Dynamics of symmetry-breaking long-ranged order in two-dimensional nonscalar systems*

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A simple model for the dynamics of a continuous spin system in contact with a temperature bath is developed from a generalization of the Glauber model. The dependence of magnetization on time is then found in an approximation which becomes exact at temperatures far from T_c. For nonscalar ($N > 1$) systems in contact with a temperature bath at $T < T_c$, symmetry-breaking magnetization will develop and persist for a time τ, after which it is destroyed by transverse fluctuations. τ is $\propto \Omega T_c/T$ for $d > 2$, while $\ln(\tau) \propto T_c/T$ for $d = 2$ (d is the spatial dimensionality and Ω is the volume). As $\Omega \rightarrow \infty$, τ is finite for $d = 2$, as required by rigourous theorems. In practice, however, a τ for $d = 2$ which is comparable to the τ for $d = 3$ systems which are large but finite is found at experimentally obtainable temperatures $T^{-1} \propto T_c^{-1}\ln(10^{23})$. We estimate that proportionality factors are such that "virtually macroscopic" persistence times are obtained close to T_c. In addition, there is evidence that the nature of the $t \gg \tau$ decay of magnetism for $d = 2$ systems provides a qualitative means of distinguishing subcritical from above critical systems. No virtually persistent magnetization is found for $d = 1$.

I. INTRODUCTION

The nature of phase transitions in two-dimensional systems having nonscalar order parameters has been of interest for some time. It has been rigorously proven by Bogoliubov[1] that no persistent superfluid condensate can exist in one- or two-dimensional Bose fluids at finite temperature. Bogoliubov's proof has been extended to show the impossibility of permanent long-range order in finite-temperature one- or two-dimensional superconductors,[2] crystals,[3] and ferromagnets and antiferromagnets, when no external field is applied.[4]

The core of the Bogoliubov proof is a rigorous inequality. For magnetic systems with no easy axis, the Bogoliubov inequality states[4] that, so long as the spatial dimensionality is ≤ 2, the system will, if it is to have a nonzero magnetization in the absence of any external magnetic field, also have spin fluctuations per lattice site which are infinite in the direction transverse to the magnetization. Since the magnitude of the spin per lattice site is finite, an infinite $\langle S^2 \rangle$ is impossible. Thus, so long as the spin dimensionality (N) is greater than one (and, consequently, transverse fluctuations are a factor) permanent symmetry-breaking magnetization is prohibited in $d \leq 2$ spin systems.

It should be noted that the Bogoliubov proof applies only to nonscalar systems, the proof cannot be applied to systems with scalar order parameters such as the Ising model and fluids displaying a liquid-gas transition. In two dimensions, at least, such systems display long-range order. For one-dimensional systems, however, an argument due to Landau[5] prohibits permanent magnetization at any nonzero temperature.

Although persistent long-range order is ruled out for nonscalar $d \leq 2$ systems, there remains strong evidence for the existence of a finite critical temperature in such systems at least for the case $d = 2$. Stanley and Kaplan[6] have done numerical calculations of the statistical mechanics of the classical two-dimensional XY model and have determined a finite critical temperature, on and below which the magnetic susceptibility is divergent. Superfluidity is observed in liquid-helium films.[7] There is some controversy as to whether the singularities on the critical point for $d = 2$ nonscalar systems are power-law singularities[8] (as usual renormalization-group considerations lead one to expect) or essential singularities.[9] However, the existence of a finite-temperature critical point for nonscalar films is well established, at least when the order-parameter dimensionality (which we shall denote by N) is 2. It should be pointed out that work by Polyakov and by Migdal[10] indicates that there may be no nonzero critical temperature when $d = 2$ and N exceeds 2.

Given a nonzero critical temperature for at least some $d = 2$, $N > 1$ systems, the remaining unanswered question is what happens below T_c. In this paper we shall deal with part of this problem by considering the dynamics of a system in contact with a temperature bath. The specific system which will be studied is the "continuous spin" model developed by Wilson[11] and applied with considerable success, to critical-phenomena problems. It will be assumed that the spin system is coupled to a temperature bath through a local interaction. By "local" we mean that spins are changed one at a time; the temperature bath will not simultaneously change spins at different locations. After also assuming detailed balance we obtain a generalization to continuous spin systems of the Glauber[12] model for Ising-model dynamics. For this paper we will use the simplest possible dynamical model which satisfies the generalized Glauber equations. The particular model used has a convenient "universal" property; the full class of generalized Glauber models reduces to this model near the critical point. In a subsequent paper more complicated dynamical models will be considered. The fundamental results of this paper remain intact.

It should not be entirely unanticipated that a dynamical study will yield interesting results. A standard problem in experimental work on systems near the critical point is the long time required for thermal equilibrium to set in. For example, fluids in contact with temperature baths close to, but not on the liquid-gas critical point will typically require times on the order of hours before equilibrium is established. Liquid-gas systems have a scalar ($N=1$) order parameter and, at temperatures near but not on the critical point, will have a large but submacroscopic correlation length. By Bogoliubov's inequality, the transverse spin-spin correlation function, for $N>1$ subcritical systems with symmetry-breaking order, will have an infinite position space range once thermal equilibrium sets in. For symmetry-breaking order to be destroyed by transverse fluctuations in $d=2$, $N>1$ systems, it is important that equilibrium be established to the point where the transverse correlations have a very long range. The range required rises exponentially with $1/T$. It should not be surprising that the same physics which, in critical slowing down, says that long-ranged correlations take a long time to develop will also imply that a singificant amount of time will elapse before transverse fluctuations in subcritical $d=2$, $N>1$ systems will be large enough to destroy magnetization. This makes possible a "virtual ferromagnetism," for $d=2$, $N>1$ systems—magnetization which endures for times which are, for laboratory purposes, infinite.

We shall deal, in this paper, with a kind of hysteresis experiment. A spin system at a high temperature is subjected to an external magnetic field. This magnetic field may be small, it is only needed to single out a direction in which magnetization may subsequently develop. The external magnetic field is then removed and the spin system is placed in contact with a temperature bath at near or subcritical temperatures. We examine the subsequent time dependence of the magnetization in an approximation which amounts to a mean-field approximation with first-order corrections. This approximation—like the mathematically similar Bogoliubov[13] approximation for a weakly interacting low-temperature Bose gas—has the useful feature of satisfying Bogoliubov's inequality once thermal equilibrium obtains. We may self-consistently check the appropriateness of the approximation by observing when the first-order corrections become important. What is found is that for $d=2$, $N>1$ systems, at low enough temperatures the mean-field approximation is good for a substantial time, during which significant magnetization develops and perisists. In this period the $d=2$ system is qualitatively similar to a bulk ferromagnet. Ultimately, however, as thermal equilibrium is approached, transverse fluctuations will develop to the point where they begin to destroy the magnetization. At this point, the mean-field approximation begins to break down. For subsequent times, we may use Bogoliubov's inequality to determine a lower bound for the time-dependent transverse fluctuations. It is found that, after magnetization and the mean-field approximation break down, transverse fluctuations will remain large enough to prohibit magnetization from redeveloping. The conclusion is that, at low temperatures, the mean-field theory with first-order corrections gives a good figure for the persistence time of symmetry-breaking magnetization. For $d=2$, $N>1$ systems of infinite volume, we find that the persistence time τ is given by

$$\ln\tau \propto T_c/T$$

for $T \ll T_c$, where T_c is the critical temperature. By comparison, we find the persistence time in a $d=3$, $N>1$ system of large but finite volume to be

$$\tau \infty \Omega$$

for subcritical temperatures. Thus we see that virtually macroscopic persistence times are found at readily obtainable temperatures T such that

$$\ln(\Omega)=\ln(10^{23}) \propto T_c/T\,.$$

The proportionality constants will be estimated in Sec. III. It should be noted, at this juncture, that no virtual ferromagnetism of the type just described for $d=2$, $N>1$ is found at nonzero temperatures for one-dimensional nonscalar systems. It should also be pointed out that, although our primary interest is in those two-dimensional systems which have a nonzero critical point, the results quoted in this paragraph are valid for all N—even when there is no $d=2$ critical point. We are considering the properties of the approach to thermal equilibrium and not of thermal equilibrium itself. If, for some N, no critical point exists, one may substitute the mean-field critical point for T_c in the results of this paragraph.

For those systems which do have a critical point, a relevant question is how the time behavior of magnetization will reveal it. Although the approximations of this paper are developed for temperature baths which are not too close to T_c, they do give qualitatively reasonable results in the vicinity of the critical point for $d=2$, $N=1$ systems and for $d=3$ systems. It is, therefore, of interest to summarize what behavior they predict in the critical region for $d=2$, $N>1$ systems. What is found is that, as the critical point is approached from below, the persistence time τ goes to zero. When T is close to T_c (and, consequently, τ is ignorable), the manner in which the magnetization ultimately decays with time provides the qualitative way to distinguish the subcritical region from the above critical one. We find that, so long as the temperature is below the critical point, symmetry-breaking magnetization will ultimately (after time τ) decay at a slower than exponential rate. Above T_c, magnetization decays exponentially with time. The time constant for this exponential decay goes to infinity as T_c is approached from above. In Sec. III we show that the slower than exponential subcritical decay is consistent with Stanley and Kaplan's[6] finding that, when statistical equilibrium is ultimately established, the magnetic susceptibility is infinite below T_c.

The work in this paper does not contradict that of Kosterlitz and Thouless.[14] Kosterlitz and Thouless specialized to such $d=2$, $N=2$ systems as the XY model. They found that, below a certain temperature such systems could be characterized by their resistence to "stirring." That is, the creation of vortexlike configurations is free-energy unfavorable at low temperatures and favorable at high temperatures. Such a situation does not, of course, rule out the system also exhibiting a more traditional ferromagnetic low-temperature behavior.

In Sec. II we will develop a simple dynamical model for continuous spin systems in contact with a temperature bath. In Sec. III this model will be used to find the time dependence of the magnetization. The results summarized in this Introduction are developed in Sec. III.

II. DYNAMICS OF A CONTINUOUS SPIN SYSTEM

In this section, the reasoning of Glauber[12] will be used to develop a dynamical model for continuous spin systems. Let us begin by reviewing the dynamics of a simple system consisting of two levels. Transitions from one level to the other occur through interaction with a temperature bath. So long as the temperature bath is either quantum mechanical or very large, we must describe the two-level system statistically. Let P_1 and P_2 denote the probabilities of occupying the respective levels. We use a to denote the rate at which the temperature bath induces transitions from level 1 to level 2 and b to denote the transition rate from 2 into 1. Then

$$\dot{P}_1 = -aP_1 + bP_2\,, \quad \dot{P}_2 = -bP_2 + aP_1\,. \tag{2.1}$$

It may be seen that (2.1) conserves total probability P_1+P_2.

Dynamics controlled by a temperature bath must be such that when equilibrium obtains, probabilities are given by statistical mechanics. Thus when $\dot{P}_1=\dot{P}_2=0$

$$P_1/P_2 = e^{-(E_1-E_2)/kT}\,, \tag{2.2}$$

where E_1 and E_2 are the energies of level 1 and 2, respectively, and T is the temperature of the temperature bath. We see that (2.1) and (2.2) imply a condition on a and b:

$$a/b = e^{(E_1-E_2)/kT}\,. \tag{2.3}$$

For example, let us suppose that the temperature bath consists of a gas of photons and that level 1 has the lower energy. Then the transition rate from 1 to 2 is given by the matrix element squared multiplied by the number of photons of energy E_2-E_1

$$a = M[1/(e^{(E_2-E_1)/kT}-1)]\,.$$

We see that (2.3) immediately implies stimulated emission:

$$b = M[1/(e^{(E_2-E_1)/kT}-1)+1]$$

as, of course, was originally discovered by Einstein.

The generalization of (2.1) and (2.3) to spin systems is quite straightforward. The energy "levels" of a multiple spin system are the various spin configurations. The energy of a spin configuration, $\underline{S}(\underline{x})$, is $H[\underline{S}]$. H is the Hamiltonian and is a functional of $\underline{S}(\underline{x})$. We will assume that the temperature-bath-induced interaction is local. That is, transitions out of a given configuration $\underline{S}(\underline{x})$ will be into configurations $\underline{S}(\underline{x})+\delta_{\underline{x},\underline{y}}\underline{\Delta}$, which differ only by having a single spin at some location $\underline{y}$ changed by some amount $\underline{\Delta}$. We will label the generalization of the photon occupation number by

$$B(E[\underline{S},\underline{\Delta}]/kT\,,$$

where E is the energy difference between the configurations $\underline{S}$ and $\underline{S}+\underline{\Delta}$:

$$E[\underline{S},\underline{\Delta}] \equiv H[\underline{S}(\underline{x})+\delta_{\underline{x},\underline{y}}\underline{\Delta}] - H[\underline{S}(\underline{x})]\,. \tag{2.4}$$

We shall make the ansatz that the generalization of the matrix element M depends only on the magnitude of the spin change $\underline{\Delta}$. We label this generalized matrix element $\overline{M}(\Delta)$. Thus the transition rate from a configuration $\underline{S}(\underline{x})$ into a configurations $\underline{S}(\underline{x})+\delta_{\underline{x},\underline{y}}\underline{\Delta}$ is

$$B(E[\underline{S},\underline{\Delta}]/kT)M(\Delta)$$

and the generalization of the two-level dynamical equation (2.1) is

$$\frac{dP[\underline{S}(\underline{x})]}{dt}=\sum_{\underline{y}}\sum_{\underline{\Delta}}M(\Delta)\{-P[\underline{S}(\underline{x})]B(E[\underline{S},\underline{\Delta}]/kT)+P[\underline{S}(\underline{x})+\delta_{\underline{x},\underline{y}}\underline{\Delta}]B(-E[\underline{S},\underline{\Delta}]/kT)\}\,, \tag{2.5}$$

where $P[\underline{S}(\underline{x})]$ is the time-dependent probability for a given spin configuration $\underline{S}(\underline{x})$. By summing both sides of (2.5) over all possible configurations, it may be seen that total probability is conserved.

If we assume detailed balance, then the generalization of the temperature bath condition (2.3) is

$$B(E[\underline{S},\underline{\Delta}]/kT)/B(-E[\underline{S},\underline{\Delta}]/kT)=e^{-E[\underline{S},\underline{\Delta}]/kT}. \tag{2.6}$$

For example, Glauber chose to study a dynamical model which satisfied (2.6) with a B which depends on its argument as

$$B(z)=2e^{-z/2}/(e^{z/2}+e^{-z/2}).$$

For the purpose of studying continuous spin systems, the simplest possible realization of (2.5) and (2.6) is obtained when the generalized matrix element $M(\Delta)$ is very short ranged. In this limit, we may expand the terms in (2.5) which are enclosed by curly braces in a power series in Δ. If we label the second moment of M by M_2,

$$M_2\equiv\sum_{\underline{\Delta}}M(\Delta)(\Delta_i)^2$$

(where i is a component of the generalized vector $\underline{\Delta}$) and if we choose the normalization $B(0)=1$ then (2.5) and (2.6) reduce to the simple form

$$\frac{dP[\underline{S}]}{dt}=\tfrac{1}{2}M_2\sum_{\underline{y}}\sum_{i=1}^{N}\frac{\delta^2P}{[\delta S_i(\underline{y})]^2}+\frac{1}{KT}\frac{\delta P}{\delta S_i(\underline{y})}\frac{\delta H}{\delta S_i(\underline{y})}+\frac{1}{kT}\frac{\delta^2 H}{[\delta S_i(\underline{y})]^2}P+O(M_4), \tag{2.7}$$

where S_i denotes the i component of the N-dimensional vector $\underline{S}$ and M_4 is the fourth moment of M. For the remainder of this paper we will choose units of time which set $\frac{1}{2}M_2=1$. This choice of units of time, when coupled with the choices of units of distance and field which are made at the beginning of Sec. III is tantamount to setting equal to unity the characteristic time for short-wavelength ($\approx$ 1 lattice constant) spin-spin correlations to approach thermal equilibrium [see Eq. (3.15)].

(2.7) is the simplest possible dynamical equation which both conserves probability and which has

$$P[\underline{S}]\propto e^{-H[\underline{S}]/kT}$$

as its equilibrium configuration. It should be pointed out that, not only is the assumption of a short ranged $M(\Delta)$ a perfectly reasonable assumption at any temperature for at least some dynamical models, it has the convenient property of being an appropriate approximation near the critical point for all dynamical models which satisfy (2.5) and (2.6). In the critical region $P[\underline{S}]$ and factors of $e^{-H/kT}$ are slowly varying functionals of the local spin; near T_c M will always be relatively short ranged and, therefore, ignoring terms of $O(M_4)$ and higher is a valid assumption. Thus, if one were to take the first moment of (2.5) and obtain an equation for the time derivative of the magnetization, one would find that the leading critical behavior would be described by the $O(M_2)$ terms.

Halperin, Hohenberg, and Ma[15] have studied the dynamics of continuous spin systems using a formalism in which the spins S were time dependent (a "Heisenberg" formalism as opposed to the "Schrödinger" formalism of the present paper). It may be shown that the short ranged $M(\Delta)$ assumption is equivalent to their case A. Thus the mean value of $\dot{S}(x,t)$ is

$$M_2[\delta H/\delta S(x,t)]$$

and the mean value of $\dot{S}(x,t)\dot{S}(x',t')$ is

$$M_2\delta(x-x')\delta(t-t').$$

Halperin, Hohenberg, and Ma also consider several variations which, while important near the critical point, are not of interest here.

It is convenient to rewrite (2.7) in terms of the Fourier transform, $\underline{\sigma}$ of the position space spin field $\underline{S}$:

$$\underline{\sigma}(\underline{k})\equiv\sum_{\underline{x}}\underline{S}(\underline{x})\frac{e^{i\underline{k}\cdot\underline{x}}}{\sqrt{\Omega}}\,,$$

where Ω is the system's volume. In terms of σ, (2.7) becomes

$$\frac{dP[\underline{\sigma}]}{dt}=\sum_{\underline{k}}\sum_{i=1}^{N}\left(\frac{\delta^2 P}{\delta\sigma_i(\underline{k})\delta\sigma_i(-\underline{k})}+\frac{1}{kT}\frac{\delta P}{\delta\sigma_i(\underline{k})}\frac{\delta H}{\delta\sigma_i(-\underline{k})}+\frac{1}{kT}\frac{\delta^2 H}{\delta\sigma_i(\underline{k})\delta\sigma_i(-\underline{k})}P\right) \quad (2.8)$$

if we choose units of time to set $\frac{1}{2}M_2=1$.

III. TIME DEPENDENCE OF MAGNETIZATION

In this section we will use as a Hamiltonian the continuous spin model of Wilson.[11]

$$\frac{H[\sigma]}{kT}=\sum_{\underline{k}}(\mu+k^2)\underline{\sigma}(\underline{k})\cdot\underline{\sigma}(-\underline{k})+\frac{\lambda}{\Omega}\sum_{\underline{k}_1}\sum_{\underline{k}_2}\sum_{\underline{k}_3}[\sigma(\underline{k}_1)\cdot\sigma(\underline{k}_2)]\times[\underline{\sigma}(\underline{k}_3)\cdot\sigma(-\underline{k}_1-\underline{k}_2-\underline{k}_3)] \quad (3.1)$$

For stability purposes, λ must be positive; μ may be either positive or negative. In (3.1) $\underline{\sigma}(\underline{k})$ is the Fourier transform of the position space spin field, $\underline{S}(\underline{x})$. In general $\underline{\sigma}$ is an N-dimensional vector. We will denote the dimensionality of $\underline{k}$ space by d. We have chosen units of spin to set the coefficient of the "kinetic energy" (k^2) term equal to one. In addition there is a maximum magnitude for the momenta $\underline{k}$. This corresponds roughly to the Brillouin-zone edge. We will choose units of distance to set the $\underline{k}$-space cutoff equal to one.

Note that the temperature T has been incorporated into the definition of the parameters μ and λ. At the end of this section we will restore explicit temperature dependence in order to express the most important results of this work in an experimentally testable form.

At this point it is worth recording the form that Bogoliubov's inequality takes for a continuous spin system. The reader is referred to Mermin[16] for more detail. We generalize (3.1) slightly by including an external magnetic field. The component of $\underline{\sigma}$ which is parallel to this field will be labeled by σ_P

$$\frac{H}{kT}=\sum(\mu+k^2)\,\underline{\sigma}\cdot\underline{\sigma}+\frac{\lambda}{\Omega}\sum\sum\sum(\underline{\sigma}\cdot\underline{\sigma})^2+\sqrt{\Omega}\,h\,\sigma_P(0)\;.$$

If equilibrium statistical mechanics obtains, then there will be a spatially uniform mean magnetization $\bar{\sigma}$ in the direction of h. Denoting any component of $\underline{\sigma}(\underline{k})$ which is the transverse to the h field by $\sigma_T(\underline{k})$, Mermin's extension of the Bogoliubov inequality states that

$$\langle\sigma_T(\underline{k})\sigma_T(-\underline{k})\rangle\geq\frac{1}{(h/\bar{\sigma})\sqrt{\Omega}+2k^2(1+D/\bar{\sigma}^2)}\;, \quad (3.2)$$

where D is analogous to the depletion in a superfluid:

$$D\equiv\sum\langle\underline{\sigma}(\underline{k})\cdot\underline{\sigma}(-\underline{k})\rangle-\bar{\sigma}^2. \quad (3.3)$$

Let us now turn to the problem of investigating the time dependence of a system with the Hamiltonian (3.1). In principle we could attempt to discuss the dynamics of such a system by defining a time-dependent Hamiltonian-like functional $\tilde{H}_t[\underline{\sigma}]$:

$$\tilde{H}_t[\underline{\sigma}]\equiv-kT\ln(P[\underline{\sigma}])$$

and rewriting (2.8) as a hierarchy of equations for the coefficients, in $\tilde{H}_t$, of various powers of $\underline{\sigma}$. As statistical equilibrium is approached $\tilde{H}_t$ will approach H. A formulation in terms of $\tilde{H}_t$ would not be very illuminating, however. In particular, it has no strong dependence on the spatial dimensionality d. What depends strongly on d is not $\tilde{H}_t$, but how the parameters of $\tilde{H}_t$ add up to determine expectation values of P. We find it more useful to examine these moments of P directly.

We may obtain an equation for the time dependence of the magnetization by multiplying both sides of (2.8) by $\underline{\sigma}(0)$—the zero Fourier component of the spin—and then functionally integrating over all $\underline{\sigma}(\underline{k})$. After several integrations by parts we obtain

$$\frac{d\langle\underline{\sigma}(0)\rangle}{dt}=-\left\langle\frac{\delta(H/kT)}{\delta\underline{\sigma}(0)}\right\rangle, \quad (3.4)$$

where, in general, $\langle f[\sigma]\rangle$ denotes a moment of the time-dependent probability P:

$$\langle f[\underline{\sigma}]\rangle\equiv\int_{\underline{\sigma}}P[\underline{\sigma}]f[\underline{\sigma}] \quad (3.5)$$

and $\int_\sigma$ is a functional integral over all $\underline{\sigma}(\underline{k})$.

As stated in the Introduction, we will be concerned with a kind of hysteresis experiment. A system is initially lined up by applying an external magnetic field h, which is then turned off. For a system with such a history we must distinguish between the component of spin which is parallel to the original magnetic field (which shall be denoted by σ_P) and the transverse spin components. The dynamical equations tell us that, for the hysteresis experiment, there will be no anisotropies besides that between parallel and transverse directions. In particular, there is no need to distinguish between the $N-1$ different transverse directions. We will, therefore, denote the component of spin in any one transverse direction by a single symbol σ_T.

Whatever magnetization the system can retain will be in the direction parallel to the formerly applied magnetic field. Thus (3.4) may be rewritten as

$$\langle\sigma_T(0)\rangle=0$$

and

$$\frac{d\langle\sigma_P(0)\rangle}{dt} = -2\mu\bar{\sigma} - \frac{4\lambda}{\Omega}\bar{\sigma}^3 - \frac{4\lambda}{\Omega}\bar{\sigma}\sum_{\underline{k}}[3G_P(\underline{k}) + (N-1)G_T(\underline{k})] - \frac{4\lambda}{\Omega}\sum_{\underline{k}_1}\sum_{\underline{k}_2}[F_P(\underline{k}_1,\underline{k}_2) + (N-1)F_T(\underline{k}_1,\underline{k}_2)] \ , \tag{3.6}$$

where, for notational brevity, we define time-dependent quantities:

$$\bar{\sigma} \equiv \langle\sigma_P(0)\rangle \ ; \tag{3.7}$$

second moments of σ

$$G_P(\underline{k}) \equiv \langle\sigma_P(\underline{k})\sigma_P(-\underline{k})\rangle - \bar{\sigma}^2\delta_{\underline{k},0} \ , \quad G_T(\underline{k}) \equiv \langle\sigma_T(\underline{k})\sigma_T(-\underline{k})\rangle \ ; \tag{3.8}$$

and third moments of σ

$$F_T(\underline{k}_1,\underline{k}_2) \equiv \langle\sigma_P(\underline{k}_1)\sigma_T(\underline{k}_2)\sigma_T(-\underline{k}_1-\underline{k}_2)\rangle - \bar{\sigma}G_T(\underline{k}_2)\delta_{\underline{k}_1,0} \ ,$$

$$F_P(\underline{k}_1,\underline{k}_2) \equiv \langle\sigma_P(\underline{k}_1)\sigma_P(\underline{k}_2)\sigma_P(-\underline{k}_1-\underline{k}_2)\rangle - 3\delta_{\underline{k}_2,0}\bar{\sigma}G_P(\underline{k}_1) - \delta_{\underline{k}_1,0}\delta_{\underline{k}_2,0}\bar{\sigma}^3 \ .$$

To deal with all of (3.6), we would need to also write equations for the time derivative of $\langle\sigma^2\rangle$ and $\langle\sigma^3\rangle$. These, in turn, would involve factors of $\langle\sigma^4\rangle$ and $\langle\sigma^5\rangle$ which would necessitate still more equations. A simple approximation to (3.6), however, is to make a mean-field approximation and ignore second and higher moments of P. We then have

$$\frac{d\bar{\sigma}}{dt} = -2\mu\bar{\sigma} - \frac{4\lambda}{\Omega}\bar{\sigma}^3 \ . \tag{3.9}$$

The reader is reminded that, for stability reasons, λ must be positive. In the mean-field approximation, systems of any d or N will have an above critical region given by $\mu > 0$ and a subcritical region given by $\mu < 0$. Thus (in the mean-field approximation), when μ is positive, the only root to $d_t\bar{\sigma} = 0$ is $\bar{\sigma} = 0$. Whatever initial value $\bar{\sigma}$ had, it will ultimately decay to zero as $e^{-2\mu t}$. When μ is negative, however, the root $\bar{\sigma} = 0$ is unstable. In the negative μ region, the stable root to $d_t\bar{\sigma} = 0$ is $\bar{\sigma} = (-\mu\Omega/2\lambda)^{1/2}$. This is ultimately approached exponentially as

$$\bar{\sigma} - (-\mu\Omega/2\lambda)^{1/2} \propto e^{4\mu t}$$

when $\bar{\sigma}$ is close to $(-\mu\Omega/2\lambda)^{1/2}$. For $\mu < 0$, then, if the initial value of $\bar{\sigma}$ is even slightly nonzero, $\bar{\sigma}$ will ultimately approach $(-\mu\Omega/2\lambda)^{1/2}$. The characteristic time to establish this magnetization is $\sim 1/|4\mu|$. We see that, as we pass deeper into the subcritical region by making μ more negative, the magnetization, $(-\mu\Omega/2\lambda)^{1/2}$, grows and the time needed to develop it, $(\sim 1/|4\mu|)$, shrinks.

In order to test the appropriateness of the mean-field description we now examine the two-spin correlation functions $\langle\sigma^2\rangle$. One may establish, by considering the hierarchy of equations for $d_t\langle\sigma\rangle$, $d_t\langle\sigma^2\rangle$, $d_t\langle\sigma^3\rangle, \ldots$, that if $\langle\sigma^2\rangle$ is smaller than $\langle\sigma\rangle$ by a factor $\gamma \ll 1$, then, self-consistently,

$$\langle\sigma^m\rangle \sim \gamma^m\langle\sigma\rangle \ .$$

If we find that $\langle\sigma^2\rangle$ is very small compared to $\langle\sigma\rangle$ for certain times and/or certain values of μ, then, in these cases, the mean-field approximation is good. But, by multiplying both sides of (2.8) by $\sigma(\underline{k})\sigma(-\underline{k})$ and functionally integrating, we obtain

$$\frac{dG_P(\underline{k})}{dt} = 2 - 2\left\langle\sigma_P(\underline{k})\frac{\delta(H/kT)}{\delta\sigma_P(\underline{k})}\right\rangle - 2\frac{d\bar{\sigma}}{dt}\bar{\sigma}\delta_{\underline{k},0} \ , \tag{3.10}$$

$$\frac{dG_T(\underline{k})}{dt} = 2 - 2\left\langle\sigma_T(\underline{k})\frac{\delta(H/kT)}{\delta\sigma_T(\underline{k})}\right\rangle, \tag{3.11}$$

where, as defined in (3.8), G_P and G_T are, respectively, the parallel and transverse (to $\bar{\sigma}$) correlation functions. We shall, at present, keep only the leading terms in the mean-field approximation to (3.10) and (3.11). We then may check the self-consistency of the approximation by applying our results for G_P and G_T to (3.6). The mean-field theory will break down when the contribution from the G's is no longer small compared to those terms which were retained in (3.9).

In the mean-field approximation, (3.10) and (3.11) reduce to

$$\frac{dG_P(\underline{k})}{dt} = 2 - 2\left(2(\mu + k^2) + \frac{6\lambda\bar{\sigma}^2}{\Omega} + O(G)\right)G_P(\underline{k}) \ ,$$

$$\frac{dG_T(\underline{k})}{dt} = 2 - 2\left(2\mu + 2k^2 + \frac{4\lambda\bar{\sigma}^2}{\Omega} + O(G)\right)G_T(\underline{k}) \ ,$$

or, after applying (3.9),

$$\frac{dG_P}{dt} = 2 - 2\left(-\frac{d_t\bar{\sigma}}{\bar{\sigma}} + \frac{2\lambda\bar{\sigma}^2}{\Omega} + 2k^2 + O(G)\right)G_P(\underline{k}) \ , \tag{3.12}$$

$$\frac{dG_T}{dt} = 2 - 2\left(-\frac{d_t\bar{\sigma}}{\bar{\sigma}} + 2k^2 + O(G)\right)G_T(\underline{k}) \ . \tag{3.13}$$

Note that, when all time derivatives are zero, (3.13) reduces to [after making a mean-field approximation and ignoring $O(G)$ terms]

$$G_T(\underline{k}) = \frac{1}{2k^2} \ ,$$

which satisfies the Bogoliubov inequality (3.2).

We now consider the $O(G)$ corrections to (3.9):

$$\frac{d\bar{\sigma}}{dt} = -2\mu\bar{\sigma} - \frac{4\lambda\bar{\sigma}^3}{\Omega} - \frac{4\lambda\bar{\sigma}}{\Omega}\left(3\sum G_P(\underline{k}) + (N-1)\sum G_T(\underline{k})\right). \tag{3.14}$$

Let us suppose that $\bar{\sigma}$ is given by the mean-field theory plus a correction, $\Delta\bar{\sigma}$ which is presumed small. Then, for times large compared to $1/|\mu|$ the time derivatives of $\bar{\sigma}$ will be zero, to $O(\Delta\bar{\sigma})$, and the correlation functions will be given by

$$G_T(\underline{k}) \cong \frac{2(1-e^{-4k^2t})}{4k^2} + G_T^I(\underline{k})e^{-4k^2t} \ , \tag{3.15}$$

$$G_P(\underline{k}) \cong \frac{2(1-e^{-(4k^2-2\mu)t})}{4k^2-2\mu} + G_P^I(\underline{k})e^{-(4k^2-2\mu)t} \tag{3.16}$$

for $\mu < 0$. G^I denotes the initial values of the correlation functions. We assume that, prior to placing the system in contact with a subcritical temperature bath, it was initially at a high, above critical, temperature. In this case the initial correlation functions G^I do not display any strong dependence on their momentum index $\underline{k}$. The contribution of G^I to $\bar{\sigma}$ in (3.14) goes to zero with time as $\sim t^{-d/2}$ and may be ignored. Putting the remainder of (3.15) and (3.16) into (3.14) we find that, for $\mu < 0$, the deviation $\Delta\bar{\sigma}$ of $\bar{\sigma}$ from $(-\mu\Omega/2\lambda)^{1/2}$ is given by

$$\frac{d\Delta\bar{\sigma}}{dt} \cong +4\mu\Delta\bar{\sigma} - \frac{4\lambda}{\Omega}\left(\frac{-\mu\Omega}{2\lambda}\right)^{1/2} 3\sum G_P(\underline{k}) + (N-1)\sum G_T(\underline{k}) \ . \tag{3.17}$$

In the large volume limit we may replace the sums on $\underline{k}$ with d-dimensional integrals. We see that, in this limit, for $d > 2$ or for $d = 2$ and $N = 1$ the ratio of $\Delta\bar{\sigma}$ to the mean-field value of $\bar{\sigma}$ is

$$\frac{\Delta\bar{\sigma}}{(-\mu\Omega/2\lambda)^{1/2}} \xrightarrow[t\to\infty]{} \sim \lambda\left(\frac{1}{|\mu|} + (N-1)\right).$$

So long as λ is small and $|\mu|$ is large (i.e., the system is not too near the critical point) the above ratio will be small and the mean-field theory will give a good approximation to subcritical dynamics. Similarly, for $\mu > 0$, we may show that, so long as μ is large enough for

$$\lambda N/\mu \ll 1 \ ,$$

the mean-field theory will give a good approximation to above critical dynamics. The most interesting case, however, is that of subcritical ($\mu < 0$) $d = 2$, $N > 1$ systems. In this case (3.17) and (3.15) imply that the deterioration in magnetization is given by

$$\frac{\Delta\bar{\sigma}}{(-\mu\Omega/2\lambda)^{1/2}} \cong -\frac{\lambda(N-1)}{|\mu|8\pi}\ln(t4) \quad (d=2,\ N>1,\ t \gg 1/|\mu|,\ \mu \ll 0) \ . \tag{3.17′}$$

This means that the mean-field approximation is a good one for times less than a time τ given by

$$\tau = \tfrac{1}{4}\exp[8\pi|\mu|/\lambda(N-1)] \quad (d=2,\ N>1,\ \mu \ll 0) \ . \tag{3.18}$$

We see that as we move deeper into the subcritical ($\mu < 0$) region, τ grows exponentially. The rapid increase of τ with $|\mu|$ for $\mu < 0$ has more significance than simply saying that the mean-field theory is an increasingly good approximation for $d = 2$, $N > 1$ systems. During the time that the mean-field approximation is good the magnetization for an $N > 1$ film will behave like the magnetization in a $d = 3$ or in a $d = 2$, $N = 1$ ferromagnet. That is, a substantial ($\propto \sqrt{-\mu}$) magnetization will develop in a short ($\sim 1/|4\mu|$) time, so long as the initial magnetization is even slightly nonzero. This quasistable magnetization will not show signs of deteriorating until a time τ has elapsed.

By comparison, we may apply the analysis which led to (3.18) to a $d > 2$, $N > 1$ system of finite volume. If $\Omega < \infty$ then we must be careful to include the $\underline{k} = 0$ mode when we apply (3.15) to (3.14). Since $G_T(0)$ increases linearly with time, there will be a finite τ

$$\tau \sim \frac{\Omega|\mu|}{2\lambda(N-1)} \quad (d > 2,\ N > 1,\ \mu \ll 0) \tag{3.19}$$

after which the mean-field theory breaks down and bulk magnetization begins to deteriorate. Later in this section it will be shown that for $\mu \ll 0$, after the mean-field theory has broken down the magnetization will continue to decay to zero. Thus the τ's given in (3.18) and (3.19) are estimates for the persistence times of substantial symmetry-breaking magnetization. Equation (3.19) gives a figure for a macroscopic persistence time—it will be $\sim\Omega\sim 10^{23}$. Hence two-dimensional systems will display virtually macroscopic persistence time—and will have behavior indistinguishable from conventional ferromagnets—when

$$|\mu|/\lambda \gtrsim [(N-1)/8\pi]\ln(10^{23}) . \tag{3.20}$$

It is useful, at this point, to reformulate (3.18) and (3.20) in terms of temperature. Let us rewrite (3.1) to explicitly include temperature:

$$\frac{H}{kT}=\frac{1}{kT}\left(\sum(\tilde{\mu}_0+\kappa_0k^2)\underline{\sigma}\cdot\underline{\sigma}+\frac{\lambda_0}{\Omega}\sum\sum\sum(\underline{\sigma}\cdot\underline{\sigma})^2\right)$$

or, after choosing units of σ to set $\kappa_0/kT=1$ and suitably redefining $\tilde{\mu}_0$ and $\tilde{\lambda}_0$

$$\frac{H}{kT}=\sum(\mu_0+k^2)\,\underline{\sigma}\cdot\underline{\sigma}+\frac{\lambda_0 kT}{\Omega}\sum\sum\sum(\underline{\sigma}\cdot\underline{\sigma})^2, \tag{3.21}$$

The critical point is determined by setting the inverse of

$$\lim_{k\to 0}\langle\sigma(\underline{k})\sigma(-\underline{k})\rangle$$

equal to zero, with $\langle\sigma\rangle=0$ and when equilibrium statistical mechanics apply. For small λ_0 we may make a Hartree-Fock-like estimate of T_c:

$$0=2\mu_0-\frac{\lambda_0 kT_c 4}{\Omega}(N+2)\sum_{\underline{k}}\langle\sigma(\underline{k})\sigma(-\underline{k})\rangle .$$

Assuming that, on the critical point

$$\langle\sigma(\underline{k})\sigma(-\underline{k})\rangle=1/2k^{2-\eta},$$

we find that for $d=2$

$$T_c\cong\mu_0\eta(2\pi)/k\lambda_0(N+2)\quad(d=2). \tag{3.22}$$

Using (3.21) and (3.22) in place of (3.1), we find that (3.18) may be rewritten as

$$T\cong\tfrac{1}{4}\exp\left[\frac{4T_c}{T\eta}\left(\frac{N+2}{N-1}\right)\right]\quad(d=2,\ N>1,\ T\ll T_c). \tag{3.23}$$

Using Lublin's[8] renormalization-group figure of 0.21 for the η of the XY model, it may be seen that virtually macroscopic persistence times will be obtained quite close to T_c. Although more careful estimates of T_c may change the proportionality constants which appear in (3.23), the basic results remain unchanged: τ rises exponentially with $1/T$ for $T<T_c$ and virtually macroscopic persistence times — virtual ferromagnetism — are obtained below some finite fraction of T_c.

We should note that when the work of this section is repeated for $d<2$, $N>1$ systems, there will be no virtual ferromagnetism at finite temperatures. Assuming that a nonzero T_c can be found at all for the case $d<2$, the persistence time below such a T_c will be

$$\tau\propto(1/T)^{2/(2-d)}\quad(N>1,\ T\ll T_c,\ d<2). \tag{3.24}$$

This τ will not be comparable to 10^{23} except at microscopically small temperatures. It is only for the case $d=2$ — when the divergences implied by Bogoliubov's inequality are marginal, logarithmic divergences — that virtual ferromagnetism can be obtained.

In order to obtain a qualitative picture of systems close to the critical point and in order to establish that, after time τ, magnetization will decay to zero, we make use of Bogoliubov's inequality. If, in a hysteresis experiment on a system with the Hamiltonian (3.1), the time-dependent probability P is given by

$$-\ln P[\sigma]=\sum[u_2(t)+k^2]\,\underline{\sigma}\cdot\underline{\sigma}+\sum\sum\sum\frac{u_4 t}{\Omega}(\underline{\sigma}\cdot\underline{\sigma})^2+-\underline{u}_1(t)\cdot\underline{\sigma}(0)\sqrt{\Omega},$$

then it may be shown that Bogoliubov's inequality implies that

$$\frac{dG_T(\underline{k})}{dt}\geq 2-2\left[\frac{-d_t\bar{\sigma}}{\bar{\sigma}}+2k^2\left(1+\frac{D}{\bar{\sigma}^2}\right)\right]G_T(\underline{k}), \tag{3.25}$$

where the definition, (3.3), for D may be rewritten as

$$D\equiv\sum[(N-1)G_T(\underline{k})+G_P(\underline{k})]. \tag{3.26}$$

To obtain (3.25), note the Bogoliubov's inequality provides a lower bound on the time-dependent correlation function $G_T(\underline{k})$, with u_1, u_2, and u_4 replacing h, μ, and λ, respectively, in (3.2). In addition, if H were replaced with $\ln P$ in (3.4) and (3.11), then the time derivatives would be zero. The resulting pair of equations, coupled with the Bogoliubov inequality, (3.4), and (3.11) leads to (3.25).

The fact that the parameter λ in H/kT is positive means that the position space spin-spin fluctuation at a given lattice site must be finite

$$\infty>\langle\underline{S}(x)\cdot\underline{S}(x)\rangle=\frac{\bar{\sigma}^2}{\Omega}+\frac{1}{(2\pi)^d}\int d^d\underline{k}\;[G_P(\underline{k})+(N-1)G_T(\underline{k})]$$

or

$$\infty>\bar{\sigma}^2/\Omega+D/\Omega . \tag{3.27}$$

It may be seen that 3.25 implies that, for $d\leq 2$, D will grow without limit as $t\to\infty$ — and thus violate

(3.27) —unless $\bar{\sigma}$ goes to zero at $t\to\infty$. Conversely, $\bar{\sigma}$ must begin deteriorating when D becomes comparable to the upper bound for $\langle S(x)\cdot S(x)\rangle$. If the mean-field estimate of $-\mu/2\lambda$ provides a reasonable figure for this, then $\bar{\sigma}$ must begin deteriorating after the time τ.

While the manner in which $\bar{\sigma}$ decays for times $t\gg\tau$ is an academic question for $d=2$, $N>1$ systems at low enough temperatures to have virtually macroscopic τ, it is an important issue for systems near the critical point. The full treatment of systems near the critical point will require use of the renormalization group. For our present purposes, however, we may obtain a description which is, hopefully, qualitatively correct by taking (3.25) as an equality

$$\frac{dG_T(\underline{k})}{dt}=2-2\left[\frac{d_t\bar{\sigma}}{\bar{\sigma}}+2k^2\left(1+\frac{D}{\bar{\sigma}^2}\right)\right]G_T(\underline{k}). \quad (3.28)$$

In addition, we expect G_P to equal G_T when $\bar{\sigma}\to 0$. We, therefore, modify (3.12) to give

$$\frac{dG_P(\underline{k})}{dt}=2-2\left[-\frac{d_t\bar{\sigma}}{\bar{\sigma}}+\frac{2\lambda\bar{\sigma}^2}{\Omega}+2k^2\left(1+\frac{D}{\bar{\sigma}^2}\right)\right]G_P(\underline{k}). \quad (3.29)$$

(3.28) and (3.29), together with (3.14)

$$\frac{d\bar{\sigma}}{dt}=-2\mu\bar{\sigma}-\frac{4\lambda\bar{\sigma}^3}{\Omega}-\frac{4\lambda\bar{\sigma}}{\Omega}\left(3\sum G_P(\underline{k})+(N-1)\sum G_T(\underline{k})\right) \quad (3.14)$$

provide a dynamic approximation which yields qualitatively reasonable results in the vicinity of the critical point for the ferromagnetic systems: $d>2$ and $d=2$, $N=1$, except for the fact that the critical point for these equations remains at $\mu=0$. For above critical systems ($\mu>0$) $\bar{\sigma}$ will still ultimately decay to zero exponentially as $e^{-2\mu t}$. Note that, as the critical point is approached from above (as $\mu\to 0^+$) the decay time $1/2\mu$ goes to infinity. Below the critical point ($\mu<0$) for $d>2$ and for $d=2$, $N=1$, $\bar{\sigma}$ will approach a stable nonzero root to $d_t\bar{\sigma}=0$. The value of the subcritical magnetization, goes to zero as the critical point is approached from below ($\mu\to 0^-$).

Finally let us sketch the behavior of subcritical ($\mu<0$) $d=2$, $N>1$ systems under (3.28), (3.29), and (3.14). For times small compared to τ, D is small compared to $(\bar{\sigma})^2$ and these dynamical equations reduce to the already discussed equations (3.12)–(3.14). For times large compared to τ, transverse fluctuations become substantial and will, through (3.14), cause $\bar{\sigma}$ to deteriorate. The reader may examine (3.28), (3.29), and (3.14) to determine that the $t\to\infty$ solution for $d=2$, $N>1$ systems is one in which $\bar{\sigma}$ goes to zero at a less than exponential rate

$$\bar{\sigma}\xrightarrow[t\gg\tau]{}\sim\frac{1}{\sqrt{\ln t}}\quad(\mu<0,\ d=2,\ N>1) \quad (3.30)$$

and

$$D\xrightarrow[t\gg\tau]{}\frac{-\mu N}{2\lambda(N+2)}.$$

It should be emphasized that this ultimate decay cannot materialize until after a sufficient time τ has elapsed for transverse fluctuations to establish themselves.

The slower-than-exponential decay for subcritical nonscalar films is a feature which is in agreement with the finding of Stanley and Kaplan.[6] To see this, observe that, when equilibrium statistical mechanics obtain with the Hamiltonian (3.1), then the zero-momentum correlation function is given by[17]

$$[G_P(0)]^{-1}=\left\langle\frac{\delta H/kT}{\delta\sigma_P(0)}\right\rangle\Big/\bar{\sigma} \quad (3.31)$$

provided that $\bar{\sigma}=0$. But, equilibrium statistical mechanics will obtain in the limit $t\to\infty$. Thus, by (3.4), (3.31) reduces to

$$\lim_{t\to\infty}\left([G_P(0)]^{-1}=-\frac{d_t\bar{\sigma}}{\bar{\sigma}}\right) \quad (3.32)$$

provided

$$\lim_{t\to\infty}\bar{\sigma}=0.$$

With the above proviso, (3.32) is a rigorous result. We see, from (3.32), that the slower-than-exponential decay after times $t\gg\tau$ for any subcritical nonscalar film —which was predicted by using the approximate dynamical equations (3.28), (3.29), and (3.14) —implies that the subcritical susceptibility is infinite:

$$[G_P(0)]^{-1}\sim\frac{1}{t(\ln t)}\xrightarrow[t\to\infty]{}0.$$

This is the result obtained by Stanley and Kaplan[6] in their numerical calculations of the equilibrium statistical mechanics of the $d=2$ XY model.

IV. CONCLUSION

For the model system considered in this paper, we have shown that nonscalar films can, at a finite fraction of their critical temperature, develop and sustain magnetization for times which, for laboratory purposes, are infinite. In addition, using an approximation which gives qualitatively reasonable results for $d>2$ systems and for $d=2$, $N=1$ systems, we predicted that the near but subcritical region may be distinguished from the near but above critical region by the nature of the $t\to\infty$ decay of magnetization. For above critical systems the decay rate is, in the $t\to\infty$ limit, exponential with time, for subcritical systems it is slower than exponential. This result is consistent with Stanley and Kaplan's[6] calculations.

The most interesting area for future research is the near-critical problem. The prediction given here of subcritical magnetization ultimately decaying as $1/\sqrt{\ln t}$ is probably only qualitatively — in the sense of its being a slower-than-exponential decay — correct.

The prediction of virtual ferromagnetism at obtainable temperatures is a sound one, for the spin system considered in this paper. It is conceivable, however, that there exists a different type of $N>1$, $d=2$ system, governed by different dynamical equations, which will not display virtual ferromagnetism. A search for such a system may be in order, although it is probably a much less pressing matter than the study of slightly subcritical, nonscalar films.

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