum_{\epsilon}(\cdots)$, is then replaced by the operation $(4\pi i)^{-1} \int_{-\infty}^{\infty} d\epsilon \tanh[(\epsilon-\omega)/2T](\cdots)$, where ω is the real frequency entering the response vertex, and ϵ is the internal fermion frequency leaving this vertex. Similarly, to obtain the advanced regular diagram, one uses the advanced propagators: $G^{\pm A}(\epsilon - \sum\omega_j) \equiv (\epsilon - \sum \omega_j \pm \xi - i\delta)^{-1}$ for all the fermion lines, and replaces $T\sum_{\epsilon}(\cdots)$ by $-(4\pi i)^{-1}\int_{-\infty}^{\infty} d\epsilon \tanh(\epsilon/2T)(\cdots)$. As for the $n-1$ anomalous diagrams, one lets each of the $n-1$ cause vertices take turn to be the "anomalous vertex." A factor $[\tanh\epsilon_i/2T - \tanh(\epsilon_i - \omega_i)/2T]$ is associated with this vertex, where ϵ_i $(\equiv \epsilon - \sum_{j=1}^{i-1}\omega_j)$ and $\epsilon_i - \omega_i$ are the electron frequencies entering and leaving this (ith) vertex. Retarded (and advanced) propagators are assigned to the electron lines before (and after) this anomalous vertex, and $(4\pi i)^{-1}\int_{-\infty}^{\infty} d\epsilon$ is finally performed in place of the frequency sum $T\sum_\epsilon$ of the corresponding thermal Feynman diagram. All other rules of the GE method are the same as those of the thermal Feynman-diagram technique.

GE have used this real-time Feynman-diagram technique to derive a complete set of TDGL equations for describing the dynamics of gapless superconducting alloys with either very high, or very low concentrations of magnetic impurities. In the former case they have also assumed the dirty-limit condition $\tau_1 \ll \tau_s$, where τ_1 and τ_s are total- and exchange-scattering lifetimes, respectively. These restrictions have been removed by Hu and Thompson[15] to obtain a complete set of TDGL equations for gapless superconductors with arbitrary amount of magnetic and nonmagnetic scatterings, originally for the purpose of studying the dependence of the flux-flow resistivity ρ_s on the two parameters τ_1 and τ_s. The resultant set of TDGL equations has been previously presented[15] in a *normalized* form in order to reveal the relevant parameters of the system. The details of the derivation have been omitted since it does not require the inclusion of more Feynman diagrams than those considered originally by GE for the limiting cases. However, in order to facilitate the introduction of an order analysis, which is needed for our present calculation of $\langle \hat{j}_0^h \rangle$ on the same type of systems by the GE method, we shall present below the *unnormalized* form of the TDGL equations (hereafter, we shall omit the subscripts in μ_0 and T_0):

$$\overline{A}\Delta - \overline{B}|\Delta|^2\Delta - C\frac{\partial \Delta}{\partial t} + D_1 \mathfrak{D}_-^2\Delta + U\Delta = 0, \tag{10a}$$

$$U = \frac{\tau_1}{4i}\left(\int d\epsilon \frac{\epsilon}{\epsilon^2+\tau_s^{-2}}(\Gamma_\epsilon^+ - \Gamma_\epsilon^-) + i\int d\epsilon \frac{\tau_s^{-1}}{\epsilon^2+\tau_s^{-2}}(\Gamma_\epsilon^+ + \Gamma_\epsilon^-)\right) \equiv U_1 + iU_2, \tag{10b}$$

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)(\Gamma_\epsilon^+ - \Gamma_\epsilon^-) = -\frac{i}{\tau_1}\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\epsilon}{\epsilon^2+\tau_s^{-2}}\frac{\partial}{\partial t}|\Delta|^2, \tag{10c}$$

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)(\Gamma_\epsilon^+ + \Gamma_\epsilon^-) = -\frac{1}{\tau_1}\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\tau_s^{-1}}{\epsilon^2+\tau_s^{-2}}\left(\Delta\frac{\partial}{\partial t}\Delta^* - \Delta^*\frac{\partial}{\partial t}\Delta\right)$$
$$-\frac{2i}{\tau_1}\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\partial}{\partial t}(e\phi + eD\vec{\nabla}\cdot\vec{A}) - \frac{2\tau_s^{-1}}{\epsilon^2+\tau_s^{-2}}|\Delta|^2(\Gamma_\epsilon^+ + \Gamma_\epsilon^-), \tag{10d}$$

$$\vec{j} = -2iN(0)eD_1(\Delta^*\vec{\mathfrak{D}}_-\Delta - \Delta\vec{\mathfrak{D}}_+\Delta^*) - \sigma\left(\frac{i\tau_1}{4e}\vec{\nabla}\int d\epsilon(\Gamma_\epsilon^+ + \Gamma_\epsilon^-) + \frac{\partial}{\partial t}\vec{A}\right), \tag{10e}$$

$$\rho = 2N(0)e^2\left(\frac{i\tau_1}{4e}\int d\epsilon(\Gamma_\epsilon^+ + \Gamma_\epsilon^-) - \phi\right), \tag{10f}$$

where $\Delta \equiv |g|\langle\psi_\uparrow(x)\psi_\downarrow(x)\rangle$ is not yet normalized by its equilibrium value $\Delta_0 = (\overline{A}/\overline{B})^{1/2}$; $\vec{\mathfrak{D}}_\pm \equiv \vec{\nabla} \pm 2ie\vec{A}$; $N(0) \equiv m_e p_F/2\pi^2$ is the normal electron density of state at the Fermi surface, σ is the normal state conductivity; $D \equiv \frac{1}{3}v_F^2\tau_1$ is the diffusion constant with v_F denoting the Fermi velocity, and the constants $\overline{A}$, $\overline{B}$, C, and D_1 appearing in these equations are still functions of T, $\rho_s \equiv (2\pi\tau_s T)^{-1}$, and $\rho_1 \equiv (4\pi\tau_1 T)^{-1}$:

$$\overline{A} = \ln(T_c/T) + \psi(\tfrac{1}{2}+\rho_s T/T_c) - \psi(\tfrac{1}{2}+\rho_s), \tag{11a}$$

$$\overline{B} = -(4\pi T)^{-2}[\psi^{(2)}(\tfrac{1}{2}+\rho_s) + \tfrac{1}{3}\rho_s\psi^{(3)}(\tfrac{1}{2}+\rho_s)], \tag{11b}$$

$$C = (4\pi T)^{-1}\psi^{(1)}(\tfrac{1}{2}+\rho_s), \tag{11c}$$

$$D_1 = \frac{D}{4\pi T}\rho_1\sum_{n=0}^{\infty}(n+\tfrac{1}{2}+\rho_s)^{-2}(n+\tfrac{1}{2}+\rho_1)^{-1}. \tag{11d}$$

If we then introduce

$\Delta = \Delta_0 f\exp(i2e\chi)$, $u_1 \equiv -C^{-1}U_1$,

$u_2 \equiv -(2eC)^{-1}U_2$, $\vec{Q} \equiv \vec{A} - \vec{\nabla}\chi$,

$\psi_\epsilon \equiv (i\tau_1/4e)(\Gamma_\epsilon^+ + \Gamma_\epsilon^-)$, $\gamma = D^{-1}$, $C_1 = CD/D_1$,

$\xi = (D_1/\overline{B})^{1/2}/\Delta_0 = (D_1/\overline{A})^{1/2}$, $C_2 = F/\overline{B}C_1$, where

$$F \equiv \frac{1}{8T}\int_{-\infty}^{\infty}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\epsilon^2}{(\epsilon^2+\tau_s^{-2})^2}\,d\epsilon$$

$$=(4\pi T)^{-2}\rho_s^{-1}[\psi^{(1)}(\tfrac{1}{2}+\rho_s)+\rho_s\psi^{(2)}(\tfrac{1}{2}+\rho_s)] \quad (11e)$$

and

$$\zeta_\epsilon=[D(\epsilon^2+\tau_s^{-2})/2\tau_s^{-1}]^{1/2}/\Delta_0,$$

$$\lambda=(D/16\pi\sigma D_1)^{1/2}/\Delta_0,$$

$$\zeta_1=(4\pi\lambda^2\sigma/C_1\gamma)^{1/2}=(D/4C)^{1/2}/\Delta_0,$$

$$\lambda_{TF}=[8\pi N(0)e^2]^{-1/2},$$

then Eqs. (10) may be converted to the normalized equations already presented once before[15]

$$C_1\gamma\left(\frac{\partial}{\partial t}+u_1\right)f-\nabla^2 f+4e^2Q^2 f+\xi^{-2}(f^3-f)=0, \quad (12a)$$

$$C_1\gamma f^2\left(\frac{\partial}{\partial t}\chi+u_2\right)+\vec{\nabla}\cdot(f^2\vec{Q})=0, \quad (12b)$$

$$\left(\gamma\frac{\partial}{\partial t}-\nabla^2\right)u_1=C_2\xi^{-2}\frac{\partial}{\partial t}f^2, \quad (12c)$$

$$u_2=\zeta_1^2\int_{-\infty}^{\infty}d\epsilon\,\zeta_\epsilon^{-2}\psi_\epsilon, \quad (12d)$$

$$\left(\gamma\frac{\partial}{\partial t}-\nabla^2+\zeta_\epsilon^{-2}f^2\right)\psi_\epsilon=\frac{1}{4T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\times\left(-\zeta_\epsilon^{-2}f^2\frac{\partial}{\partial t}\chi+\gamma\frac{\partial}{\partial t}\phi+\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}\right), \quad (12e)$$

$$\vec{j}=\sigma\left(-\vec{\nabla}\psi-\frac{\partial}{\partial t}\vec{A}\right)-\frac{f^2\vec{Q}}{4\pi\lambda^2}, \quad (12f)$$

$$\rho=\frac{(\psi-\phi)}{4\pi\lambda_{TF}^2},\quad \psi=\int_{-\infty}^{\infty}d\epsilon\,\psi_\epsilon, \quad (12g)$$

which involve the following dimensionless functions of ρ_s and ρ_1:

$$C_1=\psi^{(1)}(\tfrac{1}{2}+\rho_s)\left(\sum_{n=0}^{\infty}\frac{\rho_1}{(n+\frac{1}{2}+\rho_s)(n+\frac{1}{2}+\rho_1)}\right)^{-1}, \quad (13a)$$

$$C_2=\frac{-[\psi^{(1)}(\frac{1}{2}+\rho_s)+\rho_s\psi^{(2)}(\frac{1}{2}+\rho_s)]}{\rho_s C_1[\psi^{(2)}(\frac{1}{2}+\rho_s)+\frac{1}{3}\rho_s\psi^{(3)}(\frac{1}{2}+\rho_s)]}, \quad (13b)$$

$$C_3\equiv\xi^2[1+(\tau_s\epsilon)^2]/\zeta_\epsilon^2$$

$$=4C_2\psi^{(1)}(\tfrac{1}{2}+\rho_s)\,[\psi^{(1)}(\tfrac{1}{2}+\rho_s)+\rho_s\psi^{(2)}(\tfrac{1}{2}+\rho_s)]^{-1}, \quad (13c)$$

$$C_4\equiv\xi^2/\zeta_1^2=\rho_s\psi^{(1)}(\tfrac{1}{2}+\rho_s)C_3, \quad (13d)$$

and

$$\kappa\equiv\frac{\lambda}{\xi}=\frac{C_1[-\psi^{(2)}(\frac{1}{2}+\rho_s)-\frac{1}{3}\rho_s\psi^{(3)}(\frac{1}{2}+\rho_s)]^{1/2}}{(16\pi\sigma D)^{1/2}\psi^{(1)}(\frac{1}{2}+\rho_s)}. \quad (13e)$$

Some of Eqs. (12a)–(12g) are needed for our later evaluation of $\vec{j}_0^h$, but first let us look at Eqs. (10) in order to establish an order analysis which shall be needed for our derivation of an expression for $\vec{j}_0^h$ in terms of the variables in Eqs. (12) using the GE method.

For generality we assume[19] $\tau_1\sim\tau_s\sim(2\pi T)^{-1}$, $\Delta\sim\Delta_0$, and introduce the dimensionless small parameter $\eta\equiv\tau_s\Delta_0$. If we then equate the magnitudes of various terms in Eq. (10a), we find

$$\overline{A}\sim\overline{B}\Delta_0^2\sim C\omega\sim D_1(\vec{k}-2e\vec{A})^2\sim U\sim\eta^2. \quad (14a)$$

Using the explicit forms in Eqs. (11b)–(11d), we conclude

$$\omega\sim D(\vec{k}-2e\vec{A})^2\sim\eta\Delta_0,\quad \vec{k}\sim 2e\vec{A}\sim v_F^{-1}\Delta_0. \quad (14b)$$

Also from Eq. (10b), we obtain the order of the "anomalous vertices" as introduced by GE[12,13]

$$\Gamma_\epsilon^\pm\sim\eta\Delta_0. \quad (14c)$$

Substituting Eqs. (14b) and (14c) into Eq. (10c) we find both sides to be $\sim\eta^2\Delta_0^2$. Similarly, every term in Eq. (10d) is $\sim\eta^2\Delta_0^2$, if we put

$$e\phi\sim\eta\Delta_0(\sim\omega), \quad (14d)$$

showing the self-consistency of the order analysis. Then from *every* term in Eqs. (10e) and (10f), we conclude

$$\vec{j}\sim N(0)e\,v_F\eta^2\Delta_0,\quad \rho\sim N(0)e\eta\Delta_0, \quad (14e)$$

which implies $\dot{\rho}\sim\vec{\nabla}\cdot\vec{j}\;[\sim N(0)e\eta^2\Delta_0^2]$, as is also required by the charge continuity equation $\dot{\rho}+\vec{\nabla}\cdot\vec{j}=0$.

The above order analysis does not merely establish the self-consistency in the derivation of the TDGL equations by GE, it also provides us a systematic approach for deciding to what order one must calculate any physical quantity such as $\vec{j}_0^h$, and how one can exhaust all Feynman diagrams which contribute to a physical quantity within a given order. The order of an individual Feynman diagram is easily analyzed as follows: Every electron propagator is regarded to contribute a factor of τ_s $[\sim\tau_1\sim(2\pi T)^{-1}]$; each impurity (dotted) line a factor $[N(0)\tau_s]^{-1}$; each energy (i.e., frequency) integral a factor τ_s^{-1}, and each electron-momentum integral $\int d^3p\sim N(0)\tau_s^{-1}$ (assuming that the contribution comes from a spherical shell around the Fermi surface of a thickness $\sim T_c$, otherwise the contribution may be larger, as when evaluating certain Δ-independent diagrams which contribute even to the normal state). The order of any bare vertex function follows from our analysis which leads to Eqs. (14), and the factor $[\tanh\epsilon_i/2T - \tanh(\epsilon_i - \omega_i)/2T]$ associated with any anomalous vertex is of order $(\omega_i/2T)\cosh^{-2}(\epsilon_i/2T) \sim \eta^2$, when only linear response is of interest. The lowest order contribution of a Feynman diagram is when all external frequencies and momenta are set equal to zero. When it is necessary to include higher-order contributions (such as when the leading order vanishes), then one must expand the electron propagators with respect to all external momenta $\vec{k}_i$ and frequencies ω_i. Then the appearance of $\vec{k}_i$ to the nth power increases the nominal order of the diagram by a factor η^n, while the appearance of ω_i to nth power increases the nominal order by η^{2n}. In order to help the readers appreciate the usefulness of this systematic order analysis, we note that this order analysis actually implies (as we shall see below) that to evaluate $\vec{j}_0^h$, it will not be sufficient to evaluate every contributing Feynman diagram to its lowest order, even when the leading-order contribution is nonvanishing!

To see up to what order we must evaluate $\vec{j}_0^h$, we first use the definition $\vec{j}_0^h \equiv \vec{j}^E - (\mu/e)\vec{j}$, and the equations of continuity: $\partial\epsilon/\partial t + \vec{\nabla}\cdot\vec{j}^E = \vec{j}\cdot\vec{E}$ and $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{j} = 0$, to obtain

$$\frac{\partial\tilde{\epsilon}}{\partial t} + \vec{\nabla}\cdot\vec{j}_0^h = \vec{j}\cdot\vec{E}, \tag{15}$$

where $\tilde{\epsilon} \equiv \epsilon - (\mu/e)\rho$. We remind the reader that here μ means the equilibrium μ_0 of Sec. I and is therefore independent of space and time. Now using Eqs. (14) we find the right-hand side of Eq. (15) to be of order $N(0)\eta^3\Delta_0^3$, which implies that we must evaluate $\tilde{\epsilon}$ and $\vec{j}_0^h$, respectively, to the orders

$$\tilde{\epsilon} \sim N(0)\eta^2\Delta_0^2, \quad \vec{j}_0^h \sim N(0)v_F\eta^3\Delta_0^2. \tag{16}$$

In Sec. III we shall apply the GE method to derive an expression for $\vec{j}_0^h$ in terms of f, $\vec{Q}$, χ, u_1, ψ_ϵ, and ϕ of Eqs. (12) to the order of Eq. (16). We shall not in this paper attempt to derive a corresponding expression for $\tilde{\epsilon}$ to the order of Eq. (16), in order to explicitly verify Eq. (15), because in looking into this, we find that this task is actually more tedious than it might appear to be. This is because the lowest-order nonvanishing contribution to $\tilde{\epsilon}$ is actually of order $N(0)\Delta_0^2$, so that to verify Eq. (15) one must actually extend the TDGL equations to next order in η^2. Thus, Eq. (15) is not really a royal road for obtaining $\vec{j}_0^h$, even in the linear-response limit.

III. DERIVATION OF AN EXPRESSION FOR $\vec{j}_0^h$

In this section we apply the GE method to derive an expression for $\vec{j}_0^h$ in terms of the variables in the complete set of TDGL equations (12). The $\vec{j}_0^h$ vertex, according to Eq. (9), is

$$(2m)^{-1}[(\vec{p}-\vec{k}-e\vec{A})(\epsilon - e\phi) + (\vec{p}-e\vec{A})(\epsilon-\omega-e\phi)], \tag{17}$$

where $(\vec{p},\epsilon)$ denotes the momentum and frequency of the electron line leaving this vertex, and $(\vec{p}-\vec{k}, \epsilon-\omega)$ denotes those of the electron line entering this vertex. However, in as much as we ignore any Hall-effect related contributions, Eq. (17) may be approximated by

$$\vec{v}_F(\epsilon - \tfrac{1}{2}\omega - e\phi), \tag{18}$$

where $\vec{v}_F$ is a vector parallel to $\vec{p}$ but has a magnitude equal to the Fermi velocity. We shall see that the $-\vec{v}_F e\phi$ term in Eq. (18) is the only contribution to $\vec{j}_0^h$ that is explicitly ϕ dependent, and that this term is important for maintaining the gauge invariance of $\vec{j}_0^h$. It is clear that this term simply gives an additive contribution to $\vec{j}_0^h$ which is just $-\phi$ times the current density given by Eq. (10e). Since we shall limit ourselves in this paper to *linear* heat-current responses, we use only the first term in Eq. (10e). Then using the identity $\sigma = 2eN(0)D$, we may write this contribution as

$$(\sigma/8\pi e^2 T)[\,i2e\phi N_{2,1}(\Delta^*\vec{\mathfrak{D}}_- - \Delta\vec{\mathfrak{D}}_+\Delta^*)], \tag{19}$$

where for later convenience we have introduced the systematic notation

$$N_{\mu,\nu} \equiv \rho_1^\nu \sum_{n=1}^{\infty} (n+\tfrac{1}{2}+\rho_s)^{-\mu}(n+\tfrac{1}{2}+\rho_1)^{-\nu}. \tag{20}$$

Next we evaluate the contribution to $\vec{j}_0^h$ from the first two terms in Eq. (18). According to the GE method, this contribution may be further split into the regular part and the anomalous part. Just as in GE's derivation of the current expression, we expect on physical ground that there is no regular contribution to $\vec{j}_0^h$ that is not explicitly dependent on Δ and Δ^*.[20] This is because in the normal state a heat flow only results from a deviation of the distribution function from equilibrium. Diagrammatically this is represented by the anomalous diagrams. Thus the lowest nonvanishing contribution to the regular part is the diagram depicted in Fig. 1(a). This diagram must still be intrepreted as two identical diagrams: once as the retarded regular diagram, and once as the advanced regular diagram. The inclusion of impurity averaging is standard,[14] and amounts to renormalizing the Δ and Δ^* vertices for the

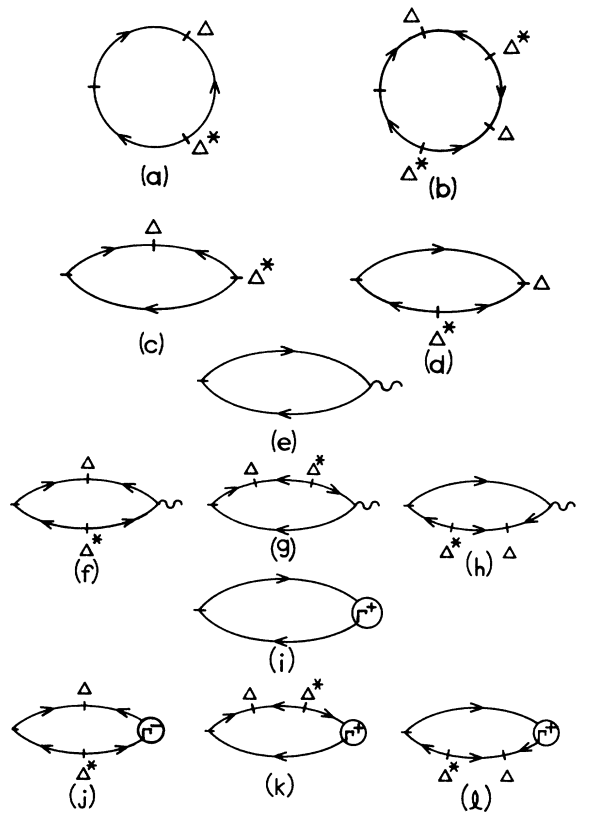

FIG. 1. Diagrams which can possibly contribute to the calculation of $\vec{j}^h_0 + \phi\vec{j}$ up to the desired order [Eq. (16)], before impurity averaging. The left-most vertex stands for the factor $(2\epsilon - \omega)\vec{v}$, which includes a factor of 2 for spin sum. The first two diagrams are "regular" diagrams as defined by Gor'kov and Éliashberg, and should be used once as retarded diagrams, and once as advanced diagrams. The rest of the diagrams are "anomalous" diagrams, each having the right-most pointed vertex being the anomalous vertex. A wavy line stands for the factor $-e\vec{v}\cdot\vec{A}$. The $\Gamma^{\pm}$ factors (each defined with an extra minus sign) are the anomalous vertex parts introduced by Gor'kov and Éliashberg (see Fig. 2). Omitted in this collection of likely contributing diagrams are some Δ-independent regular diagrams which clearly must vanish on physical grounds (see Sec. II), and those diagrams which may be obtained from any of the depicted diagrams by inserting regular $-e\vec{v}\cdot\vec{A}$ vertices for as many times as are permitted by the order analysis.

present diagram. After some straightforward algebra, it is found that the leading nonvanishing contribution from this diagram is linear in ωk:

$$\frac{\sigma}{8\pi e^2 T}\Big[(N_{2,1}-\rho_s N_{3,1})[\vec{\mathfrak{D}}_-(1)-\vec{\mathfrak{D}}_+(2)]\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right) - \rho_1^{-1}(N_{1,2}-\rho_s N_{2,2}) \times\left(\vec{\mathfrak{D}}_-(1)\frac{\partial}{\partial t_2}+\vec{\mathfrak{D}}_+(2)\frac{\partial}{\partial t_1}\right)\Big]\Delta(1)\Delta^*(2)\Big|_{1=2}, \qquad (21)$$

where the notation $N_{\mu,\nu}$ is that introduced in Eq. (20), and one should note that all $N_{\mu,\nu}$ are finite in either the dirty limit ($\rho_1 \gg 1$ and ρ_s) or the pure exchange-scattering limit ($\tau_1=\tau_s$, $\rho_1=\frac{1}{2}\rho_s$), but can become small for $\nu\neq 0$ if $\rho_1 \ll 1$. Owing to the natural restriction $\rho_1 \geq \frac{1}{2}\rho_s$, the last case is possible only in the limit of low magnetic impurity concentrations ($\rho_s \ll 1$) in a pure sample. Since Eq. (21) is already of the order prescribed for $\vec{j}^h$ in Eq. (16), there is no need to extend this contribution to higher powers in ω or $\mathfrak{D}^2_{\pm}$. That the expression in Eq. (21) depends explicitly on ω, and therefore vanishes for a static Δ, is expected from physical viewpoint, which also explains why one obtains vanishing contributions from the ring diagrams (11 of them after impurity averaging) involving two Δ and two Δ^* vertices [Fig. 1(b)], since to evaluate these diagrams to within the order of interest [Eq. (16)], one should drop all ω dependences in the electron propagators (and then expand them to first power in $\vec{v}\cdot\vec{k}_i$, or insert correspondingly a $-\vec{v}\cdot e\vec{A}$ vertex).

Next we consider the anomalous part. All anomalous diagrams which can possibly contribute to $\vec{j}^h_0$ within the desired order [Eq. (16)] are depicted in Figs. 1(c)–1(l). Using the order analysis of Sec. II, we first realize that Figs. 1(c) and 1(d) are already of nominal order $N(0)v_F\eta^3\Delta_0^2$.[21] Thus in evaluating these diagrams we may neglect the external-frequency dependences in the electron propagators. The result is then

$$\frac{\sigma}{8\pi e^2 T}\Big[\rho_1^{-1}(\rho_s N_{2,1}+N_{1,2}-N_{1,1}) \times\left(\vec{\mathfrak{D}}_-(1)\frac{\partial}{\partial t_2}+\vec{\mathfrak{D}}_+(2)\frac{\partial}{\partial t_1}\right)\Delta(1)\Delta^*(2)\Big|_{1=2} - \rho_1^{-1}[1-\rho_s\psi^{(1)}(\tfrac{1}{2}+\rho_s)]\vec{\nabla}\frac{\partial}{\partial t}|\Delta|^2\Big]. \qquad (22)$$

As an illustration of the real-frequency diagrammatic method of GE, we have presented more details of the derivation of Eq. (22) in Appendix A. Next we consider Fig. 1(e) which has a nominal order $N(0)v_F\eta\Delta_0^2$. We must therefore evaluate this diagram to third power in the k dependences of the electron propagators, but we may still neglect their ω dependences due to our limitation to linear responses. [For the same reason we do not allow the vertex $e\phi$ to appear in Fig. 1(e) or any other anomalous diagrams for which the anomalous vertex factor is already proportional to ω.] Without expanding the electron propagators with respect to $\vec{v}\cdot\vec{k}$, it can be shown that Fig. 1(e) gives a vanishing contribution to all powers in $\vec{k}$, since after performing $\int d^3p = N(0)\int d\xi$, one is left with the integration of an odd function of ϵ. Similarly one can show that insertion of one, two, or three more $-\vec{v}\cdot e\vec{A}$ vertices into Fig. 1(e) does not change it into a nonvanishing contribution. Furthermore, one can show that Fig. 1(f) vanishes identically, and Fig. 1(g) exactly cancels out Fig. 1(h). [Each of Figs. 1(f)–1(h) are actually three diagrams after impurity averaging.] Thus Figs. 1(e)–1(h), though their nominal orders suggest that they should be included in our consideration, actually do not make any contribution to $\vec{j}_0^h$.

We are then left with the anomalous Figs. 1(i)–1(l) involving the anomalous vertex parts $\Gamma_\epsilon^\pm$ defined by GE.[12,13] Figure 1(i) may seem simpler, but it turns out that Figs. 1(j)–1(l) are more straightforward to evaluate. We therefore consider the latter diagrams first. Recalling from Eq. (14c) that $\Gamma_\epsilon^\pm \sim \eta\Delta_0$, it is seen that these diagrams are already of nominal order $N(0)v_F\eta^3\Delta_0^2$. We thus can expand the electron propagators to first power in $\vec{v}\cdot\vec{k}$ (or insert correspondingly a $-\vec{v}\cdot e\vec{A}$ vertex) and at the same time drop all ω dependences in the propagators. Impurity averaging converts every diagram into three diagrams besides renormalizing the Δ vertices, and the relation $\Gamma_\epsilon^-(+\epsilon) = \Gamma_\epsilon^+(-\epsilon)$ [cf. Eqs. (10c) and (10d)] may be used to combine Fig. 1(j) with 1(k) and 1(l). The combined result of Figs. 1(j)–1(l) is

$$\left(\frac{\sigma}{8\pi e^2 T}\right)\rho_1^{-1} \int_{-\infty}^{\infty} d\epsilon \left(-\frac{\epsilon^2 \tau_s^{-1}(\epsilon^2 + \tau_s^{-2} + \tau_1^{-1}\tau_s^{-1} + \frac{1}{2}\tau_1^{-2})}{(\epsilon^2 + \tau_s^2)^2(4\epsilon^2 + \tau_1^{-2})} (\Delta^* \vec{\mathfrak{D}}_- \Delta - \Delta \vec{\mathfrak{D}}_+ \Delta^*)(\Gamma_\epsilon^+ + \Gamma_\epsilon^-) \right.$$
$$\left. - i\, \frac{\epsilon(\epsilon^2 \tau_1/\tau_s + \frac{1}{2}\tau_s^{-2})}{(\epsilon^2 + \tau_s^{-2})^2} (\vec{\nabla}|\Delta|^2)(\Gamma_\epsilon^+ - \Gamma_\epsilon^-) + i\, \frac{\epsilon[\epsilon^2(4\tau_1/\tau_s + 1) + \tau_s^{-2}]}{4(\epsilon^2 + \tau_s^{-2})^2} \vec{\nabla}[|\Delta|^2(\Gamma_\epsilon^+ - \Gamma_\epsilon^-)] \right). \tag{23}$$

Finally, let us look at Fig. 1(i). It is easy to see that the leading nonvanishing order of this diagram is $N(0)v_F\eta\Delta_0^2$, so we must also evaluate its next-order correction which is a factor η^2 smaller. This means that not only must we expand the electron propagators up to the third power in k (or inserting into the diagram up to three $-\vec{v}\cdot e\vec{A}$ vertices), but we must also evaluate the anomalous vertex part $\Gamma_\epsilon^\pm$ to next order in η^2. Recalling that $\Gamma_\epsilon^\pm$ is defined by GE through an integral equation as represented by Fig. 2,[12,13] it is clear that to find the next-order correction $\delta\Gamma_\epsilon^\pm$ one must extend the integral equation depicted in Fig. 2 to next order in η^2, which includes, among other things, evaluating the kernels $Q_1 - Q_3$ up to order $k^2|\Delta|^2$ or $|\Delta|^4$. This is a formidable task though it can be done in principle. Fortunately, as we shall see below, we actually do not need to find the quantities $\delta\Gamma_\epsilon^\pm$ if we are only interested in evaluating $(\vec{j}_0^h)_{av}$, which already contains the desired information about α and S_D. Only when one wishes to study the *local* behavior of $\vec{j}_0^h$ and $\tilde{\epsilon}$, or to explicitly verify Eq. (15), must one then find the quantities $\delta\Gamma_\epsilon^\pm$. The reason is that only the gradients of $\delta\Gamma_\epsilon^\pm$ appear in $\vec{j}_0^h$ with vanishing averages. This fact further supports our earlier statement that Eq. (15) is not a royal road for determining $\vec{j}_0^h$. In terms of $\delta\Gamma_\epsilon^\pm$, we may now present the contribution to $\vec{j}_0^h$ from Fig. 1(i):

$$\left(\frac{\sigma}{8\pi e^2 T}\right)\left(\frac{-i}{2\rho_1}\vec{\nabla}\int d\epsilon\, \epsilon[(\Gamma_\epsilon^+ - \Gamma_\epsilon^-) + (\delta\Gamma_\epsilon^+ - \delta\Gamma_\epsilon^-)] - i\,\frac{9}{10}\,\frac{D}{4\pi T\rho_1^2}\vec{\nabla}\nabla^2 \int d\epsilon\, \epsilon(\Gamma_\epsilon^+ - \Gamma_\epsilon^-)\right). \tag{24}$$

Equations (19) and (21)–(24) together, constitute our final expression for $\vec{j}_0^h$. Let us first check the

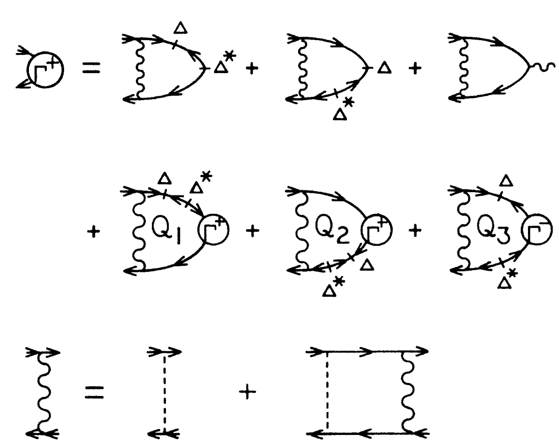

FIG. 2. Integral equation introduced by Gor'kov and Éliashberg to define the anomalous vertex part Γ^+, whose arguments are ϵ, $\vec{k}$, ω or ϵ, $\vec{r}$, t. The corresponding equation for Γ^- is obvious.

gauge invariance of this expression by noting that under a gauge transformation

$$\Delta \to \Delta e^{2i\Lambda}, \quad \vec{A} \to \vec{A} + \vec{\nabla}\Lambda,$$

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \quad \Gamma_\epsilon^+ - \Gamma_\epsilon^- \to \Gamma_\epsilon^+ - \Gamma_\epsilon^-,$$

$$\Gamma_\epsilon^+ + \Gamma_\epsilon^+ \to \Gamma_\epsilon^+ + \Gamma_\epsilon^- + \frac{2ie}{\tau_1}\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\partial \Lambda}{\partial t},$$

so that

$$\psi \equiv \frac{i\tau_1}{4e}\int d\epsilon\,(\Gamma_\epsilon^+ + \Gamma_\epsilon^-) \to \psi - \frac{\partial \Lambda}{\partial t},$$

which may all be easily verified by using Eqs. (10). It is then clear that the gradient terms in Eqs. (23) and (24) are already gauge invariant, and that under a gauge transformation,

$$\vec{j}_0^h \to \vec{j}_0^h + \frac{\sigma}{8\pi e^2 T}\Big[(\tfrac{1}{2}N_{2,1} - \rho_s N_{3,1}) + (\tfrac{1}{2}\rho_1)(N_{1,1} - \rho_s N_{2,1} - \rho_s N_{2,2}) - \pi\int d\epsilon\,\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\epsilon^2\tau_s^{-1}(\epsilon^2 + \tau_s^{-2} + \tau_1^{-1}\tau_s^{-1} + \frac{1}{2}\tau_1^{-2})}{(\epsilon^2 + \tau_s^{-2})^2(4\epsilon^2 + \tau_1^{-2})}\Big](4ie\dot{\Lambda})(\Delta^*\vec{\mathfrak{D}}_-\Delta - \Delta\vec{\mathfrak{D}}_+\Delta^*). \tag{25}$$

Evaluating the ϵ integral by a contour integration method, we obtain $\rho_s N_{3,1} + (\rho_s/2\rho_1)N_{2,2} - \frac{1}{2}N_{2,0}$. Substituting this result into Eq. (25), and using the identity

$$N_{2,1} + \rho_1^{-1}N_{1,1} = N_{2,0} + (\rho_s/\rho_1)N_{2,1}, \tag{26}$$

we finally establish the gauge invariance of $\vec{j}_0^h$.

Our next step is to rewrite $\vec{j}_0^h$ in terms of the normalized quantities of Eqs. (12). For example, using $\Delta = \Delta_0 f\exp(i2e\chi)$, $\vec{Q} = \vec{A} - \vec{\Delta}\chi$, it may be shown that

$$[\vec{\mathfrak{D}}_-(1) - \vec{\mathfrak{D}}_+(2)]\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)\Delta(1)\Delta^*(2)\Big|_{1=2} = \Delta_0^2\left(\vec{\nabla}\frac{\partial f^2}{\partial t} - 4\dot{f}\vec{\nabla}f + (4e)^2\vec{Q}\dot{\chi}f^2\right).$$

In this way, we obtain a more compact expression for $\vec{j}_0^h$:

$$\vec{j}_0^h = \frac{\sigma\Delta_0^2}{4\pi e^2 T}\Big((L_M + L_S)(4e^2 f^2\vec{Q}\dot{\chi} - \vec{\nabla}f\dot{f}) + 4\pi T\int_{-\infty}^{\infty} d\epsilon\,\frac{\epsilon^2(2\epsilon^2 + 2\tau_s^{-2} + 2\tau_s^{-1}\tau_1^{-1} + \tau_1^{-2})}{(\epsilon^2 + \tau_s^{-2})^2(4\epsilon^2 + \tau_1^{-2})\tau_s}\,4e^2 f^2\vec{Q}\psi_\epsilon + L_M 4e^2 f^2\vec{Q}\phi - L_u u_1\vec{\nabla}f^2\Big) + \vec{\Xi}, \tag{27}$$

where

$$L_M \equiv N_{2,1} = [\rho_1/(\rho_1 - \rho_s)^2][\psi(\tfrac{1}{2} + \rho_s) - \psi(\tfrac{1}{2} + \rho_1)] + [\rho_1/(\rho_1 - \rho_s)]\psi^{(1)}(\tfrac{1}{2} + \rho_s) \tag{28}$$

and

$$L_S \equiv N_{2,0} - 2\rho_s N_{3,1} - (\rho_s/\rho_1)N_{2,2} = [1 + \rho_1\rho_s/(\rho_1 - \rho_s)^2]\psi^{(1)}(\tfrac{1}{2} + \rho_s) - [\rho_1\rho_s/(\rho_1 - \rho_s)^2]\psi^{(1)}(\tfrac{1}{2} + \rho_1) + [\rho_1\rho_s/\rho_1 - \rho_s)]\psi^{(2)}(\tfrac{1}{2} + \rho_s) \tag{29}$$

have already been defined in I for expressing our result on S_D near H_{c2}, while

$$L_u = \frac{1}{4}\,\frac{(1 + 6\tau_1/\tau_s)\psi^{(1)}(\frac{1}{2} + \rho_s) - (1 - 10\tau_1/\tau_s)\rho_s\psi^{(2)}(\frac{1}{2} + \rho_s) - (1 - 2\tau_1/\tau_s)\rho_s^2\psi^{(3)}(\frac{1}{2} + \rho_s)}{1 + \rho_s\psi^{(2)}(\frac{1}{2} + \rho_s)/\psi^{(1)}(\frac{1}{2} + \rho_s)}, \tag{30}$$

and we have used $\vec{\Xi}$ to denote the sum of all gradient terms

$$\vec{\Xi} \equiv \frac{\sigma\Delta_0^2}{8\pi e^2 T}\,\vec{\nabla}\left(L_A\frac{\partial f^2}{\partial t} + L_B f^2 u_1\right) - \frac{\sigma}{e^2\tau_s}L_C\vec{\nabla}(u_1 + \delta u_1 + \tfrac{9}{5}D\tau_1\nabla^2 u_1), \tag{31}$$

where

$$L_A = N_{2,1} - \rho_s N_{3,1} - \rho_1^{-1}[1 - \rho_s\psi^{(1)}(\tfrac{1}{2} + \rho_s)]$$
$$= [\rho_1^2/(\rho_1 - \rho_s)^3][\psi(\tfrac{1}{2} + \rho_s) - (\tfrac{1}{2} + \rho_1)] + [\rho_1^2/(\rho_1 - \rho_s)^3]\psi^{(1)}(\tfrac{1}{2} + \rho_s) + \tfrac{1}{2}[\rho_1\rho_s/(\rho_1 - \rho_s)]\psi^{(2)}(\tfrac{1}{2} + \rho_s) - \rho_1^{-1}[1 - \rho_s\psi^{(1)}(\tfrac{1}{2} + \rho_s)], \tag{32}$$

$$L_B = \psi^{(1)}(\tfrac{1}{2}+\rho_s) + \frac{\tau_s}{\tau_1}\frac{3\psi^{(1)}(\frac{1}{2}+\rho_s)+5\rho_s\psi^{(2)}(\frac{1}{2}+\rho_s)+\rho_s^2\psi^{(3)}(\frac{1}{2}+\rho_s)}{1+\rho_s\psi^{(2)}(\frac{1}{2}+\rho_s)/\psi^{(1)}(\frac{1}{2}+\rho_1)}\,, \tag{33}$$

$$L_C = \frac{1-\rho_s\psi^{(1)}(\frac{1}{2}+\rho_s)}{1+\rho_s\psi^{(2)}(\frac{1}{2}+\rho_s)/\psi^{(1)}(\frac{1}{2}+\rho_s)}\,, \tag{34}$$

and

$$\delta u_1 \equiv i\frac{\tau_1\tau_s}{4L_C}\int d\epsilon\,(\delta\Gamma_\epsilon^+ - \delta\Gamma_\epsilon^-)\,. \tag{35}$$

We remark that in bringing about u_1 in Eqs. (27) and (31), we have used the fact that all weighted ϵ integrals of $\Gamma_\epsilon^+ - \Gamma_\epsilon^-$ are proportional to each other [which may be easily verified by using Eq. (10c)] and therefore can all be expressed in terms of $u_1 \equiv -C^{-1}U_1$, where U_1 is defined as the first part of Eq. (10b). Equations (27)–(34) constitute the main result of this section.

IV. PROOF OF EXACT CANCELLATION OF T^{-1} DIVERGENCES BETWEEN $\vec{j}_0^h$ AND $\vec{\mathcal{E}}\times\vec{m}$ CONTRIBUTIONS TO S_D IN THE LOW-FIELD LIMIT

We turn to the calculation of the Ettingshausen coefficient α and the transport entropy S_D in the low-field limit ($B\simeq 0$) by using Eqs. (2)–(4) and (27). To evaluate $\vec{j}_0^h$, the TDGL equations of Eqs. (12) must first be solved and then substituted into Eq. (27). For an arbitrary value of B below H_{c2}, one must seek a solution of Eqs. (12) corresponding to a two-dimensional triangular vortex lattice moving with a uniform velocity $v=(\mathcal{E})_{av}/B$ in the x direction, when the external magnetic field is in the z direction, and an applied electric field $(\mathcal{E})_{av}$ is assumed to be in the y direction. Then it should be clear from Eqs. (2) and (3) that

$$\begin{aligned}\alpha &= (j_x^h)_{av}/(\mathcal{E}_y)_{av} = (j_x^h)_{av}/vB\,,\\ S_D &= \phi_0(j_x^h)_{av}/vBT\,.\end{aligned} \tag{36}$$

In the low-field limit when the vortices are very far apart, we can consider an isolated vortex line and realize that it actually occupies an area ϕ_0/B. In terms of such an isolated-vortex-line solution of the TDGL equations we have

$$(j_x^h)_{av} = (B/\phi_0)\int dx\,dy\,j_x^h\,,$$

so that

$$\begin{aligned}\alpha\big|_{B\to 0} &= (v\phi_0)^{-1}\int dx\,dy\,(\hat{v}\cdot\vec{j}^h)\,,\\ S_D\big|_{B\to 0} &= (vT)^{-1}\int dx\,dy\,(\hat{v}\cdot\vec{j}^h)\,.\end{aligned} \tag{37}$$

It is clear that if $\vec{j}^h$ should turn out to approach a finite limit as $T\to 0$, then $S_D\propto T^{-1}$ in this limit and the third law of thermodynamics would be violated. In view of Eq. (4), we have two contributions to S_D:

$$S_D = S_D^0 + \delta S_D^{mag}\,, \tag{38}$$

and in the low-field limit,

$$S_D^0 = (vT)^{-1}\int dx\,dy\,(\hat{v}\cdot\vec{j}_0^h)\,, \tag{39}$$

$$\delta S_D^{mag} = (vT)^{-1}\int dx\,dy\,(\hat{v}\cdot\vec{\mathcal{E}}\times\vec{m})\,. \tag{40}$$

The purpose of the present section is to show that both S_D^0 and δS_D^{mag} diverge like T^{-1} in the low-temperature limit, but *the two divergences cancel each other exactly*, leaving a physical linear T dependence for S_D in the limit of $T\to 0$.

We start by eliminating χ, ψ_ϵ, and ϕ from Eq. (27) in favor of the gauge-invariant quantities

$$P_\epsilon \equiv \psi_\epsilon + \frac{1}{4T}\cosh^{-2}\!\left(\frac{\epsilon}{2T}\right)\chi,\quad P\equiv\int d\epsilon\,P_\epsilon = \psi+\chi\,.$$

The result is

$$\vec{j}_0^h = \frac{\sigma\Delta_0^2}{4\pi e^2 T}\Bigg((L_M+L_S)\vec{\nabla}f\,(\vec{v}\cdot\vec{\nabla})f - 4\pi T\int_{-\infty}^{\infty} d\epsilon\,\frac{\epsilon^2(2\epsilon^2+2\tau_s^{-2}+2\tau_1^{-1}\tau_s^{-1}+\tau_1^{-2})}{(\epsilon^2+\tau_s^{-2})^2(4\epsilon^2+\tau_1^{-2})\tau_s}\,(2ef)^2\vec{Q}P_\epsilon + L_M(2ef)^2\vec{Q}P - L_u u_1\vec{\nabla}f^2\Bigg) + \vec{\Xi}\,, \tag{41}$$

where we have replaced $\dot{f}$ by $(-\vec{v}\cdot\vec{\nabla})f$ for a steady flux flow, and we have ignored the difference between ϕ and ψ due to $\lambda_{TF}\sim P_F^{-1}$ being much smaller than any other characteristic length of the problem.[22] Using Eq. (12e), and the fact that $2e\chi=\theta$ for an isolated vortex line, we obtain the equation and the boundary condition for P_ϵ:

$$\left(\gamma\frac{\partial}{\partial t} - \nabla^2 + \zeta_\epsilon^{-2}f^2\right)P_\epsilon = \frac{1}{4T}\cosh^{-2}\!\left(\frac{\epsilon}{2T}\right)\times\left(\gamma\frac{\partial}{\partial t}P + \frac{\partial}{\partial t}\,\vec{\nabla}\cdot\vec{Q}\right), \tag{42}$$

$$P_\epsilon \to \frac{v}{8eT}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\sin\theta}{r} \quad \text{as } r\to 0\,, \tag{43}$$

where (r,θ,z) are the cylindrical coordinates around a vortex core. For linear flux flow, Eq. (42) may still be simplified to

$$\nabla^2 P_\epsilon - \zeta_\epsilon^{-2} f^2 P_\epsilon = 0\,. \tag{44}$$

Before we study the low-temperature behavior of Eq. (41), we must still recall that the TDGL equations (12), and Eq. (27) are all derived under the gaplessness condition $\tau_s\Delta_0 \ll 1$. In order for this condition to cover the point $T=0$, we must restrict ourselves to the limit of high magnetic impurity concentrations when $\rho_s \gg 1$. We thus lose ρ_s as a free parameter but the ratio τ_s/τ_1 remains arbitrary. In this limit we have[15] $\zeta_\epsilon = \zeta_1$ $(\equiv\zeta)$, $P_\epsilon = (1/4T)\cosh^{-2}(\epsilon/2T)P$, so that Eq. (41) reduces to

$$\vec{j}_0^h = \frac{\sigma\Delta_0^2}{4\pi e^2 T}\{(L_M+L_S)[\vec{\nabla} f(\vec{v}\cdot\vec{\nabla})f + (2ef)^2\vec{Q}P] - L_u u_1 \vec{\nabla} f^2\} + \vec{\Xi} \quad (\text{for } \rho_s \gg 1)\,, \tag{45}$$

where

$$\nabla^2 P - \zeta^{-2} f^2 P = 0\,, \quad P \to (v/2e)(\sin\theta/r) \tag{46}$$

as $r\to 0$. Furthermore, in the limit $\rho_s \gg 1$, it can be shown that

$$L_M \to \rho_s^{-1}\left(\frac{2\tau_1/\tau_s}{(1-2\tau_1/\tau_s)^2}\ln\frac{2\tau_1}{\tau_s} + \frac{1}{1-2\tau_1/\tau_s}\right) \propto T\,, \tag{47}$$

$$L_s \to (6\rho_s^3)^{-1}(1+2\tau_1/\tau_s + 2\tau_1^2/\tau_s^2) \propto T^3\,, \tag{48}$$

$$L_u \to \rho_s^{-1} \propto T\,, \tag{49}$$

and also

$$C_2 \to L_M/2\rho_s \propto T^2\,, \tag{50}$$

which appears in the equation for u_1 [Eq. (12c)] and is defined by Eq. (13b). Then it becomes clear that $\int \vec{j}_0^h dx\,dy$ indeed behaves like a constant in the limit $T\to 0$, but only because of the appearance of L_M in Eq. (41). (We have assumed $\int \vec{\Xi}\,dx\,dy = 0$, which will be verified in Appendix B.) Thus to prove that there is an exact cancellation between the T^{-1} divergences in S_D^0 and δS_D^{mag}, we need only show that

$$\int dx\,dy\left(\frac{\sigma\Delta_0^2 L_M}{4\pi e^2 T}[(\vec{v}\cdot\vec{\nabla} f_0)^2 + (2ef_0)^2(\vec{v}\cdot\vec{Q}_0)P] + \vec{v}\cdot\vec{\mathcal{E}}\times\vec{m}_0\right) = 0\,, \tag{51}$$

where in view of our limitation to linear responses, we have replaced f, $\vec{m}$, and $\vec{Q}$ by their equilibrium values f_0, m_0, and $\vec{Q}_0$. While we expect Eq. (51) to be valid at all values of B (in order to uphold the third law of thermodynamics at all B), the proof given below is valid only in the $B\to 0$ limit.

First we use the definition of λ just below Eq. (11e) to establish that

$$\sigma\Delta_0^2 L_M/4\pi e^2 T = 1/16\pi e^2\lambda^2\,. \tag{52}$$

We also need the following identities:

$$\int [(\vec{v}\cdot\vec{\nabla})f_0]^2 dx\,dy = \pi v^2 \int (\nabla f_0)^2 r\,dr \equiv \pi v^2 C_\gamma\,, \tag{53}$$

$$\begin{aligned}\vec{\mathcal{E}}\times\vec{m}_0 &= [-\vec{\nabla}(P-\vec{v}\cdot\vec{Q}_0) - \vec{v}\times(\vec{\nabla}\times\vec{Q}_0)]\times\vec{m}_0\\ &= (P-\vec{v}\cdot\vec{Q}_0)\vec{j}_0 - \vec{\nabla}\times[(P-\vec{v}\cdot\vec{Q}_0)\vec{m}_0]\\ &\quad + \vec{v}[\vec{\nabla}\cdot(\vec{Q}_0\times\vec{m}_0) + \vec{Q}_0\cdot\vec{j}_0]\,,\end{aligned}$$

where use has been made that $\vec{\nabla}\times\vec{m}_0 = \vec{j}_0$, and $\vec{\nabla}\cdot\vec{m}_0 = 0$. Then using $\vec{j}_0 = (4\pi)^{-1}\vec{\nabla}\times(\vec{\nabla}\times\vec{Q}_0) = -f_0^2\vec{Q}_0/4\pi\lambda^2$,[23] we find

$$\begin{aligned}\vec{v}\cdot(\vec{\mathcal{E}}\times\vec{m}_0) = &-(4\pi\lambda^2)^{-1}\\ &\times\{f^2(\vec{v}\cdot\vec{Q}_0)P - f_0^2[(\vec{v}\cdot\vec{Q}_0)^2 - v^2 Q_0^2]\}\\ &- \vec{v}\cdot\vec{\nabla}\times[(P-\vec{v}\cdot\vec{Q}_0)m_0] + v^2\vec{\nabla}\cdot(\vec{Q}_0\times\vec{m}_0)\,.\end{aligned}$$

If we further use

$$\int f_0^2(\vec{v}\cdot\vec{Q}_0)^2 dx\,dy = \tfrac{1}{2}v^2\int f_0^2 Q_0^2 dx\,dy\,,$$

$$\int \vec{\nabla}\times[(P-\vec{v}\cdot\vec{Q}_0)\vec{m}_0]\,dx\,dy = 0$$

(since $P-\vec{v}\cdot\vec{Q}_0$ is regular at $r=0$),[23] the left-hand side of Eq. (51) becomes

$$\left(\frac{v}{4e\lambda}\right)^2 C_\gamma - v^2\int dx\,dy\left(\frac{1}{8\pi\lambda^2}f_0^2 Q_0^2 - \vec{\nabla}\cdot(\vec{Q}_0\times\vec{m}_0)\right).$$

We then use $f_0^2\vec{Q}_0/\lambda^2 = -\vec{\nabla}\times\vec{b}_0$ to convert the second term to

$$+\tfrac{1}{4}v^2\left(\int_0^\infty b_0^2 r\,dr + \frac{1}{2e}[b_0(0) - 2H_{c1}]\right),$$

where we have put $4\pi m_0 = b_0 - H_{c1}$, which follows from the defining equation[1] $\vec{\nabla}\times\vec{m}_0 = \vec{j}_0$ and the boundary condition $m_0 \to -H_{c1}/4\pi$ as $r\to\infty$. We have also used the boundary condition[23] $\vec{Q}_0 \to -(2er)^{-1}\hat{e}_\theta$ as $r\to 0$ in order to integrate a pure divergence term. We thus find that to prove Eq. (51) is equivalent to proving the following identity:

$$C_\gamma = 2e\lambda^2[2H_{c1} - b_0(0)] - (2e\lambda)^2\int_0^\infty b_0^2(r)r\,dr\,,$$

which has already been established before [cf. Ref. 23, Eq. (56)].

This completes our proof in the low-field limit that there is an exact cancellation of T^{-1} divergences between S_D^0 and δS_D^{mag}, implying that S_D as calculated by using the heat-current expression Eq. (4) does obey the condition $S_D \to 0$ as $T \to 0$, as required by the third law of thermodynamics.

V. EVALUATION OF S_D FOR THE LOW-FIELD LIMIT AND ANALYSIS OF RESULTS FOR BOTH $B \simeq H_{c2}$ AND $B \simeq 0$

We may now evaluate S_D in the low-field limit for gapless superconductors with arbitrary amounts of magnetic and nonmagnetic impurities. The starting equation is just Eq. (41) integrated over space and without the L_M-dependent terms since, as we have shown in Sec. IV, such terms are exactly canceled by the $\vec{\mathcal{E}} \times \vec{m}$ contribution in the heat current-expression, Eq. (4). Thus

$$(\vec{j}^h)_{av} = \frac{\sigma\Delta_0^2}{4\pi e^2 T}\bigg(L_s \vec{\nabla} f_0 (\vec{v}\cdot\vec{\nabla}) f_0 - 4\pi T \int_{-\infty}^{\infty} d\epsilon\, W(\epsilon)(2ef_0)^2 \vec{Q}_0 P_\epsilon - L_u u_1 \vec{\nabla} f_0^2\bigg)_{av}, \tag{54}$$

where

$$W(\epsilon) \equiv \frac{\epsilon^2(2\epsilon^2 + 2\tau_s^{-2} + 2\tau_s^{-1}\tau_1^{-1} + \tau_1^{-2})}{(\epsilon^2 + \tau_s^{-2})^2(4\epsilon^2 + \tau_1^{-2})\tau_s}. \tag{55}$$

Now in view of Eqs. (37), (52), and $\lambda^{-2} = 2eH_{c2}/\kappa^2$, it is clear that

$$S_D\big|_{B\simeq 0} = \frac{\phi_0}{4\pi T}\frac{H_{c2}}{2\kappa^2}\,\Phi_1(\rho_s, \rho_1, \kappa), \tag{56}$$

where

$$\Phi_1 = (\pi L_M)^{-1} \times \int dx\, dy \bigg(L_s(\hat{v}\cdot\vec{\nabla} f_0)^2 - \frac{4\pi T}{v}\int d\epsilon\, W(\epsilon)(2ef_0)^2 \times (\hat{v}\cdot\vec{Q}_0)P_\epsilon - L_u \frac{u_1}{v}\hat{v}\cdot\vec{\nabla} f_0^2\bigg). \tag{57}$$

The evaluation of the first term has been done in Eq. (53), and for $\kappa \gg 1$ the constant $C_\gamma = 0.2791$ has already been evaluated previously.[24] To evaluate the third term, we need to find u_1 by solving Eq. (12c) (to first order in v), which has already been carried out by Gor'kov and Kopnin.[18] However, because of the existence of a subtle point in connection with this quantity (to be elaborated in Appendix B), we shall go through the solution once again below: Dropping the $\gamma(\partial/\partial t)u_1$ term in Eq. (12c) because it is second order in v, and noting that $\partial f^2/\partial t = -\vec{v}\cdot\vec{\nabla} f_0^2 = -\cos\theta\, df_0^2/dr$, where $\theta \equiv \hat{r}\cdot\hat{v}$, it is clear that we have $u_1 = v\cos\theta\, g(r)$, where

$$\frac{d}{dr}\bigg(\frac{1}{r}\frac{d}{dr}(rg)\bigg) = C_2\xi^{-2}\frac{df_0^2}{dr}. \tag{58}$$

This equation may be integrated straightforwardly, and if the integration constants are taken such that $g \to 0$ as $r \to \infty$ and is regular at origin, then the solution is

$$g(r) = -C_2\xi^{-2}r^{-1}\int_0^r (1 - f_0^2) r\, dr. \tag{59}$$

This is the solution given by Gor'kov and Kopnin.[18] In Appendix B we shall see that this solution requires a subtle modification in order not to give a spurious contribution to $(\vec{j}^h)_{av}$ and S_D, but this modification does not affect our present evaluation of the third term in Eq. (56), so we can just use Eq. (59). Then

$$\int dx\, dy\, u_1(\hat{v}\cdot\vec{\nabla}) f_0^2 = v\int r\, dr\, d\theta\bigg(\cos^2\theta\, g(r)\frac{df_0^2}{dr}\bigg) = \pi v\bigg[rg(f_0^2 - 1)\big|_0^\infty - \int_0^\infty (f_0^2 - 1)\bigg(\frac{1}{r}\frac{d}{dr}rg\bigg) r\, dr\bigg]$$
$$= -\pi v C_2 \xi^{-2}\int_0^\infty (f_0^2 - 1)^2 r\, dr = -\pi v C_2, \tag{60}$$

where use has been made of an identity derived previously [Ref. 24, Eq. (39)].

To evaluate the second term in Eq. (57), it is necessary to find P_ϵ by solving Eqs. (43) and (44). Before we do that, we shall first transform this term into a simpler form by using Eqs. (43), (44), and $\nabla^2\vec{Q}_0 - \lambda^{-2} f_0^2 \vec{Q}_0 = 0$. First we note that

$$(\zeta_\epsilon^{-2} - \lambda^{-2})\int f_0^2(\hat{v}\cdot\vec{Q}_0)P_\epsilon\, dx\, dy = \int [\hat{v}\cdot\vec{Q}_0\nabla^2 P_\epsilon - P_\epsilon\nabla^2(\hat{v}\cdot\vec{Q}_0)]dx\, dy = \int_0^{2\pi}\bigg(P_\epsilon\frac{\partial}{\partial r}(\hat{v}\cdot\vec{Q}_0) - (\hat{v}\cdot\vec{Q}_0)\frac{\partial}{\partial r}P_\epsilon\bigg) r\, d\theta\bigg|_{r\to 0}.$$

For $r \to 0$, it can be shown that

$$(\hat{v}\cdot\vec{Q}_0) \to \sin\theta[(2er)^{-1} - \tfrac{1}{2}b_0(0)r],$$
$$P_\epsilon \to v\sin\theta(1/4T)\cosh^{-2}(\epsilon/2T)$$
$$\times[(2er)^{-1} - H_{c2}C_\epsilon r],$$

where the constant C_ϵ is yet unknown. Then the last integral above may be evaluated in terms of C_ϵ as

$$2\pi v(2e\xi)^{-2}(1/4T)\cosh^{-2}(\epsilon/2T)\,[C_\epsilon - b_0(0)/2H_{c2}].$$

In the limit $\kappa \gg 1$, we have $\lambda \gg \zeta_\epsilon$ and $H_{c2} \gg b_0(0)$, then

$$\int (2ef_0)^2(\hat{v}\cdot\vec{Q}_0)P_\epsilon\,dx\,dy = 2\pi v\frac{1}{4T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\frac{\zeta_\epsilon^2}{\xi^2}C_\epsilon\left(\frac{\zeta_\epsilon}{\xi}\right), \quad (61)$$

where the function $C_\epsilon(\zeta_\epsilon/\xi)$ may be obtained by solving

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{x^2} - \frac{\xi^2}{\zeta_\epsilon^2}f^2(x)\right)p_\epsilon = 0,$$
$$p_\epsilon \to x^{-1} - C_\epsilon x \quad \text{as } x \to 0, \quad (62)$$

with $x = r/\xi$. Putting Eqs. (53), (60), and (61) into Eq. (57), we finally obtain

$$\Phi_1|_{\kappa\to\infty} = L_M^{-1}\left[L_sC_\gamma + L_uC_2 + 2\pi\int_{-\infty}^{\infty} d\epsilon\, W(\epsilon)\,\frac{\zeta_\epsilon^2}{\xi^2}C_\epsilon\left(\frac{\zeta_\epsilon}{\xi}\right)\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\right]. \quad (63)$$

Equation (62) has already been solved previously by numerical method,[24] so, in principle, we could evaluate Eq. (63) exactly. However, in order to reduce the amount of numerical work involved, we shall be contented with an approximate evaluation, the accuracy of which has already been established.[15,23] In this approximation we put $f(r) = r/(r^2 + \delta\xi^2)^{1/2}$ with $\delta = 2.83$ being chosen to give exact results for C_ϵ in the limit of high magnetic and nonmagnetic impurity concentrations when $\rho_1 \gg \rho_s \gg 1$. [In this limit C_ϵ reduces to C_E evaluated before, see Ref. 24, Eq. (46).] Then Eq. (62) may be solved analytically[15,23]

$$p_\epsilon = \frac{(r^2+\delta\xi^2)^{1/2}}{r\delta^{1/2}}\,\frac{K_1((r^2+\delta\xi^2)^{1/2}/\zeta_\epsilon)}{K_1(\delta^{1/2}\xi/\zeta_\epsilon)}, \quad (64)$$

$$C_\epsilon = \frac{\xi}{2\delta^{1/2}\zeta_\epsilon}\,\frac{K_0(\delta^{1/2}\xi/\zeta_\epsilon)}{K_1(\delta^{1/2}\xi/\zeta_\epsilon)}. \quad (65)$$

In view of Eq. (13c), we define $x = \tau_s\epsilon$, $x_0 = (\delta C_3)^{1/2}$, then

$$\Phi_1|_{\kappa\to\infty} = L_M^{-1}\Bigg(L_sC_\gamma + L_uC_2 + \pi x_0^{-1}\int_{-\infty}^{\infty} dx\,\frac{x^2(\tau_s^2/\tau_1^2 + 2\tau_s/\tau_1 + 2 + 2x^2)}{(1+x^2)^2(\tau_s^2/\tau_1^2 + 4x^2)} \times \frac{(1+x^2)^{1/2}K_0(x_0/(1+x^2)^{1/2})}{K_1(x_0/(1+x^2)^{1/2})} \times \cosh^{-2}(\pi\rho_s x)\Bigg). \quad (66)$$

Equation (66), which clearly depends on the two parameters ρ_s and τ_s/τ_1, has been evaluated numerically. The combination $t^{-2}\Phi_1$ is then plotted in Fig. 3 with respect to

$$t \equiv \exp[\psi(\tfrac{1}{2}) - \psi(\tfrac{1}{2} + \rho_s)] \quad (67)$$

for various values of τ_s/τ_1. We note that if in Eq. (67) we replace $\rho_s \equiv (2\pi\tau_s T)^{-1}$ by $(2\pi\tau_s T_c)^{-1}$, then t would become the ratio T_c/T_{c0},[11] where T_{c0} and T_c denote the transition temperatures before and after magnetic impurities are introduced into the system, respectively. Because of the gaplessness condition $\tau_s\Delta_0(T) \ll 1$, the replacement of t by

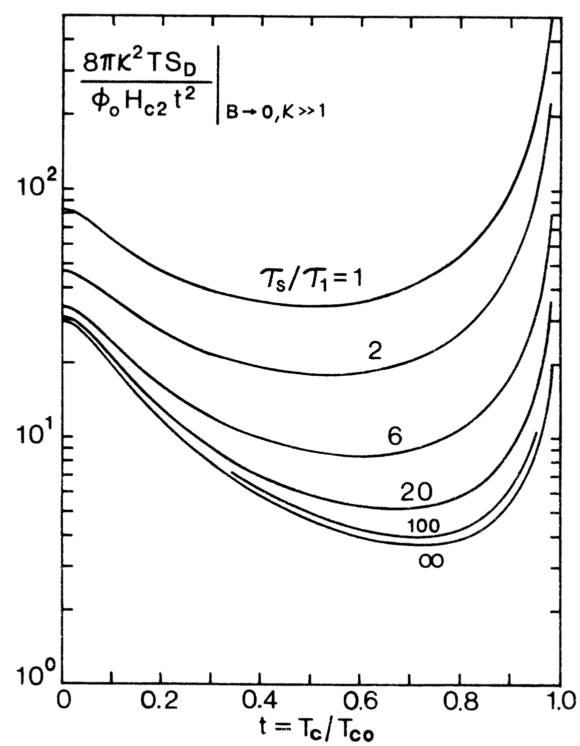

FIG. 3. Plotted vs $t \equiv \exp[\psi(\frac{1}{2}) - \psi(\frac{1}{2}+\rho_s)] \simeq T_c/T_{c0}$ (the reduced transition temperature), for various values of τ_s/τ_1 (the ratio of exchange- to total-scattering lifetimes of the conduction electrons), is the combination $t^{-2}\Phi_1|_{\kappa\gg1}$, where Φ_1 is defined in Eq. (56), and is just the transport entropy S_D calculated in the low-field limit $B \simeq 0$, (only for $\kappa \gg 1$), normalized by $(\Phi_0/4\pi T)(H_{c2}/2\kappa^2)$.

T_c/T_{c0} is usually a very good approximation, unless one wishes to study the $T\to 0$ behavior of Φ_1 when T_c/T_{c0} is very small but still finite as in Sec. IV. Also, if one intends to extrapolate Eq. (66) to the region $\tau_s\Delta_0 \gtrsim 1$, it is then necessary to keep the difference between t and T_c/T_{c0}.

The corresponding results for S_D in the limit $B\simeq H_{c2}$ have already been presented previously [Ref. 1, Eqs. (7)–(10)]. We can write it in a form parallel to Eq. (56):

$$S_D\big|_{B\simeq H_{c2}} = (\phi_0/T)\,|M|\,\Phi_2(\rho_s,\rho_1)\ , \tag{68}$$

where

$$-4\pi M = (H_{c2}-B)/[(2\kappa^2-1)1.16+1]$$
$$\simeq (H_{c2}-B)/(2\kappa^2\times 1.16)\quad \text{for } \kappa\gg 1\ , \tag{69}$$

$$\Phi_2 = L_s/L_M\ , \tag{70}$$

and use has been made of Eq. (13e), as well as $2eD_1H_{c2}=\bar{A}$. Comparing Eqs. (68) and (69) with Eq. (56), we see that if $\Phi_2 = 1.16\times\Phi_1$, we would have a linear B dependence for $S_D|_{\kappa\gg1}$ down to $B=0$. The ratio of 1.16 Φ_1/Φ_2 is therefore a good measure of the deviation of $S_D(B)$ from linearity, with a value bigger than one indicating a positive curvature. In Fig. 4 we have plotted $t^{-2}\Phi_2$ with respect to t for the same set of values of τ_s/τ_1 as in Fig. 3, and in Fig. 5 the ratio $1.16\Phi_1/\Phi_2$ has been plotted in the same way. From Fig. 5 we see that $S_D(B)$ is predicted to have a positive curvature for most values of the two parameters t and τ_s/τ_1, except in the limit of $\tau_s\approx\tau_1$ and $t\ll 1$, corresponding to a near-critical amount of exchange scattering, combined with almost no ordinary scattering—a situation most likely unrealizable in practice. Figure 5 also shows that $S_D(B)$ has a larger upward curvature for dirtier systems (i.e., for smaller τ_1 at fixed τ_s), especially in the low magnetic-impurity concentration limit.

In the following we give analytic results for the limiting cases. For $\rho_s\gg 1$, $t\sim 0.140\,\rho_s^{-1}\ll 1$, corresponding to high concentrations of magnetic impurities, we use Eqs. (47) and (48) to find

$$\Phi_2 = \frac{1}{6\rho_s^2}\left(1+\frac{2\tau_1}{\tau_s}+\frac{2\tau_1^2}{\tau_s^2}\right)$$
$$\times\left(\frac{2\tau_1/\tau_s}{(1-2\tau_1/\tau_s)^2}\ln\frac{2\tau_1}{\tau_s}+\frac{1}{1-2\tau_1/\tau_s}\right)^{-1}$$
$$=\begin{cases} 8.46\,t^2 & \text{for } \tau_1\ll\tau_s\ll(2\pi T)^{-1}\ ,\\ 110.0\,t^2 & \text{for } \tau_1=\tau_s\ll(2\pi T)^{-1}\ .\end{cases}$$

On the other hand, in this limit Φ_1 may be directly evaluated from Eq. (57) by using $P_\epsilon=(1/4T)\times\cosh^{-2}(\epsilon/2T)P$, so that

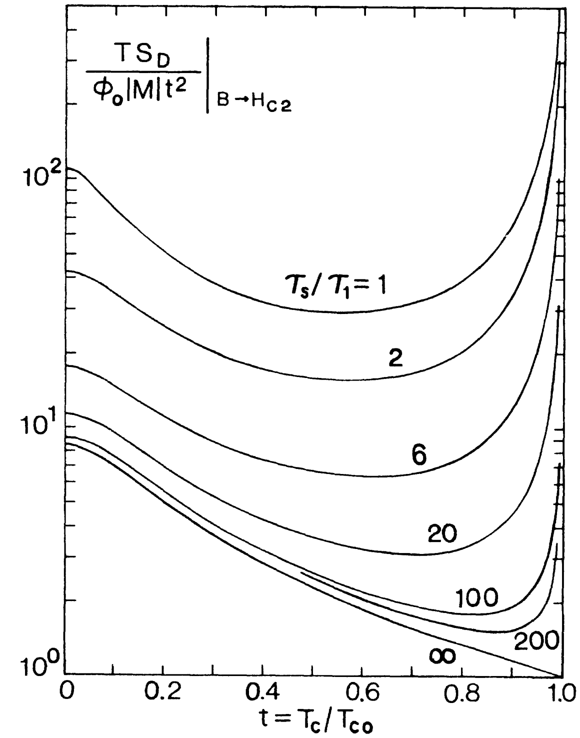

FIG. 4. Plotted in the same way as in Fig. 3 is the combination $t^{-2}\Phi_2$, where Φ_2 is defined in Eq. (68), and is just the transport entropy S_D calculated in the high field limit $B\simeq H_{c2}$, normalized by ϕ_0/T times the magnitude of the equilibrium magnetization $|M|=(H_{c2}-B)/4\pi[1.16(2\kappa^2-1)+1]$.

$$\Phi_1 = L_M^{-1}\left\{L_s\left[C_\gamma+\frac{2\zeta^2}{\xi^2}\,C_E\left(\frac{\zeta}{\xi}\right)\right]+L_uC_2\right\}$$
$$=\begin{cases} 29.1\,t^2 & \text{for } \tau_1\ll\tau_s\ll(2\pi T)^{-1}\ ,\\ 83.0\,t^2 & \text{for } \tau_1=\tau_s\ll(2\pi T)^{-1}\ ,\end{cases}$$

where use has been made of Eqs. (47)–(50) and our earlier results on C_E.[24]

Next we consider the limit $\rho_s\approx 0.203\,(1-t)\ll 1$, corresponding to very low concentrations of magnetic impurities. In this limit $L_s=\psi^{(1)}(\tfrac12)$ is independent of ρ_1, while

$$L_M = [\psi(\tfrac12)-\psi(\tfrac12+\rho_1)]/\rho_1+\psi^{(1)}(\tfrac12)$$
$$=\begin{cases} \psi^{(1)}(\tfrac12) & \text{for } \rho_1\gg 1\ ,\\ -\tfrac12\psi^{(2)}(\tfrac12)\rho_1 & \text{for } \rho_1\ll 1\ ,\end{cases}$$

so that in this limit,

$$\Phi_2=\begin{cases} 1 & \text{for } \rho_1\gg 1\gg\rho_s,\\ 5.79\,(\tau_1/\tau_s)(1-t)^{-1} & \text{for } \rho_1,\rho_s\ll 1\ .\end{cases}$$

As for the behavior of Φ_1 in the limit $\rho_s\ll 1$, we note that in Eq. (66), L_s/L_M is just Φ_1, $L_u\approx(\tfrac18\pi^2)\times(1+6\tau_1/\tau_s)$, $C_2/L_M\approx 0.293(1-t)^{-1}$, while in the x

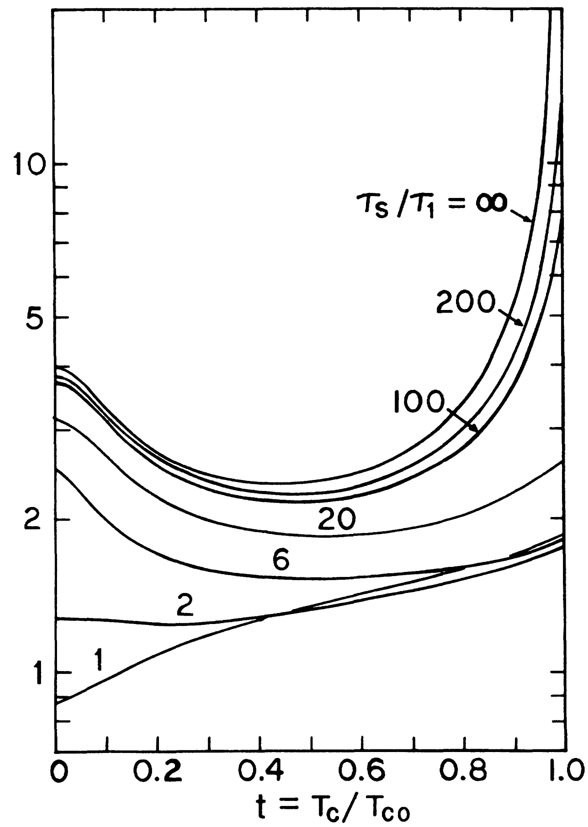

FIG. 5. Plotted in the same way as in Figs. 3 and 4 is the ratio $1.16\,\Phi_1|_{\kappa>>1}/\Phi_2$, which is just the ratio of the transport entropy S_D calculated in the limit $B \simeq 0$ assuming $\kappa >> 1$, and its would-be value if the linear field dependence predicted for S_D near H_{c2} is extrapolated down to $B = 0$.

integral, one may replace the factor $\cosh^{-2}(\pi\rho_s x)$ by 1, and use $x_0 \approx 4.05\,(1-t)^{-1/2}$ for $\rho_1 \gg 1$ and $\approx 1.68\,(\tau_s/\tau_1)^{1/2}$ for $\rho_1 \ll 1$. Thus for $\rho_1 \gg 1$ it may be shown that the second term dominates in Eq. (66), and

$$\Phi_1|_{\kappa\to\infty} = 0.362(1-t)^{-1} \quad \text{for } \rho_1 \gg 1 \gg \rho_s \ ,$$

but when both ρ_1 and ρ_s are small in comparison with one, then all three terms in Eq. (66) become equally important, and we have

$$\Phi_1 \propto (1-t)^{-1} \quad \text{for } \rho_1, \rho_s \ll 1 \ ,$$

where the constant of proportionality (still a function of τ_s/τ_1) can only be obtained numerically.

VI. CONCLUSIONS

In this paper we have calculated microscopically the transport entropy S_D of vortices in gapless superconductors containing arbitrary amounts of magnetic and nonmagnetic impurities, by using a new general prescription for calculating heat-current responses in *magnetic* conductors developed recently by the author. This new prescription differs from the old one existing in the literature for nonmagnetic materials, by a new additive contribution to the local heat-current response $\vec{j}^h$, equal to the cross product of the local electric field $\vec{\mathcal{E}}$ and the local magnetization $\vec{m}$. The present calculation, plus a previous work of the author,[1] demonstrated in both the high- and low-field limits and for arbitrary ratio of exchange- to ordinary-scattering lifetimes, that this new additive contribution to $\vec{j}^h$ is necessary in order for the calculated S_D to approach zero as the absolute temperature T approaches zero, as is required by the third law of thermodynamics. It is worth emphasizing that when both $\vec{\mathcal{E}}$ and $\vec{m}$ are space dependent, then the space average of $\vec{\mathcal{E}} \times \vec{m}$ does not in general equal to the cross product of the space averages. This implies that an earlier simpler prescription proposed by Maki[7] for removing the unphysical divergences of a calculated S_D at low temperatures is valid only in the vicinity of H_{c2} when $\vec{\mathcal{E}}$ is essentially uniform in space.[1] Thus the exact proof that $S_D \to 0$ as $T \to 0$ presented in this paper for the *low*-field limit, which has not been achieved by anybody previous to us, should constitute a strong support for the correctness of our new prescription for calculating heat-current responses in any *magnetic* conductors, including type-II superconductors.

Combining our high- and low-field results on S_D, under the gaplessness condition $\tau_s\Delta_0 \ll 1$, we predict that S_D will first rise linearly as the average magnetic induction B is lowered below H_{c2}, and then it will bend upward for practically all concentrations of magnetic and nonmagnetic impurities as B is further lowered toward zero, but S_D will always reach a finite value at $B = 0$, for any fixed $T \neq 0$. The exact amount of upward bending depends on the concentrations of magnetic and nonmagnetic impurities, but is generally larger for dirtier systems. These results are summarized in Figs. 3–5. While experimentally there have not been measurements of S_D on gapless superconductors containing paramagnetic impurities, there do exist measurements of $S_D(B)$ on superconductors with a finite energy gap. A very recent such measurement by deLange and Otter[25] for the full range of fields $0 \leqslant B \leqslant H_{c2}$, does show essentially the same type of field dependence of S_D predicted here for gapless superconductors, suggesting that the dynamic properties of type-II superconductors with and without gaps are at least qualitatively similar, in spite of the fact that the dynamic equations governing type-II superconductors with a nonvanishing gap are drastically more complicated than those for gapless type-II conductors. Nevertheless, a direct test for the present theory by measuring $S_D(B)$ in superconductors with paramagnetic impurities remains worthwhile, particularly with regard to the verification of the predicted increase of the upward curvature of $S_D(B)$ as one increases the amount of nonmagnetic scatterings in the system, which may very well be a more general feature pertaining also to superconductors with a finite gap.

In the remaining part of this section, we present a plausible identification of the physical meaning of the anomalous quantity u_1 (or $U_1 = -Cu_1$ before normalization) in the TDGL equations, Eqs. (12a) and (12c), which was first discovered by Éliashberg.[13] While it has been demonstrated in various places (Refs. 13, 15, 18, and the present work) that u_1 plays a very important role in the dynamics of gapless superconductors, particularly in the limit of low magnetic impurity concentrations, so far the physical meaning of u_1 seems to have not been discussed beyond the general association of the anomalous terms with a nonequilibrium distribution of quasiparticles.[26,27] Some insight into this question has been revealed by our Eqs. (27) and (31), which show that the gradient of u_1 appears as a contribution to the local heat-current density $\vec{j}^h$ [i.e., the third term in Eq. (31)]. Since this term does not depend on the superconductive order parameter explicitly, its existence can be reconciled only by identifying u_1 as being proportional to the local temperature deviation from equilibrium δT. The proportionality constant between u_1 and δT may then be determined by setting the third term of Eq. (31) equal to $-K_n \vec{\nabla}\delta T$, where the normal state (electronic) thermoconductivity K_n may be related to the normal-state electric conductivity σ by the Wiedemann-Franz law $K_n/\sigma = (\pi^2/3e^2)T$, which is exactly valid for the systems considered here, since our model includes only elastic scattering. In this way we obtain

$$u_1 = (\pi/6\rho_s L_c)\,\delta T\,, \tag{71}$$

or, because $U_1 = -Cu_1$ and Eqs. (11c), (34), we have

$$U_1 = -\frac{\pi\tau_s}{12}\,\frac{\psi^{(1)}(\frac{1}{2}+\rho_s) + \rho_s\psi^{(2)}(\frac{1}{2}+\rho_s)}{1-\rho_s\psi^{(1)}(\frac{1}{2}+\rho_s)}\,\delta T\,. \tag{72}$$

There exist two internal-consistency checks on this identification. First, from Eq. (59) and $u_1 = v\cos\theta g(r)$, we note the long-range nature of u_1 (i.e., $u_1 \propto \cos\theta/r$ even for $r \gg \lambda$), which implies that it cannot be of electromagnetic origin, since any electromagnetic disturbance in a superconductor should be screened off within a distance λ or ξ_ϵ with $\epsilon \sim T$. Identifying u_1 as a temperature disturbance is then supported by noticing that the (r,θ) dependence of u_1 agrees precisely with a corresponding behavior of δT outside a vortex core obtained previously by Clem[16] in a phenomenological theory. (Note that the overall sign also agrees.) As our second check on the identification $u_1 \propto \delta T$, let us see whether the $U_1\Delta$ term in Eq. (10a) may be attributed to a variation of temperature in the corresponding equilibrium Ginzberg-Landau equation, which is just Eq. (10a) without the terms $\partial\Delta/\partial t$ and $U\Delta$. While the coefficients $\overline{B}$, C, and D_1 all depend on T except in the limit of high concentrations of magnetic impurities, it is not difficult to convince oneself, using the order analysis of Sec. II, that the dominant T dependence is in the $\overline{A}\Delta$ term. Setting $(d\overline{A}/dT)(\delta T)\Delta$ equal to $U_1\Delta$, we find

$$U_1 = -(1/T)\left[1-\rho_s\psi^{(1)}(\tfrac{1}{2}+\rho_s)\right]\delta T\,. \tag{73}$$

Comparing this equation with Eq. (72), we find them to agree only in the limit of high concentrations of magnetic impurities when both expressions reduce to

$$U_1 = -(\tfrac{1}{3}\pi^2\tau_s^2 T)\,\delta T \quad (\text{when } T_c \ll T_{c0})\,. \tag{74}$$

The discrepancy between Eqs. (72) and (73) in the general case is not fully understood yet, but we do not think that it is a serious alarm to our association of u_1 with δT, since one already has a similar difficulty when one tries to identify $\psi \equiv \int \psi_\epsilon\, d\epsilon$ as the electrochemical potential. Referring to Eqs. (12b), (12d), (12f), and (12g), we see that a different weighted average of ψ_ϵ, namely, u_2, appears in the equation for the phase χ of the order parameter, while it is ψ which appears in the equations for $\vec{j}$ and ρ. We have $u_2 = \psi$ only when the magnetic impurity concentration is high. Apparently, when a gapless superconductor with a not-very-high concentration of magnetic impurities is driven out of equilibrium, by say, an external electromagnetic perturbation, then its elementary-excitation distribution function is more complicated than can be described by a local equilibrium distribution, which is characterized by only two space-dependent parameters $\delta T(\vec{r})$ and $\delta\mu(\vec{r})$. Thus any attempt to identify δT and $\delta\mu$ in the equations for $\vec{j}^h$ and $\vec{j}$, can only be taken in an effective sense, unless the magnetic impurity concentration is high, as we have cautiously stated in Sec. I.

APPENDIX A

In this appendix we illustrate the real-time Feynman-diagram method of Gor'kov and Éliashberg, for studying *dynamic* properties of superconductors at *finite* temperatures, by giving more details of our evaluation of two diagrams contributing to $\langle \hat{j}_0^h \rangle$. They are Figs. 1(c) and 1(d) which involve Δ and Δ^* as the anomalous vertices. Using the Feynman rules reviewed in Sec. II, we evaluate Fig. 1(c) as

$$\frac{N(0)}{4\pi i}\int_{-\infty}^{\infty} d\epsilon \int d\xi \, \frac{d\Omega}{4\pi}\, \frac{(2\epsilon-\omega)\,\vec{v}\,(\omega_2/2T)\cosh^{-2}(\epsilon/2T)\tilde{\Delta}^R(1)\,\Delta^*(2)}{(\epsilon-\xi+\vec{v}\cdot e\vec{A}+\frac{1}{2}i\tau_1)[\epsilon-\omega_1+\xi-\vec{v}\cdot(\vec{k}_1-e\vec{A})+i/2\tau_1][\epsilon-\omega-\xi+\vec{v}\cdot(\vec{k}+e\vec{A})-\frac{1}{2}i\tau_1]}\,,$$

where $\omega=\omega_1+\omega_2$; $\vec{k}=\vec{k}_1+\vec{k}_2$; $\tilde{\Delta}^{R(A)}$ and $\tilde{\Delta}^{*R(A)}$ denote renormalized vertices which, to the lowest order in η needed here, is just the bare vertices Δ and Δ^* multiplied by the factor $(2\epsilon \pm i/\tau_1)/(2\epsilon\pm 2i/\tau_s)$.[13] Note that we can include the effects of the vector potential $\vec{A}$ by including it in the electron propagators, because the commutator of $\vec{k}$ (or ω) and $\vec{A}(\vec{r})$ is negligible within the order of interest. In view of the ω_2 factor in the anomalous-vertex factor and that only linear response is considered here, we may drop all other ω dependences. Then we can expand the integral to first power in $\vec{k}$ or $e\vec{A}$, perform the ξ integral (by contour integration) and the angular average, to obtain

$$\frac{-i\sigma}{2e^2}\int_{-\infty}^{\infty} d\epsilon\, \frac{\epsilon}{\epsilon+i/\tau_s}\left(\frac{\vec{k}_1-2e\vec{A}}{2\epsilon+i/\tau_1}+\frac{\vec{k}}{i/\tau_1}\right)\frac{\omega_2}{4T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)\Delta(1)\,\Delta^*(2)\,.$$

Figure 1(d) may be evaluated similarly, giving precisely the above result with the replacement $i\to -i$, $\omega_2\to\omega_1$, and $(\vec{k}_1-2e\vec{A})\to(\vec{k}_2+2e\vec{A})$. The two contributions may be combined (after the substitution $\epsilon\to-\epsilon$ in the second contribution), and the ϵ integrations may be performed by contour method (after an integration by parts)

$$\int_{-\infty}^{\infty} d\epsilon\, \frac{\epsilon}{\epsilon+i/\tau_s}\,\frac{1}{4T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)=1-\rho_s\,\psi^{(1)}(\tfrac{1}{2}+\rho_s),$$

$$\int_{-\infty}^{\infty} d\epsilon\, \frac{\epsilon}{(\epsilon+i/\tau_s)(2\epsilon+i/\tau_1)}\,\frac{1}{4T}\cosh^{-2}\left(\frac{\epsilon}{2T}\right)=\frac{i}{4\pi T}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2}+\rho_s)(n+\frac{1}{2}+\rho_1)}\left(1-\frac{n+\frac{1}{2}}{n+\frac{1}{2}+\rho_s}-\frac{n+\frac{1}{2}}{n+\frac{1}{2}+\rho_1}\right)\,.$$

Then using the definition for $N_{\mu,\nu}$ in Eq. (20), we can finally cast our result into the form given in Eq. (22).

APPENDIX B

In Secs. III and V, we have assumed that the gradient term $\vec{\Xi}$ of $\vec{j}_0^h$ [cf. Eqs. (27) and (31)] give null contributions to $(\vec{j}_0^h)_{av}$ and S_D. From the physical point of view, this assumption might be easily acceptable, but to prove it mathematically turns out to involve an interesting subtle point which we elaborate in this appendix.

The subtlety actually arises only from the second, third and fourth terms in Eq. (31), so let us first show that the first and last terms have null space averages. We notice from Eq. (12c) that in the linear-response limit $\nabla^2 u_1 \propto \partial f^2/\partial t$, so both the last term and the first term of Eq. (31) give rise to the following integral:

$$\int dx\,dy\,\vec{\nabla}(v\cdot\vec{\nabla})f_0^2=\hat{v}\int dx\,dy\,(\hat{v}\cdot\vec{\nabla})^2 f_0^2$$
$$=\hat{v}\pi\int_0^{\infty}\left[\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d}{dr}f_0^2\right)\right]r\,dr=0\,.$$

Now let us consider the simplest of the three middle terms in Eq. (31), viz., the term in direct proportion to $-\vec{\nabla}u_1$. If we use the solution for u_1 given by Gor'kov and Kopnin [i.e., Eq. (59) and $u_1=v\cos\theta g(r)$], we find

$$-\int\vec{\nabla}u_1\,dx\,dy=-\hat{v}\int(\hat{v}\cdot\vec{\nabla})\,u_1\,dx\,dy$$
$$=-\hat{v}\pi\int_0^{\infty}\left(\frac{1}{r}\,\frac{\delta}{\delta r}rg\right)r\,dr$$
$$=\hat{v}\pi C_2\,\xi^{-2}\int_0^{\infty}(1-f_0^2)\,r\,dr$$
$$=\hat{v}\pi C_2\,(\ln\kappa+0.497)\neq 0\,, \qquad \text{(B1)}$$

where use has been made of Ref. 24, Eqs. (14), (18), and (28). It is easy to see that this result is necessarily unphysical. In our earlier analysis of the order of each diagram, we have already noted that this contribution is larger than the others by a factor η^{-2}. Its $\ln\kappa$ dependence also makes it dominant over the other terms. Then, because this contribution does not have a factor Δ_0^2 in front, it does not even vanish when T is let to approach T_c. In the conclusion section of the main text, we argue that u_1 has the physical interpretation of being proportional to the temperature deviation from equilibrium [cf. Eq. (71)]. If this is further confirmed, it then becomes very clear that we must have $(-\vec{\nabla}u_1)_{av}=0$, or else the isolated vortex line under consideration would be subject to a nonvanishing average temperature gradient in the $-\hat{v}$

direction, which would of course drive a heat flow in the direction $\hat{v}$. This dilemma is resolved by noting the long-range nature of u_1 (i.e., $u_1 \propto \cos\theta/r$ even for $r \gg \lambda$), which implies that there is really no "isolated vortex limit" for u_1. We must therefore regard the vortex line under consideration to be located at the center of a unit cell. Since u_1 is odd with respect to the transformation $\vec{r} \to -\vec{r}$, it must vanish on the cell boundary. If we adopt a Wigner-Seitz approximation, then we must solve Eq. (58) with the boundary condition $g(r) \to 0$ as $r \to R$, where $R = (\phi_0/\pi B)^{1/2} \gg \lambda$ is the radius of the circular unit cell. Then the solution is only slightly different from that of Eq. (59):

$$g(r) = -C_2 \xi^{-2} r^{-1} \int_0^r (1 - f_0^2 - \Theta) r\, dr\,, \qquad \text{(B2)}$$

where

$$\Theta \equiv 2R^{-2} \int_0^R (1 - f_0^2) r\, dr\,. \qquad \text{(B3)}$$

Because $\Theta \propto (\xi/R)^2 \ln \kappa \ll 1$, it is negligible in most calculations involving u_1, such as in all the earlier results for flux-flow resistivity in the low-field limit.[15,18] However, if Eq. (B2) in place of Eq. (59) is used in evaluating $\int dx\, dy(-\vec{\nabla} u_1)$, the result is no longer that given in Eq. (B1), but is exactly zero, as it must be from a physical point of view.

Using the new solution for u_1 in Eqs. (B2) and (B3), it is not difficult to show that the second term in Eq. (31) also has a vanishing space integration. Since we did not yet derive an expression for δu_1, it is not possible for us to explicitly verify that $\int dx\, dy\, (-\vec{\nabla}\delta u_1)$ also vanishes. However, in view of the fact that δu_1 is a higher order correction to u_1, it should be safe to assume that the fourth term in Eq. (31) also does not contribute to $(\vec{j}_0^h)_{av}$ and S_D. We shall be contented with this unverified assertion in this work, since the derivation of δu_1 could amount to doubling the present effort, as we have explained in Sec. III. It would seem reasonable to postpone this task until one studies the local distribution of heat flow, where the gradient terms become important.

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[1]C.-R. Hu, Phys. Rev. B 13, 4780 (1976).

[2]A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957) [Sov. Phys.-JETP 5, 1174 (1957)].

[3]H. B. Callen, *Thermodynamics* (Wiley, New York, 1960), Chap. 17.

[4]This amounts to ignoring $\tau_{tr}\omega_c$ and T/E_F with respect to 1, where τ_{tr} is the transport lifetime, $\omega_c \equiv eH/m^*c$ is the cyclotron frequency, T is the temperature, and E_F is the Fermi energy. Using $H \lesssim H_{c2}$, it may be shown that for the gapless superconductors considered here, $\tau_{tr}\omega_c$ is at most of the order $(\tau_s\Delta_0)(\Delta_0/E_F)$. Thus both conditions are easily satisfied whenever $T_c \ll E_F$.

[5]For a review on flux-flow phenomena, see, for example, Y. B. Kim and M. J. Stephen, in *Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1969), Vol. 2, Chap. 19.

[6]C. Caroli and K. Maki, Phys. Rev. 164, 591 (1967).

[7]K. Maki, Phys. Rev. Lett. 21, 1755 (1968); J. Low Temp. Phys. 1, 45 (1969).

[8]The boundary condition that $m \equiv 0$ outside the sample can not be met only if the sample is multiply connected, with a persistent current flowing around one of its "holes." Such a situation does not concern us here and may be safely excluded from our consideration.

[9]See, for example, A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971), p. 228. The substitution $t = -i\tau$ is not made in our Eq. (9), because we shall eventually use the *real-time* diagram technique of Gor'kov and Éliashberg for evaluating causal response functions at finite temperatures (see Refs. 12 and 13).

[10]So far $\hat{j}^E$ has been called the energy current operator as if ϵ is the energy density. More properly $\hat{j}^E$ should be called the magnetic enthalpy current operator, and ϵ the magnetic enthalpy density, so that $u \equiv \epsilon + \vec{m}\cdot\vec{b}$, which satisfies the thermodynamic relation $\delta u = T\delta s + \mu\delta n + \vec{b}\cdot\delta\vec{m}$, may be identified as the internal energy. Then we can understand the relation at nonequilibrium situations: $u = \langle \hat{h}_T \rangle - (-\vec{m}\cdot\vec{b}) - \rho\phi$, viz., the internal energy is simply the total energy minus all the external energies. This further explains why we should let $\hat{\epsilon} \equiv \hat{h}_T - \hat{\rho}\phi$.

[11]A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. 39, 1781 (1960) [Sov. Phys.-JETP 12, 1243 (1961)].

[12]L. P. Gor'kov and G. M. Éliashberg, Zh. Eksp. Teor. Fiz. 54, 612 (1968) [Sov. Phys.-JETP 27, 328 (1968)].

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[14]For an explicit definition of τ_1 and τ_s in terms of impurity potentials, see, for example, Y. Baba and K. Maki, Prog. Theor. Phys. 44, 1431 (1970).

[15]C.-R. Hu and R. S. Thompson, Phys. Rev. Lett. 31, 217 (1973). In the *linear-response* limit, the TDGL equations derived in this reference and reproduced in Eqs. (10) and (12) here may also be derived by a method which avoids the enumeration of diagrams. See, for example, A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 64, 1096 (1973) [Sov. Phys.-JETP 37, 557 (1973)].

[16]J. R. Clem, Phys. Rev. Lett. 20, 735 (1968).

[17]G. S. Cohen and G. Rickayzen, J. Phys. F 3, 1582 (1973).

[18]L. P. Gor'kov and N. B. Kopnin, Zh. Eksp. Teor. Fiz. 60, 2331 (1971) [Sov. Phys.-JETP 33, 1251 (1971)].

[19]The assumption does not exclude the limiting cases when $\tau_1 \ll \tau_s$, when $2\pi\tau_s T \ll 1$, or when $2\pi\tau_1 T \ll 1$. Instead, it allows for the inclusion of all relevant

Feynman diagrams, some of which may become unimportant in one or more of the limiting cases. Similarly, the conclusion $\omega \sim e\phi \sim \eta\Delta_0$ obtained in Eqs. (14b) and (14d) does not exclude the limiting case ω, $e\phi \ll \eta\Delta_0$ either, but in fact implies that the validity of Eqs. (10) is not restricted by the latter condition.

[20]To explicitly verify this statement by evaluating all relevant Feynman diagrams up to the order prescribed in Eq. (16), one must be careful not to use the approximate vertex equation (18) in place of the exact equation (19) in any diagram with three or less electron propagators, for which $\int d^3p$ cannot be replaced by $N(0)\int d\xi$.

[21]To illustrate our systematic order analysis using these two diagrams, we note that the heat current vertex gives a factor v_F/τ_s, the three electron propagators together give a factor τ_s^3, the internal frequency sum and the internal momental integral together give a factor $N(0)\tau_s^{-2}$, the two Δ vertices give a factor Δ_0^2, the anomalous vertex factor gives a factor η^2, and finally one must still count in another factor of η due to the expansion of the product of all electron propagators to first power in $\vec{v}\cdot\vec{k}$ (or due to the insertion of an extra $-\vec{v}\cdot e\vec{A}$ vertex) in order to attain a nonvanishing angular integration. Multiplying all contributing factors together, one obtains the final-order estimate of $N(0)v_F\eta^3\Delta_0^2$ for these two diagrams.

[22]R. S. Thompson and C.-R. Hu, Phys. Rev. Lett. 27, 1352 (1971).

[23]C.-R. Hu and R. S. Thompson, Phys. Rev. B 6, 110 (1972).

[24]C.-R. Hu, Phys. Rev. B 6, 1756 (1972).

[25]O. L. deLange and F. A. Otter, Jr., J. Low Temp. Phys. 18, 31 (1975).

[26]A. Schmid and G. Schön, J. Low Temp. Phys. 20, 207 (1975).

[27]N. B. Kopnin, Zh. Eksp. Teor. Fiz. 69, 364 (1975).
This reference has also proposed a magnetic correction term to the heat current expression using an argument which does not seem to be complete. Thus he obtained for the heat current: $\vec{q}=\vec{j}_\epsilon-\vec{j}'_\epsilon$ with $\vec{\nabla}\cdot\vec{j}'_\epsilon=\vec{j}_0\cdot\vec{E}$, which is solvable only when $\vec{\nabla}\times\vec{E}=-\partial\vec{b}/\partial t=0$. More generally, one must add to the left-hand side a term $-\vec{m}_0\cdot(\partial\vec{b}/\partial t)$, where $\vec{\nabla}\times\vec{m}_0\equiv\vec{j}_0$. Then $\vec{j}'_\epsilon=-\vec{\mathcal{E}}\times\vec{m}_0$, and his expression for $\vec{q}$ would agree with our equation (4). Furthermore, the first term in his continuity equation, $T^{-1}(\partial\bar{\epsilon}/\partial t)_\mu+\vec{\nabla}\cdot(\vec{q}/T)=w/T$, would be changed to $\partial s/\partial t$, where $T\delta s=\delta\epsilon-\mu\delta n+\vec{m}_0\cdot\delta\vec{b}$, exactly the entropy continuity equation obtained in Ref. 1 by the present author, except for the restriction of m to m_0, which is valid in the linear response limit only.