Theory of dynamical conductivity of interacting electrons*

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Starting from the current-current correlation function $Q(\omega)$, and taking into account the fact that the current operator (or the total momentum operator) commmutes with the total Hamiltonian of the interacting electrons in the absence of impurities, we generate a series of diagrams for the dynamical conductivity. One essential feature of this approach is that at finite external frequency ω, each of these diagrams vanishes in the limit of zero impurity concentration. We give a formal expression for the conductivity $\sigma(\omega)$ at $\omega\tau \gg 1$ (τ is the electron scattering lifetime), and we evaluate explicity $\sigma(\omega)$ for ω and $\tau^{-1} \ll E_F$, the Fermi energy. Our results in the limit $\omega \to 0$ are in complete agreement with those obtained previously.

I. INTRODUCTION

In the study of the transport properties of an electron gas, the interaction among the electrons plays an important role.[1] It can come in through the quasiparticle properties, such as renormalization of the mass and the g factor, or it can manifest itself in collective phenomena such as the plasmon-assisted magneto-optical transition.[2] Theoretical treatments of the effects of electron-electron interaction necessarily involve approximations, the most popular being some version of the random-phase approximation. However, it has been pointed out[3] that when the external perturbation is spatially uniform, it couples only to the center-of-mass degrees of freedom. Since the interparticle interaction involves only the relative coordinates, it cannot influence any observable properties in this situation. This is a very general argument which shows that in the response to a uniform field, electron-electron interaction effect can only come in through the breaking of the translation symmetry by impurities, phonons, etc. Conventional theoretical approaches do not explicitly take this fact into consideration. There is, therefore, no guarantee that the interaction effects cancel out in the proper limit. This can lead to erroneous conclusions. In the study of plasmon-assisted optical transition, for example, Ting and Quinn[2] have found that many terms in a given order of perturbation theory cancel out yielding a final cross section which is proportional to impurity concentration. For higher-order perturbation terms, the summation of all processes to obtain the correct limiting value for a vanishingly small concentration of impurities becomes a very tedious procedure.

In view of these considerations, it is desirable to formulate a study of the transport properties which has the built-in property that interaction effects cancel out in the absence of impurities. In this paper we present such an approach. The idea is first to separate the center-of-mass and internal degrees of freedom, which are coupled only through impurities. The Hamiltonian which describes the interacting electron gas in the presence of randomly distributed impurities is given by

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i>j} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} + \sum_{j,a} e^{i\vec{q}\cdot(\vec{r}_j - \vec{R}_a)} u(\vec{q}) , \tag{1}$$

where $\vec{p}_i$ and e are the momentum and charge of an electron with mass m and coordinate $\vec{r}_i$. $\vec{R}_a$ denotes the coordinate of one of the randomly distributed impurities. $u(\vec{q})$ is the electron-impurity scattering potential in momentum space. We now define the center-of-mass degree of freedom variable $\vec{P}, \vec{R}$ and the relative electron variables $\vec{p}_i', \vec{r}_i'$ in the conventional manner.

$$\vec{P} = \sum_i \vec{p}_i, \quad \vec{R} = \frac{1}{N} \sum_i \vec{r}_i ,$$

and

$$\vec{p}_i' = \vec{p}_i - \vec{P}/N, \quad \vec{r}_i' = \vec{r}_i - \vec{R} ,$$

where N is the number of electrons. It is easy to verify that $\vec{P}$ and $\vec{R}$ are canonical conjugate variables and that they commute with the relative variables $\vec{p}_i', \vec{r}_i'$. However, we do not gain an extra degree of freedom because the relative variables are no longer independent. In fact

$$[\vec{r}_j', \vec{p}_i'] = i(\delta_{ij} + 1/N) .$$

Since we are only interested in extended systems ($N \to \infty$), the term of order $1/N$ can be neglected in the above equation; we can then use the conventional way of quantization to treat the $N+1$ degrees of freedom. The Hamiltonian can now be written in the form

$$H = \frac{\vec{P}^2}{2m} + \sum_i \frac{\vec{p}_i'^2}{2m} + \sum_{i>j} \frac{e^2}{|\vec{r}_i' - \vec{r}_j'|} + \sum_{j,a} e^{i\vec{q}\cdot(\vec{r}_j' + \vec{R} - \vec{R}_a)} u(\vec{q}) . \quad (2)$$

This form brings out very clearly the role of impurity in coupling the internal and center-of-mass degrees of freedom. If now a uniform external field is applied, it will introduce an additional term in H involving only the center-of-mass variables. Thus if we develop the linear-response function starting with this representation of the Hamiltonian, electron-electron interaction effects are guaranteed to cancel out in the limit of vanishing concentration of impurities. This is an important advantage of the present approach.

In the past, there have been many studies of the dynamical conductivity of an electron gas.[4] In these studies, electron-electron interaction effects are usually neglected or treated in a phenomenological fashion. The memory-function method developed by Mori[5] and applied by Götze and Wölfe[4] to the study of transport properties of electron gas is very powerful. However, its validity in the high-frequency limit has not been established, and it is also not clear whether electron-electron interaction effects are fully incorporated in this approach.

In this paper, we shall apply our new formalism to study the response of an interacting electron gas to a time-dependent uniform electric field. The dc conductivity has been studied in considerable detail in Ref. 1, by a diagrammatic approach. It was concluded there that one can still retain the quasiparticle concept, but the current carried by the individual quasiparticle is determined by its group velocity. Although no explicit result is given for the finite frequency conductivity in Ref. 1, it was also conjectured that in the limit of high frequency, the effect of collisions with impurities became negligible, and the current carried by the quasiparticle should be determined by the phase velocity. One would also expect this result from the principle of Galilean invariance as usually assumed in the development of Fermi-liquid theory.[6] In the subsequent sections, we shall derive an expression for the finite frequency-dependent conductivity $\sigma(\omega)$. In Sec. II, an equation of motion method is developed for evaluating the current-current correlation function. In Sec. III, a formal expression for the conductivity $\sigma(\omega)$ at $\omega\tau \gg 1$ will be given. In Secs. IV and V, we explicitly evaluate $\sigma(\omega)$ at $\omega, \tau^{-1} \ll E_F$ for noninteracting and interacting electrons, respectively. All these calculations are performed at zero temperature. Section VI contains a summary and discussion of our results.

II. CURRENT-CURRENT CORRELATION FUNCTION

In the presence of a time-dependent electric field $\vec{E} = \vec{E}_0 e^{i\omega t}$, the coupling between the electric field and electrons is of the form

$$H_{\text{int}} = -i \frac{e}{m\omega} \vec{E} \cdot \sum_i \vec{p}_i = -i \frac{e}{m\omega} \vec{E} \cdot \vec{P} , \quad (3)$$

where $\vec{P}$ is the momentum of the center of mass of the system. Therefore, the electric field only couples to the center-of-mass degree of freedom. From linear-response theory the conductivity is related to the current-current correlation function defined as[7]

$$Q_{\alpha\beta}(t) = -i\Theta(t)\langle[P_\alpha(t), P_\beta(0)]\rangle , \quad (4)$$

where $\Theta(t) = 1$ if $t > 0$ and $\Theta(t) = 0$ if $t < 0$. α and β are indices which label the x, y, and z directions of the center-of-mass momentum $\vec{P}$. There is a very important property of the current-current correlation function $Q_{\alpha\beta}(t)$, namely, that in the absence of impurities, H and P commute with each other [from Eq. (2)] and $Q_{\alpha\beta}(t) = 0$. This fact, however, is not easy to demonstrate explicitly if one uses the diagrammatic method to evaluate $Q_{\alpha\beta}(t)$ defined in Eq. (4) directly for an interacting electron gas. What we will do in the following is to use the equation of motion method first, and then to use the diagrammatic method to calculate $Q_{\alpha\beta}(t)$. Differentiating $Q_{\alpha\beta}(t)$ twice with respect to t yields

$$i\frac{d^2 Q_{\alpha\beta}(t)}{dt^2} = \frac{d}{dt}\{\delta(t)\langle[P_\alpha(t), P_\beta(0)]\rangle\} + \delta(t)\langle[\dot{P}_\alpha(t), P_\beta(0)]\rangle - \Theta(t)\langle[\dot{P}_\alpha(0), \dot{P}_\beta(-t)]\rangle . \quad (5)$$

It is rather easy to see that the first term on the right-hand side of the above equation vanishes. From now on we will assume that the polarization of the external field is along the x direction. Denoting $Q_{xx} = Q$, and transforming Eq. (5) into ω space, we have

$$Q(\omega) = (i/\omega^2)\langle[\dot{P}_x(0), P_x(0)]\rangle + I(\omega)/\omega^2 , \quad (6a)$$

where

$$I(\omega) = -i\int_{-\infty}^{\infty} dt\, e^{i\omega t}\Theta(t)\langle[\dot{P}_x(t), \dot{P}_x(0)]\rangle . \quad (6b)$$

Making use of Eq. (2), we can express $\dot{P}_x$ as

$$\dot{P}_x = -i[P_x, H] = -i\sum_{\vec{q},a} q_x e^{-i\vec{q}\cdot\vec{R}_a} u(\vec{q})\rho_q , \quad (6c)$$

where $\rho_q = \sum_j e^{i\vec{q}\cdot\vec{r}_j}$ is the density operator. The first term on the right side of Eq. (6a) is just equal to $-I(0)/\omega^2$. The above formulas are valid

for any impurity configuration. To obtain the current-current correlation function for the real system, one needs to average over a random distribution of the impurity configuration. Thus (6a) becomes

$$\overline{Q}(\omega) = (1/\omega^2)\overline{I}(\omega)\,, \tag{7}$$

where $\tilde{I}(\omega) = \overline{I}(\omega) - \overline{I}(0)$. The bar above $Q(\omega)$ and $I(\omega)$ denotes the average with respect to impurity configurations.

III. HIGH-FREQUENCY CONDUCTIVITY

In this section we will use the approach developed in Sec. II to discuss the high-frequency conductivity of an interacting electron system. The conductivity $\sigma(\omega)$ is related to $\overline{Q}(\omega)$ as[7,8]

$$\sigma(\omega) = \frac{i}{\omega}\frac{Ne^2}{m} + \frac{i}{\omega}\left(\frac{e}{m}\right)^2 \overline{Q}(\omega)\,. \tag{8}$$

Substituting Eq. (7) into the above expression yields the equation

$$\sigma(\omega) = \frac{i}{\omega}\frac{Ne^2}{m} + \frac{i}{\omega^3}\left(\frac{e}{m}\right)^2 \tilde{I}(\omega)\,. \tag{9}$$

From now on we will use the standard diagrammatic method to evaluate $\tilde{I}(\omega)$, which is just the Fourier transform of the force-force correlation function. The diagram to lowest order in the impurity concentration is shown in Fig. 1. The solid line in the figure represents the electron Green's function, the crosses denote the electron-impurity scattering, and the dashed line between two crosses is a result of averaging over impurity configuration. It can be interpreted as the electron being scattered from the same impurity atom. Thus each dashed line gives rise to one extra power of impurity concentration n. The wavy line here stands for the external photon, and the open square labeled by Γ is the vertex function due to electron-electron interactions. It is interesting to note that we no longer have the well-known current vertex here. What we have is the "force vertex." The external photon can generate particle-hole pairs in the system only in the presence of an impurity line as shown in the figure. We refer the reader to Ref. 8 for details of the rules associated with evaluation and drawing of the diagrams.

In the high-frequency or low-impurity-concentration limit, i.e., for $\omega\tau \gg 1$, the diagram in Fig. 1 is the dominant term in $\overline{I}(\omega)$, it is given by the expression

$$\overline{I}_1(\omega) = n\sum_{\vec{q}} q_x^2 |u(\vec{q})|^2 S(\vec{q},\omega)\,, \tag{10}$$

where

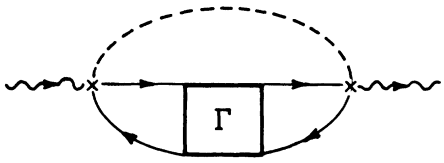

FIG. 1. The lowest-order diagram for $\overline{I}(\omega)$ in the electron-impurity scattering potential. Here the solid line represents the Green's function. The cross means the electron-impurity interaction and the dashed line between two crosses means that the electron is scattered from the same impurity atom. The wavy line here stands for the external photon. The open square labeled by Γ is the vertex function due to electron-electron interactions.

$$S(\vec{q},\omega) = -i\int_{-\infty}^{\infty} dt\, e^{i\omega t}\Theta(t)\langle[\rho_q(t)\rho_{-q}(0)]\rangle$$

is the density-density correlation function of the electrons and n is the impurity concentrations. In (10), $S(\vec{q},\omega)$ need only be evaluated in the absence of impurities that still incorporate full electron-electron interaction. An extensive study of $\overline{I}_1(\omega)$ given in (10) has been made in Ref. 4 using the random-phase approximation for $S(\vec{q},\omega)$. When ω is greater than the plasmon frequency, higher-order processes beyond the random-phase approximation are of interest. For example, in a semiconductor, one can have plasmon-assisted free-carrier absorption, i.e., a photon is absorbed, creating an electron-hole pair plus a plasmon excitation. The diagrammatic representation for $\overline{I}(\omega)$ of these processes are represented by graphs (a)–(d) in Fig. 2. There the double wavy line represents the dynamically screened Coulomb interaction. Within the plasmon-pole approximation,[9] it is just the plasmon propagator. The vertical dashed line indicates where the imaginary part of the graph should be taken. All these processes can occur only in the presence of impurities (or some other mechanism, like phonons, which destroys the translational invariance). Because the impurity is required to absorb momentum, the final-state momentum of each of these processes is not conserved. This conclusion is in agreement with an earlier prediction,[2] using a somewhat more complicated approach. The evaluation of these diagrams is straightforward[2] and will not be given here. In the following, we will employ the diagrammatic method to evaluate the low-frequency ($\omega \ll E_F$) conductivities for both noninteracting and interacting electrons. The comparison of our results with well-known expressions for these quantities acts as a test of this new approach.

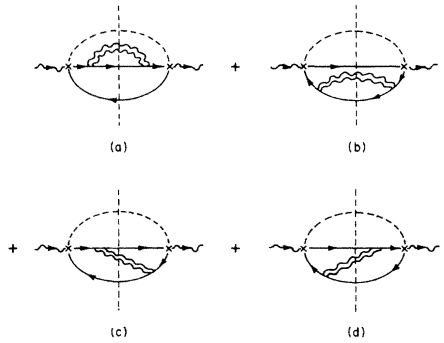

FIG. 2. To lowest order in impurity concentration, the cross section for plasmon-assisted free-carrier absorption is proportional to the imaginary part of $\bar{I}(\omega)$ including the contribution from these diagrams. The double wavy line stands for the plasmon propagator and the vertical dashed line indicates where the imaginary part should be taken.

IV. CONDUCTIVITY FOR NONINTERACTING ELECTRONS

In the low-frequency limit, it is well-known[8] that an infinite set of diagrams to all orders of electron-impurity scattering has to be summed to produce results similar to that derived from transport equations. The evaluation of the force-force correlation function is considerably simpler than the corresponding calculation of the current-current correlation function, even though it still requires summation of a large set of diagrams. In this section we shall calculate the conductivity in the absence of electron-electron interaction, deferring the full calculation for an interacting electron gas to Sec. V. This will bring out more clearly the major role of the electron-electron interaction in the renormalization of various observable parameters.

In the diagrammatic expansion for $\bar{I}(\omega)$ for the noninteracting electrons we will include contributions represented by the diagrams (a) and (b) in Fig. 3. The shaded circle in Fig. 3(b) is the photon-electron-impurity vertex and is diagrammatically represented by Fig. 3(c). The shaded square here is the vertex entirely due to electron-impurity scattering and is represented by Fig. 3(d). The selection of the diagrams here are dictated by the criteria that the dominant vertex corrections are for electron-hole pairs of small momentum transfer. Diagrams illustrated in Fig. 3(e) are not included in the present calculation, just as in the standard calculation of current-current correlation function.[8] This is based on the argument that these diagrams involve integrations over the region of momentum far removed from the Fermi surface and are expected to give little contributions to $\bar{I}(\omega)$.

We now proceed to evaluate the contributions

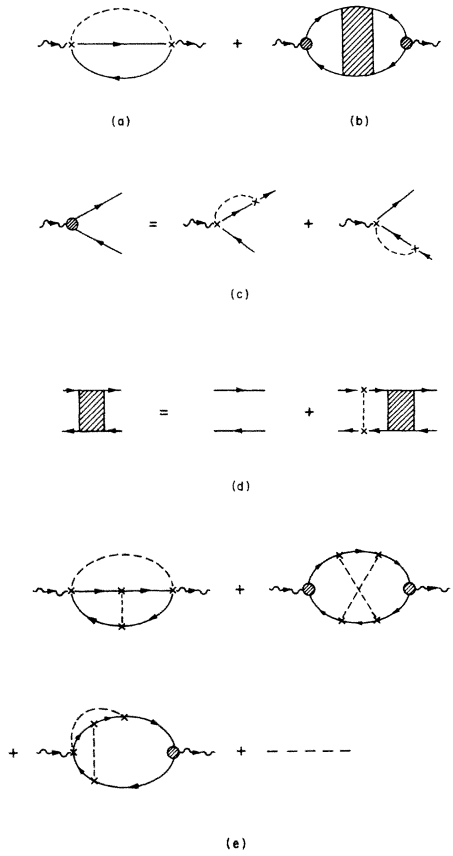

FIG. 3. Diagrams for $I(\omega)$ in the absence of electron-electron interactions. The photon-electron impurity vertex $\Lambda(\omega, \omega', \vec{p})$ is represented by (c). The vertex $W(\vec{k}, \vec{q}, \omega', \omega)$ due to the electron-impurity scattering is sketched in (d). The class of graphs illustrated in (e) are not included in the present calculation, because they give very little contribution to $\bar{I}(\omega)$.

from Figs. 3(a) and 3(b). The explicit expression for the contribution to $\bar{I}(\omega)$ from Fig. 3(a) is

$$I_{3(a)}(\omega) = 2n \sum_{\vec{p},\vec{p}} |p_x - p_x'|^2 |u(\vec{p} - \vec{p}')|^2 \times \frac{1}{2\pi i} \int d\omega' G(\vec{p}', \omega' + \omega) G(\vec{p}, \omega') , \tag{11}$$

where the factor 2 on the right-hand side of the above equation comes from adding the contribution of both spins. The Green's function is defined as[8]

$$G(\vec{p}, \omega) = \frac{1}{\omega - \xi_p + i\omega/2\tau|\omega|} , \tag{12}$$

and

$$\tau^{-1} = [nmp_0/(2\pi)^2] \int |u(\theta)|^2 d\Omega .$$

We also assume here that the scattering lifetime satisfies the condition $\tau^{-1} \ll p_0^2/2m$. The energy ξ_p is defined by $\xi_p = p^2/2m - p_0^2/2m$ and p_0 is the Fermi momentum. Carrying out the momentum integration[8] first and then integrating over ω', we can rewrite Eq. (11) in the form

$$\tilde{I}_{3(a)}(\omega) = -i(mp_0^3/3\pi^2)(\omega/\tau_{tr}), \tag{13}$$

where $\tilde{I}_{3(a)}(\omega)$ is connected with $\bar{I}_{3(a)}(\omega)$ through Eq. (7). Before evaluating Fig. 3(b), we first evaluate the photon-electron-impurity vertex $\Lambda(\omega, \omega, \vec{p})$ as shown in Fig. 3(c).

$$\Lambda(\omega, \omega', \vec{p}) = n \sum_{\vec{q}} q_x |u(\vec{q})|^2 G(\vec{p}+\vec{q}, \omega'+\omega) + n \sum_{\vec{q}} q_x |u(\vec{q})|^2 G(\vec{p}-\vec{q}, \omega').$$

It can be shown that $\Lambda(\omega, \omega', \vec{p})$ has the following form:

$$\Lambda(\omega, \omega', \vec{p}) = (ip_x/2\tau_{tr})\left(\frac{\omega'+\omega}{|\omega'+\omega|} - \frac{\omega'}{|\omega'|}\right),$$

where τ_{tr} is the transport lifetime and is defined as

$$\tau_{tr}^{-1} = \frac{nmp_0}{(2\pi)^2} \int |u(\theta)|^2 (1-\cos\theta)\, d\Omega.$$

The expression for the contribution to $\bar{I}$ coming from Fig. 3(b) is of the form

$$\bar{I}_{3(b)}(\omega) = -\frac{2}{2\pi i} \int d\omega' \sum_{\vec{p}} V(\omega, \omega', \vec{p}) G(\vec{p}, \omega'+\omega) \times G(\vec{p}, \omega') \Lambda^*(\omega, \omega', \vec{p}), \tag{14}$$

where $V(\omega, \omega', \vec{p})$ is represented by Fig. 4. Its evaluation is identical to that described in Ref. 8. The result for $V(\omega, \omega', \vec{p})$ can be written down immediately,

$$V(\omega, \omega', \vec{p}) = \Lambda(\omega, \omega', \vec{p})\left(1 + \frac{i}{\tau_1} \frac{\Theta(\omega'+\omega)\Theta(-\omega')}{\omega + i/\tau_{tr}}\right), \tag{15}$$

where $\tau_1^{-1} = \tau^{-1} - \tau_{tr}^{-1}$. Integrating over ξ_p first and then ω' in Eq. (14), we obtain

$$\tilde{I}_{3(b)}(\omega) = -\frac{mp_0^3}{3\pi^2} \frac{\omega}{\omega + i/\tau_{tr}} \left(\frac{1}{\tau_{tr}}\right)^2. \tag{16}$$

Substitutions of Eqs. (13) and (16) into Eq. (9) yields as the expression for $\sigma(\omega)$

$$\sigma(\omega) = e^2 N/m(1/\tau_{tr} - i\omega). \tag{17}$$

This expression for $\sigma(\omega)$ is completely in agreement with the classical Drude formula. It is valid for $\omega \ll E_F$ and $\tau_{tr}^{-1} \ll E_F$. The dc conductivity (or resistivity) has also been studied by a number of authors.[4,10,11]

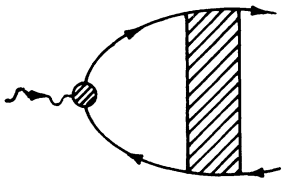

FIG. 4. The diagram which represents the vertex $V(\omega, \omega', \vec{p})$.

V. CONDUCTIVITY FOR INTERACTING ELECTRONS

In this section we consider the conductivity of a fully interacting electron gas. The effect of electron-electron interaction enters into the conductivity calculation in three different ways: (i) The electron-impurity potential $u(\vec{q})$ can be screened by electrons and their mutual interactions. This problem has been studied previously[1] and will not be discussed here again. From now on $u(\vec{q})$ will be regarded as the screened electron-impurity potential. (ii) The Green's function is renormalized by the self-energy correction due to electron-electron interactions. (iii) The vertex function can also be renormalized by electron-electron interactions. Of course, it is impossible to take the electron-electron interaction into account exactly. What will be done in the following is to treat this interaction in a similar spirit as Langer[1] did for the calculation of the dc conductivity. In the calculation of $\bar{I}(\omega)$ in this section, we will include contributions from Figs. 3(a) and 3(b) just as Sec. IV, except that all the Green's functions are renormalized and the electron-impurity potentials are now screened. In addition we include the contribution from Fig. 5(a), which is the dominant vertex correction to electron-hole pair of small momentum transfer due to both electron-electron interaction and electron-impurity interaction. The open square labeled by Γ is the vertex function due to electron-electron-impurity scattering and Γ_0 is the electron-electron interaction vertex without electron-impurity scattering.[13]

We now discuss details of the evaluation of the contribution to $\bar{I}(\omega)$ from Figs. 3(a), 3(b), and 5(a). The expression associated with Figs. 3(a) and 3(b) are still given by Eqs. (11) and (14), respectively. The Green's function G in these expressions is now given by the expression

$$G(\vec{p}, \omega) = \frac{1}{\omega - \xi_p - \Sigma(\vec{p}, \omega) + i\omega/2\bar{\tau}|\omega|}. \tag{18}$$

$\Sigma(\vec{p}, \omega)$ is the self-energy due to electron-electron

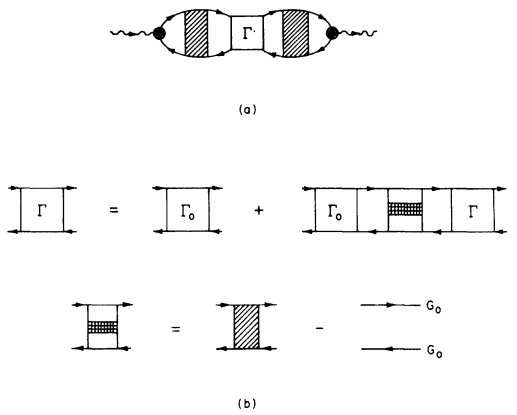

FIG. 5. Diagram (a) stands for the vertex correction to $\bar{\Gamma}(\omega)$ due to electron-electron interactions. Diagram (b) represents the integral equation which has to be satisfied by Γ. Γ_0 and G_0 are the vertex and the Green's function without renormalization by the electron-impurity scattering potential.

interaction and its imaginary part for $\omega = 0$ is zero.[12] $\bar{\tau}$ is the electron-scattering lifetime which will be discussed later. Define the quasiparticle energy ϵ_p as

$$\epsilon_p = \xi_p + \Sigma(\vec{p}, \epsilon_p) \tag{19}$$

and then expand $\Sigma(\vec{p}, \omega)$ near $\omega = \epsilon_p$; we have

$$\Sigma(\vec{p}, \omega) = \Sigma(\vec{p}, \epsilon_p) + (\omega - \epsilon_p)\frac{\partial\Sigma(\vec{p}, \omega)}{\partial\omega}\bigg|_{\omega=\epsilon_p} . \tag{20}$$

Substituting Eqs. (18) and (19) into Eq. (17), we have

$$G(\vec{p}, \omega) = \frac{N_F}{\omega - \epsilon_p + i\omega/2\tau^*|\omega|} , \tag{21}$$

where

$$N_F = \left(1 + \frac{\partial\Sigma(\vec{p}_0, \omega)}{\partial\omega}\bigg|_{\omega=0}\right)^{-1} , \quad \frac{1}{\tau^*} = \frac{N_F}{\bar{\tau}} . \tag{21a}$$

In these equations N_F is a renormalization factor and we have put $\omega = \epsilon_{p_0} = 0$. τ^* is the quasiparticle-scattering lifetime. The scattering lifetime $\bar{\tau}$ is given by[8]

$$-\frac{i\omega}{2\bar{\tau}|\omega|} = \frac{n}{(2\pi)^3}\int d^3p' |u(\vec{p} - \vec{p}')|^2 G(\vec{p}', \omega) .$$

After integrating over momentum $\vec{p}'$ and by using the last identity in Eq. (21a), we obtain the quasiparticle scattering lifetime τ^*

$$\frac{1}{\tau^*} = \frac{n}{(2\pi)^2}\frac{N_F^2 p_0^2}{u_g}\int |u(\theta)|^2 d\Omega , \tag{22}$$

where p_0 is a solution of the equation

$$p_0^2/2m - E_F - \Sigma(\vec{p}_0, 0) = 0 .$$

To a good degree of accuracy p_0 can still be regarded as the Fermi momentum of a noninteracting electron gas. u_g is the group velocity and is defined as

$$u_g = \frac{d\epsilon_p}{dp}\bigg|_{p=p_0} = N_F\left(\frac{p_0}{m} - \frac{\partial\Sigma(p_0, 0)}{\partial p_0}\right) = \frac{p_0}{m^*} ,$$

where m^* is the quasiparticle effective mass. Using the result in (21) and (22) for G, it is straightforward to evaluate the contribution to $I(\omega)$ from Fig. 3(a). It is given by

$$\tilde{I}_{3(a)}(\omega) = -i\frac{p_0^4}{3\pi^2 u_g}\frac{\omega}{\tau_{tr}^*} , \tag{23}$$

where

$$\frac{1}{\tau_{tr}^*} = \frac{n}{(2\pi)^2}\frac{N_F^2 p_0^2}{u_g}\int |u(\theta)|^2(1 - \cos\theta)\, d\Omega .$$

τ_{tr}^* defined in the above expression is the quasiparticle transport lifetime. In evaluating the contribution from Fig. 3(b), the functions $V(\omega, \omega', \vec{p})$ and $\Lambda(\omega, \omega', \vec{p})$ appearing in (14) are now given by the following expressions:

$$\begin{aligned} V(\omega, \omega', \vec{p}) &= \Lambda(\omega, \omega', \vec{p})\left(1 + \frac{i}{\tau_1^*}\frac{\Theta(\omega' + \omega)\Theta(-\omega')}{\omega + i/\tau_{tr}^*}\right) , \\ \Lambda(\omega, \omega', \vec{p}) &= \frac{ip_x}{2\tau_{tr}^* N_F}\left(\frac{\omega + \omega'}{|\omega' + \omega|} - \frac{\omega'}{|\omega'|}\right) , \end{aligned} \tag{24}$$

where τ_1^* is defined in an analogous way as τ_1 which appears in Eq. (15). Integrating over $\vec{p}$ and then ω in Eq. (14), we obtain

$$\tilde{I}_{3(b)}(\omega) = -\frac{1}{3\pi^2}\frac{p_0^4}{u_g}\frac{\omega}{\omega + i/\tau_{tr}^*}\left(\frac{1}{\tau_{tr}^*}\right)^2 . \tag{25}$$

The contribution to $I(\omega)$ from Fig. 5(a) is given by the expression

$$\bar{I}_{5(a)}(\omega) = \frac{2}{(2\pi i)^2}\int\int d\omega'\, d\omega'' \sum_{\vec{p},\vec{p}'} V(\omega', \omega, \vec{p})V(\omega'', \omega, \vec{p}')$$

$$\times G(\vec{p}, \omega' + \omega)G(\vec{p}, \omega')G(\vec{p}', \omega'' + \omega)G(\vec{p}', \omega'')\,\Gamma(\vec{p}, \vec{p}', \omega', \omega'', \omega) . \tag{26}$$

The contribution from ω' or ω'' integration comes from the region $0>\omega'>-\omega$ or $0>\omega''>-\omega$. Since we are interested in the limit $\omega \ll E_F$, we can put[14] $\omega'=\omega''=0$ in the vertex $\Gamma(\vec{p}', \vec{p}, \omega', \omega'', \omega)$ and represent it then by $\Gamma(\vec{p}', \vec{p}, \omega)$. Equation (26) can be simplified as

$$\tilde{I}_{5(a)}(\omega) = -\frac{p_0^4}{3\pi^2 u_g}\left(\frac{\omega}{\omega + i/\tau_{tr}^*}\,\frac{1}{\tau_{tr}^*}\right)^2 \alpha(\omega)\,, \tag{27}$$

where

$$\alpha(\omega) = \frac{p_0^2 N_F^2}{(2\pi)^3 u_g}\int \Gamma(\vec{p}, \vec{p}', \omega)\cos\theta\, d\Omega\,.$$

The angle θ is defined from $\vec{p}\cdot\vec{p}' = p_0^2\cos\theta$, namely, $\vec{p}$ and $\vec{p}'$ are on the Fermi surface. $\Gamma(\vec{p}, \vec{p}', \omega)$ satisfies the integral equation as shown in Fig. 5(b); this integral equation has the form

$$\Gamma(\vec{p}, \vec{p}', \omega) = \Gamma_0(\vec{p}, \vec{p}', \omega) + \frac{1}{2\pi i}\int d\omega' \sum_{\vec{k},\vec{q}} \Gamma_0(\vec{p}, \vec{k}, \omega) \times [W(\vec{k}, \vec{q}, \omega', \omega) - G_0(\vec{k}, \omega'+\omega) G_0(k, \omega')\delta(\vec{k}-\vec{q})]\,\Gamma(\vec{q}, \vec{p}', \omega)\,, \tag{28}$$

where $G_0(\vec{k}, \omega)$ is the Green's function defined in Eq. (21) by letting $\tau^* \to \infty$, or the Green's function without being renormalized by the electron-impurity scattering. The graphical representation of $W(\vec{k}, \vec{q}, \omega', \omega)$ has been sketched in Fig. 3(d). Equation (28) can be solved if we expand $\Gamma(\vec{p}, \vec{p}', \omega)$ and $\Gamma_0(\vec{p}, \vec{p}', \omega)$ in Legendre polynomials, say, $\Gamma(\vec{p}, \vec{p}', \omega) = \sum_l A_l(\omega) P_l(\cos\theta)$ and $\Gamma_0(\vec{p}, \vec{p}', \omega) = \sum_l B_l(\omega) P_l(\cos\theta)$. For $l=1$, we can show that

$$A_1(\omega) = B_1(\omega) - \frac{i}{\omega + i/\tau_{tr}^*}\left(\frac{1}{\tau_{tr}^*}\right)\frac{p_0^2 N_F^2}{6\pi^2 u_g} B_1(\omega) A_1(\omega)\,,$$

and

$$A_1(\omega) = \frac{3}{4\pi}\int \Gamma(\vec{p}, \vec{p}', \omega)\cos\theta\, d\Omega\,,$$
$$B_1(\omega) = \frac{3}{4\pi}\int \Gamma_0(\vec{p}, \vec{p}', \omega)\cos\theta\, d\Omega\,. \tag{29}$$

Solving Eq. (29) and from Eq. (27), we have

$$\alpha(\omega) = \frac{\alpha_0(\omega)}{1 + [i/(i+\omega\tau_{tr}^*)]\,\alpha_0(\omega)}\,,$$
$$\alpha_0(\omega) = \frac{p_0^2 N_F^2}{(2\pi)^3 u_g}\int \Gamma_0(\vec{p}, \vec{p}', \omega)\cos\theta\, d\Omega\,. \tag{30}$$

$\tilde{I}(\omega)$ can be obtained by summing up Eqs. (23), (25), and (27); substitution into Eq. (9) then gives

$$\sigma(\omega) = \frac{ie^2 N}{\omega m} + \frac{e^2 N m^*}{m^2 \omega}\left(\frac{1}{(\omega\tau_{tr}^* + i)} - \frac{i\alpha(\omega)}{(\omega\tau_{tr}^* + i)^2}\right). \tag{31}$$

The problem now left is to relate $\alpha(\omega)$ or $\Gamma_0(\vec{p}, \vec{p}', \omega)$ to some physical quantities. $\Gamma_0(\vec{p}, \vec{p}', \omega)$ is the vertex function without being renormalized by the electron-impurity scattering,[12] and its properties in the limit $\omega \to 0$ have been studied quite extensively in Chap. 4 of Ref. 8. We can relate $\Gamma_0(\vec{p}, \vec{p}', 0)$ to

$$\Gamma^{\omega}(\vec{p}, \vec{p}') = \lim_{\omega\to 0}\lim_{\kappa\to 0}\Gamma_0(\vec{p}, \vec{p}', \vec{k}, \omega) = \lim_{\omega\to 0}\Gamma_0(\vec{p}, \vec{p}', \omega)\,,$$

comparing with Eq. (19.6) of Ref. 8, we have

$$\alpha_0(0) = m^*/m - 1\,. \tag{32}$$

Therefore, we can define a frequency-dependent quasiparticle mass $m^*(\omega)$ by

$$\alpha_0(\omega) = m^*(\omega)/m - 1\,.$$

Since the external frequency is restricted to the region where $\omega \ll E_F$, the Fermi energy, we can always approximate $m^*(\omega)$ by its value at the Fermi level, namely, $m^*(\omega) \simeq m^*$, or $\alpha_0(\omega) \simeq \alpha_0(0)$. In terms of the quasiparticle effective m^*, $\alpha(\omega)$ can be written as

$$\alpha(\omega) = \frac{(m^*/m - 1)(i + \omega\tau_{tr}^*)}{im^*/m + \omega\tau_{tr}^*}\,. \tag{33}$$

Substituting into Eq. (31) we have

$$\sigma(\omega) = \frac{ie^2 N}{m}\,\frac{\tau_{tr}^*}{\omega\tau_{tr}^* + i(m^*/m)}\,. \tag{34}$$

Let us point out once more that the above formula is valid only for $\omega \ll E_F$ and $(\tau_{tr}^*)^{-1} \ll E_F$. For large ω, $\tilde{I}(\omega)$ defined in Eq. (10) should be used to compute $\sigma(\omega)$. In the limit of $\omega \to 0$, we have the dc conductivity

$$\sigma(0) = (e^2 N/m^*)\tau_{tr}^*\,. \tag{35}$$

Equation (35) is in complete agreement with that obtained in Ref. 1 using a different approach. If one defines a modified relaxation time $\tau_{c.m.}$ as

$$\frac{1}{\tau_{c.m.}} = \frac{m^*}{m}\,\frac{1}{\tau_{tr}^*}\,, \tag{36}$$

it is interesting to note that Eq. (34) reduces to the classical Drude formula

$$\sigma(\omega) = \frac{e^2 N}{m(1/\tau_{\text{c.m.}} - i\omega)} . \tag{37}$$

In the next section, we shall discuss the significance of this result.

VI. DISCUSSION

We have presented an investigation of the finite frequency conductivity of an interacting electron gas in the presence of random impurities. The novel feature is the separation of the center-of-mass and internal degrees of freedom. This leads to a new set of perturbation diagrams but they can still be evaluated by standard many-body techniques. For the impurity-scattering effects we have retained the ladder diagrams which yields the dominant contribution. For the electron-electron interaction, vertex corrections are included only for electron-hole scattering involving small momentum transfer. The Green's function for electrons are renormalized by both electron-electron and electron-impurity scattering. Our result in the dc limit is completely equivalent to that of Ref. 1, and we shall not discuss its physical significance in any further detail. The result for finite frequency is new. It is valid for arbitrary values of $\omega\tau$ provided ω and $1/\tau$ are both $\ll E_F$.

It is interesting to note that the final expression for $\sigma(\omega)$ in Eq. (37) is still given by the simple Drude formula with a suitably defined relaxation time. This implies that in considering the response to a uniform field, we need only to study the classical equation of motion of the center-of-mass coordinates, which behaves as a particle with charge Ne and mass Nm. Because of the coupling of the internal degrees of freedom through the impurities, the center-of-mass relaxation time $\tau_{\text{c.m.}}$ used in the equation of motion now becomes a function of ω and is influenced by electronic interactions.

In the three-dimensional system, it is difficult to test our results for $\sigma(\omega)$ directly because of band-structure effects. However, for the two-dimensional electron gas in metal-semiconductor-oxide junction, measurements on ac conductivity have recently been made by Allen *et al.*[15] They find that the experiment data can be fitted very well with a Drude-like formula, in agreement with our theoretical results. We want to emphasize however, that the relaxation time appearing in the Drude formula should be the center-of-mass relaxation time $\tau_{\text{c.m.}}$. Thus if the value for τ used for curve fitting is derived from measurements of dc conductivity, the mass used in extracting values of τ should be the *bare* band-structure mass and not m^*, the mass renormalized by electron-electron interaction. Otherwise one would be getting the quasiparticle relaxation time τ_{tr}^*.

The idea of a center-of-mass relaxation time can be exploited to make use of the classical equation of motion for the center-of-mass degree of freedom, thus circumventing the lengthy microscopic calculations. This is similar in spirit to the memory function approach in Ref. 4, and leads to identical results for similar choice of Hamiltonians. This is particularly valuable in the presence of a dc magnetic field where microscopic first-principle calculations such as that presented in this paper became unfeasible. The study of magnetoconductivity and effect of the electronic interaction on cyclotron-resonance experiments in metal-oxide-semiconductor structure will be described in a separate publication.

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[1]J. S. Langer, Phys. Rev. 120, 714 (1960); 124, 1003 (1961).

[2]C. S. Ting and J. J. Quinn (unpublished).

[3]W. Kohn, Phys. Rev. 123, 1242 (1961).

[4]W. Götze and P. Wölfle, Phys. Rev. B 6, 1226 (1972). (See references in this paper).

[5]H. Mori, Prog. Theor. Phys. (Kyoto) 33, 423 (1965); 34, 399 (1965).

[6]D. Pines and P. Nozières, *Theory of Quantum Liquid*, (Benjamin, New York, 1973), Vol. 1.

[7]R. Kubo, J. Phys. Soc. Jpn. 12, 570 (1957).

[8]A. A. Abrikosov, L. P. Gorkov, and I. YE. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Mechanics* (Pergamon, New York, 1965), pp. 322–330.

[9]B. I. Lundquivst, Phys. Kondens. Mater. 6, 193 (1967).

[10]S. F. Edwards, Philos. Mag. 3, 1020 (1958).

[11]A. A. Abrikosov and L. P. Gorkov, Zh. Eksp. Teor. Fiz. 35, 1558 (1958); 36, 319 (1959) [Sov. Phys.-JETP 8, 1090 (1959); 9, 220 (1959)].

[12]Since we consider here that the number of impurities is small, the electron-impurity scattering effect on $\Sigma(\vec{p}, \omega)$ will be neglected. This approximation is consistent with that used in Ref. 1. For Im $\Sigma(p, 0) = 0$, see p. 173 of Ref. 8.

[13]Here Γ_0 is equal to Γ which is given by Eq. (18.3) of Ref. 8. The irreducible vertex $\Gamma^{(1)}$ in our present case should be renormalized by the electron-impurity scattering. Those Green's functions constructing $\Gamma^{(1)}$, however, involve large momentum transfers; thus the impurity effect on $\Gamma^{(1)}$ can be neglected. This approximation is in the same spirit as that described in Ref. 1.

[14]This approximation introduces an error in the irreducible vertex $\Gamma^{(1)}$ [see Eq. (18.3) of Ref. 8] by a factor of ω/E_F. In the limit $\omega \ll E_F$, it can be neglected.

[15]S. J. Allen, Jr., D. C. Tsui, and F. DeRosa, Phys. Rev. Lett. 35, 1359 (1975).