Series analysis of corrections to scaling for the spin-pair correlations of the spin-*s* Ising model: Confluent singularities, universality, and hyperscaling

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We report a detailed study of twelve-term, high-temperature series for the second moment of spin-pair correlations $\mu_2(t)$ and the specific heat $c_H(t)$ of the nearest-neighbor spin-s Ising model in zero magnetic field on the fcc lattice. Near criticality we find $\mu_2(t) = A_2(s)t(s)^{-(\gamma+2\nu)}[1 + B_2(s)t(s)^{\Delta_1} + \cdots]$, $\{t(s) = [T - T_c(s)]/T\}$, showing a confluent correction to the dominant scaling singularity. To within uncertainties the exponents have the universal (i.e., spin-independent) values $\nu = 0.638^{+0.002}_{-0.008}$ (with $\gamma = 1.250^{+0.003}_{-0.007}$) and $\Delta_1 = 0.6 \pm 0.1$. The confluent exponent Δ_1 is in reasonable agreement with the correction-to-scaling index derived from earlier analysis of the susceptibility, as predicted by renormalization-group arguments. A similar analysis of the specific heat c_H for the same model finds no detectable confluent singularities in rather noisy, high-temperature series and gives $\alpha = 0.125 \pm 0.020$ in general confirmation of earlier $s = 1/2$ estimates. With ν as quoted above the hyperscaling relation $d\nu = 2-\alpha$ at $d = 3$ requires $\alpha = 0.086^{+0.024}_{-0.006}$, so the validity of hyperscaling remains problematical.

I. INTRODUCTION

In earlier work[1,2] on the spin-s nearest neighbor fcc Ising model we have studied corrections to the dominant scaling behavior of the zero-field ($h = 0$) susceptibility at temperatures near to but greater than the critical temperature $T_c(s)$. On the basis of 12-term high-temperature expansions, we found compelling evidence for confluent singularities[3] of the form,

$$\chi(t) = A_0(s)\, t(s)^{-\gamma}[1 + B_0(s)\, t(s)^{\Delta_1} + \cdots] , \qquad (1)$$

with spin-dependent amplitudes $[B_0(\frac{1}{2}) \approx 0]$, but exponents, $\gamma = 1.250^{+0.003}_{-0.007}$, $\Delta_1 = 0.50 \pm 0.08$, which were spin independent[4] (universal) to within uncertainties. Renormalization-group arguments[3,5] suggest that the *same* correction-to-scaling index Δ_1 should in general appear in all other thermodynamic and correlation functions. In this paper we extend our previous analysis to the second moment of spin-pair correlations $\mu_2(t)$ and the specific heat per spin $c_H(t)$. The high-temperature critical behavior of these quantities defines the exponents ν (correlation length[6]) and α (specific heat). We are primarily motivated by three questions: (a) Are these exponents properly spin independent (universal) in an analysis which allows for confluent singularities? (b) Are the confluent corrections which appear characterized by the same exponent Δ_1 which appeared in (1)? And finally, (c) are the universal values of ν and α consistent with the hyperscaling relation[7] $d\nu = 2 - \alpha$?

Our work is based on previously derived series[1,2,8] for the $h = 0$ spin-pair correlations,

$$\Gamma(\vec{R}, t) \equiv s^{-2}[\langle S(\vec{r})\, S(\vec{r} + \vec{R})\rangle - \langle S\rangle^2]$$
$$\approx R^{-d+2-\eta} D(Rt^{\nu}) . \qquad (2)$$

The final member of (2) describes the expected dominant scaling behavior[7] near $T_c(s)$, where $\Gamma(\vec{R})$ develops spherical symmetry.[9] The index η describes the decay of correlations at $T = T_c(s)$ $(t = 0)$. At fixed $t \gtrsim 0$ it is known[10] that for large enough R, $\Gamma(R, t) \propto e^{-R/\xi(t)}/R^{(d-1)/2}$, defining the true correlation[6] length $\xi(t)$. This is built into (2) via $\xi(t) = \xi_0 t^{-\nu}$ and the large-x dependence of the scaling function $\ln D(x) \to -x/\xi_0 + (\frac{1}{2}d - \frac{3}{2} + \eta) \ln x + \text{const}$ as $x \to \infty$. The spherical moments of $\Gamma(\vec{R}, t)$ are defined by

$$\mu_n(t) \equiv \sum_{\vec{R}} R^n\, \Gamma(\vec{R}, t) \approx A_n(s)\, t(s)^{-\nu(2-\eta+n)} . \qquad (3)$$

The leading critical scaling form shown in (3) follows from the scaling form of correlations (2) via $\mu_n(t) \propto \xi^{2-\eta+n}$. The susceptibility sum rule identifies $\mu_0(t)$ as the (reduced) susceptibility (1), from which follows the well-known scaling law[11] $\gamma = \nu(2-\eta)$.

Wegner[3] was the first to consider the corrections to dominant critical scaling within the framework of the renormalization group. The upshot of this analysis[3,12] is inclusion of additional "irrelevant scaling fields" $\{h_k\}$ into a generalized scaling function [cf. (2)],

$$\Gamma(\vec{R}, t, \{h_k\}) \approx R^{-d+2-\eta}\,\mathfrak{D}(Rt^{\nu}, \{h_k t^{\nu|\lambda_k|}\}), \tag{4}$$

valid near criticality. The eigenvalues[12] λ_k associated with the irrelevant scaling fields are all *less* than zero.[13] Near $T_c(s)$ the h_k's are finite, $t \to 0$, and the generalized scaling function can be expanded about $x_k = h_k t^{\nu|\lambda_k|} = 0$ for all k,

$$\Gamma(\vec{R}, t\{h_k\}) \approx R^{-d+2-\eta} D(Rt^{\nu}) \times \left(1 + \sum_k h_k t^{\nu|\lambda_k|} \left.\frac{\partial \mathfrak{D}}{\partial x_k}\right|_{\{x_k=0\}} + \cdots\right) \tag{5}$$

$[D(Rt^{\nu}) = \mathfrak{D}(Rt^{\nu}, \{x_k = 0\})]$. For small t the leading corrections come from the least negative λ_k and carry a factor t^{Δ_1} with $\Delta_1 = \nu|\lambda_k|$ for the minimum $|\lambda_k|$. The substitution of (5) into the definition (3) leads to

$$\mu_n(t) \approx A_n(s)\, t^{-(\gamma + n\nu)}[1 + B_n(s)\, t^{\Delta_1} + \cdots], \tag{6}$$

of which (1) is a special case. A parallel analysis of the free energy[3,5] leads to a similar form for the specific heat. Note that the same exponent Δ_1 governs the leading corrections to scaling for all quantities.[14] Δ_1 has been calculated by renormalization group-techniques and observed experimentally; Refs. 1 and 2 contain numerous citations.

The remaining content of this paper is divided into three parts. In Sec. III we briefly introduce necessary series terminology. Sections III and IV describe our analysis of second-moment and specific-heat series, respectively. The analysis relies mainly on two methods which have been described elsewhere, the Baker-Hunter transformation[2,15] and the method of n fits.[1,16]

In our analysis of μ_2 (Sec. III) we find clear evidence of confluent singularities of the form (6) for $s > \frac{1}{2}$. For $s = \frac{1}{2}$ (only) it appears that $B_2(\frac{1}{2}) \approx 0$, and our data are consistent with the interpretation that the leading correction to scaling is just the analytic correction, in which the square bracket in (6) is replaced by $[\ \ ] = 1 + \text{const}\, t + \cdots$. Our analysis of the dominant exponent $(\gamma + 2\nu)$ shows a very weak[17] spin dependence. If we assume $\gamma = 1.250$ for all spins, then we might conclude $\nu(s = \frac{1}{2}) = 0.638$, $\nu(s = \infty) = 0.633$ with intermediate spins between these values. We are inclined, however, to attribute this spread to the subtle effects of corrections of higher order than t^{Δ_1} and conclude that the $d = 3$ Ising-correlation-length index has a universal value $\nu = 0.638^{+0.002}_{-0.008}$ in agreement with earlier $s = \frac{1}{2}$ fcc work,[18] but with somewhat wider uncertainties. This estimate reflects our bias in favor of the $s = \frac{1}{2}$ series, which show very rapid apparent convergence.[18] The correction exponent is somewhat less well determined. We conclude $\Delta_1 = 0.6 \pm 0.1$ independent of s. This is within uncertainties of (but somewhat higher than[19]) earlier estimates[1,2] based on the susceptibility and may be regarded as consistent with $s = \frac{1}{2}$ data of Tarko and Fisher[20] for the critical isotherm behavior of the sc and bcc lattices.

A similar analysis of the spin-s fcc specific-heat series (Sec. IV) is far less satisfactory. Unlike those for χ and μ_2, these series are quite irregular in the low orders available. Their ratio plots do not show smooth curvature of a type that can be well fitted[1] by confluent corrections analogous to (1) or (6). It is, thus, not surprising that the introduction of the possibility of confluent corrections does not noticeably improve or modify the results of conventional analysis.[21] One might anticipate difficulty in the specific-heat analysis because of the weakness $(\alpha \approx \frac{1}{8})$ of the expected singularity relative to background terms,

$$c_H(t)/k_B \approx C_0 t^{-\alpha}(1 + C_1 t^{\Delta_1} + \cdots) + R_0 + R_1 t + \cdots, \tag{7}$$

where R_0, R_1, etc., represent analytic corrections.[22] Note that, since $\Delta_1 > \alpha$, the first correction to scaling (easily visible in the usual analysis[21]) is actually the constant term R_0. One must search "under" R_0 for the "normal" correction varying as t^{Δ_1}. We have attempted to circumvent such difficulties both by inserting critical temperatures $T_c(s)$ derived from the better-converged series χ and μ_2 and by studying series for $\partial c_H(t)/\partial t$ in which the first correction is expected to be normal. No method of analysis we have tried gives any evidence for the existence of t^{Δ_1} corrections. We emphasize that this failure is not to be interpreted as strong evidence *against* such singularities. Indeed, if the confluent amplitude C_1 were weak enough, it is quite plausible that normal confluent corrections might be hidden in the intrinsic noisiness of the specific-heat series. Corrections not withstanding, standard analysis leads to the spin-independent estimate $\alpha = 0.125 \pm 0.020$. There are no noticeable trends with spin. The quoted uncertainties are, we feel, reasonable.

It is reassuring that γ, ν, and α appear universal with respect to spin; however, the status of hyperscaling is problematical. If $\alpha = 0.125 \pm 0.020$, then $d\nu = 2 - \alpha$ implies $\nu = 0.625 \pm 0.007$, which is just barely compatible with our direct estimate $\nu = 0.638^{+0.002}_{-0.008}$. Two further considerations make this marginal compatibility even less tenable: (a) If we give full credence to spin universality, then we are entitled to evaluate both ν and α from that *single* spin ($s = \frac{1}{2}$) which appears to determine them most precisely. Based on $s = \frac{1}{2}$ *alone* we are inclined to quote $\alpha = 0.125 \pm 0.010$, which translates (under hyperscaling) into $\nu = 0.625 \pm 0.003$, quite incompatible with the best $s = \frac{1}{2}$ estimate[18] $\nu = 0.638^{+0.002}_{-0.001}$. Furthermore, (b) if we accept the wider uncertainties and compromise on $\nu = 0.631$, $\alpha = 0.107$, we must still reconcile this with thermodynamic scaling relations like $\alpha + 2\beta + \gamma = 2$, which are generally regarded as solid. Taking $\gamma = 1.250$, this would require $\beta = 0.322$, uncomfortably higher than the most recent $s = \frac{1}{2}$ series estimates[23] ($\beta = 0.312 \pm 0.005$). It remains conceivable that some such "compromise" set[24] of $d = 3$ Ising exponents will ultimately prevail. At present and until we have a firmer understanding of those corrections affecting the series estimates of ν and α, the apparent failure of hyperscaling seems perilously close to the borderline of unambiguous resolvability.

II. DERIVATION OF HIGH-TEMPERATURE SERIES

The Hamiltonian of the nearest-neighbor spin-s Ising model is

$$-\mathcal{H}/k_B T = \frac{K}{2s^2} \sum_{\vec{r}} \sum_{\vec{\delta}} S(\vec{r}) S(\vec{r} + \vec{\delta}) + \frac{h}{s} \sum_{\vec{r}} S(\vec{r}) , \tag{8}$$

where $S(\vec{r}) = -s, 1-s, \ldots, s-1, s$. The vector $\vec{\delta}$ runs over nearest neighbors, h is a reduced magnetic field, and $K = J/k_B T$ (J is the exchange energy). High-temperature $h = 0$ series through order K^{12} for the spin-pair correlation function have previously been derived for a variety of lattices including the fcc.[8,25,26] Summation of the spin-pair correlations according to (3) gives[27] $\mu_2(t)$. The energy density is related to the nearest-neighbor correlations,

$$\epsilon(t) \equiv E/N = -\tfrac{1}{2} q J \Gamma(\vec{\delta}, t) \tag{9}$$

($q = 12$ is the coordination number for the fcc lattice). Equations (3) and (9) lead to series of the form

$$\mu_2(t) = \frac{s+1}{3s} \sum_{n=1}^{\infty} m_n(s)\, v^n \tag{10}$$

and

$$\epsilon(t) = -qJ\, \frac{s+1}{6s} \sum_{n=1}^{\infty} c_n(s)\, v^n , \tag{11}$$

where[28] $v \equiv (s+1)K/3s$. The fcc coefficients $m_n(s)$ and[29] $c_n(s)$, $1 \leq n \leq 12$, are tabulated in Tables I and II for spins $s = \frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and ∞. Data are also available for other spins and lattices.[30] Series for the specific heat are easily derived from (11),

$$c_H(t)/k_B = \frac{d\epsilon(t)}{dk_B T} = \frac{q}{2} \sum_{n=1}^{\infty} c_n(s)\, v^{n+1} . \tag{12}$$

We analyze these series in Secs. III and IV.

III. ANALYSIS OF SECOND-MOMENT SERIES

The analysis of moment series presented herein follows closely our previous analyses[1,2] of the susceptibility series. Our task is somewhat simplified in that we have available very accurate estimates for the critical point $v_c(s)$ as a function of s from our susceptibility studies. In addition to analyses based on now standard ratio and Padé methods[1,2] we have employed two more sophisticated methods of series analysis—both of which allow explicitly for confluent singularities.

The first method involves the use of a nonlinear series transformation introduced by Baker and Hunter.[15] These authors recognized that the series

TABLE I. Spin-s second-moment coefficients, $m_n(s)$ ($1 \leq n \leq 12$), to ten-place accuracy for $s = \frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and ∞.

$s=\frac{1}{2}$	$s=1$	$s=\frac{3}{2}$
12	12	12
288	288	288
4904	4972.5	4991.8592
71 904	74 574	75 332.352
968 209.6	1 032 530.006	1 051 103.047
12 336 076.8	13 573 559.93	13 938 772.08
151 187 110.6	172 041 937.8	178 347 928.8
1 800 280 955.5	2 122 240 592	2 222 108 207
20 965 809 027	25 635 764 908	27 122 693 077
239 886 483 889	304 536 941 557	325 674 029 676
2 705 501 680 746	3 568 714 268 794	3 858 552 259 544
30 150 818 448 085	41 348 367 601 860	45 209 800 563 297
$s=2$	$s=\frac{5}{2}$	$s=\infty$
12	12	12
288	288	288
5000.18	5004.544 261	5014.08
75 658.8	75 830.141 39	76 204.8
1 059 139.482	1 063 367.465	1 072 636.263
14 097 870.47	14 181 831.05	14 366 514.79
181 115 982.5	182 581 839.7	185 818 480.5
2 266 301 187	2 289 791 053	2 341 867 895
27 786 214 668	28 140 252 835	28 928 482 461
335 187 249 762	340 283 302 888	351 678 571 269
3 990 137 885 682	4 060 909 182 631	4 219 862 830 858
46 978 318 506 593	47 933 357 150 937	50 088 042 687 788

TABLE II. Spin-s energy density coefficients, $c_n(s)$ ($1 \le n \le 12$), to ten-place accuracy for $s=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and ∞.

$s=\frac{1}{2}$	$s=1$	$s=\frac{3}{2}$
1.000 000 000	1.000 000 000	1.000 000 000
4.000 000 000	4.000 000 000	4.000 000 000
21.666 666 67	27.375 000 00	28.988 266 67
133.333 333 3	187.500 000 0	202.965 333 3
886.133 333 3	1360.792 188	1516.134 997
6288.177 777	10 386.993 75	11 852.648 39
46 930.746 03	82 247.341 32	95 833.954 00
363 098.4127	670 875.9314	796 679.8085
2 885 527.334	5 604 565.731	6 773 008.730
23 418 295.41	47 740 012.91	58 651 670.48
193 335 209.7	413 249 018.3	515 770 867.2
1 618 886 140	3 625 849 863	4 594 900 964
$s=2$	$s=\frac{5}{2}$	$s=\infty$
1.000 000 000	1.000 000 000	1.000 000 000
4.000 000 000	4.000 000 000	4.000 000 000
29.681 666 67	30.045 355 10	30.840 000 00
209.633 333 3	213.135 673 5	220.800 000 0
1585.808 521	1623.053 424	1706.115 265
12 525.646 19	12 889.093 58	13 708.443 43
102 225.9888	105 617.6776	113 683.1609
857 134.8465	890 473.8058	967 384.7822
7 345 101.537	7 663 389.171	8 404 961.797
64 084 459.16	67 131 347.07	74 294 913.48
567 595 543.1	596 874 386.8	666 290 478.9
5 091 639 467	5 374 187 891	6 049 312 572

analysis for a function $f(x)$ composed solely of confluent branch-cut singularities, as in

$$f(x)=\sum_{l=1}^{M} A_l(1-yx)^{-\gamma_l} ,$$

could be greatly simplified by noting that the first N coefficients of the expansion of $f(x)$ in powers of x determine the first N coefficients in powers of $\zeta=\ln(1-yx)$ of the related function

$$\mathfrak{F}(\zeta)=\sum_{l=1}^{M}[A_l/(1-\gamma_l\zeta)] .$$

This latter function is a sequence of simple poles, and is ideally suited for analysis by direct Padé approximants.

The second method, four fits, has been described in detail elsewhere,[1,2,16] and we only briefly outline it herein. From Eq. (6) the second moment is expected to behave as ($\tilde{A}_2 \equiv A_2/K$)

$$K^{-1}\mu_2(t)\approx \tilde{A}_2(s)\, t^{-(\gamma+2\nu)}[1+B_2(s)\, t^{\Delta_1}+\cdots] \qquad (13)$$

in the critical region. Were μ_2 *exactly* of this form it would be completely described by the five parameters $\tilde{A}_2(s)$, $B_2(s)$, $(\gamma+2\nu)$, Δ_1, and $v_c(s)$. In the method of four fits we obtain a sequence of estimates for $\tilde{A}_2(s)$, $B_2(s)$, Δ_1, and $v_c(s)$ (or $\gamma+2\nu$) by specifying $\gamma+2\nu$ [or $v_c(s)$] and *assuming* the neglected terms in Eq. (13) to vanish. Then the above four parameters are determined from the series coefficients m_{n+2}, m_{n+1}, m_n, and m_{n-1}. Of course, if μ_2 were exactly given by a pair of confluent singularities, the sequence of estimates would be independent of n. In practice the apparent convergence of the sequence of estimates is determined by (i) the accuracy of the estimate for $\gamma+2\nu$ or $v_c(s)$ used, and (ii) the magnitude of the neglected terms in Eq. (13).[31]

The values of the critical exponent ν estimated from naive ratio and Padé analyses of the second-moment series depend upon spin in very much the same manner that similar estimates for the susceptibility exponents γ were previously found to depend on spin.[1,2,8] That is, the spin-$\frac{1}{2}$ series produces ratio plots with very little curvature and yields estimates for ν in the range $\nu=0.638-0.639$. As the spin value increases, the ratio plots become increasingly curved and apparently extrapolate to considerably lower values, e.g., for $s=\infty$, $\nu\simeq 0.625\pm 0.005$. In the case of the susceptibility, the curvature of the ratio plots for higher spin values was successfully understood in terms of a universal value of the critical exponent γ associated with the dominant singularity which is somewhat masked by confluent corrections, again with universal exponent $\gamma-\Delta_1$, but with amplitude which is quite large at $s=\infty$, and gradually diminishes to zero as s decreases to $s=\frac{1}{2}$.

As noted in the Introduction, we expect from renormalization-group arguments that such corrections are also present in the second moment and that the same correction exponent Δ_1 is involved. Further, from the similarity of the spin effect found in ratio analysis of μ_2 to that found in ratio analysis of χ, we expect the corrections to diminish in importance as s decreases to $s=\frac{1}{2}$. The confluent-singularity analyses discussed below satisfactorily confirm this picture.

A. Baker-Hunter analysis

The results of the Baker-Hunter analysis will be described first. Given an accurate value for $v_c(s)$ from susceptibility analysis, one can obtain estimates for the exponents and amplitudes in Eq. (13) for $\mu_2(t)$. In fact, we have performed Baker-Hunter analyses for each value of spin using a range of values of $v_c(s)$ about the "best" value from susceptibility analysis. We present herein the Baker-Hunter analysis for the central value of $v_c(s)$ for $s=\frac{1}{2}$, $\frac{5}{2}$, and ∞. The variation in estimates due to small changes in assumed value of $v_c(s)$ are also discussed. Finally for $s=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and ∞ the results for ν, $\tilde{A}_2(s)$, $B_2(s)$, and Δ_1, obtained using the best value of $v_c(s)$ from susceptibility analysis, are summarized.

In Table III we list the results of Padé analysis of the Baker-Hunter transformed series for $s=\frac{1}{2}$. The estimate, $v_c^{-1}=9.7944$, was employed in the transformation.[32] The four entries in Table III listed for the $[L/M]$ approximants represent, in order from top to bottom, the correlation length exponent ν, the amplitude of the dominant singularity Δ_1, in Eq. (13), the correction-to-scaling exponent $\tilde{A}_2$, and the amplitude of the leading correction term $\tilde{A}_2 B_2$. (The estimates for ν were obtained using $\gamma=1.250$.) The convergence of higher-order estimates to $\nu=0.638$ is very good. Similarly the correction exponent is consistently in the range $\Delta_1=1.0-1.1$ (which is consistent with an analytic background correction; see below). The amplitude of the dominant singularity is found to be $\tilde{A}_2=11.10$ with good precision, and the correction-term amplitude is in the range $\tilde{A}_2 B_2=0.93\pm0.10$.

These estimates are, of course, biased in that a particular value of $v_c(\frac{1}{2})$ is assumed. However, the results are not significantly sensitive to small variations in the assumed value of the critical point. In particular, for critical points in the range $9.794\leq v_c(\frac{1}{2})^{-1}\leq 9.795$ ν varies in the range $\nu\simeq 0.637-0.640$. Additionally, the smallness of B_2 and the proximity of Δ_1 to unity are unchanged by such variations in the assumed value of the critical points.

In Tables IV and V we present similar analyses for $s=\frac{5}{2}$ (γ assumed to be 1.250), and $s=\infty$ (γ assumed to be 1.247).[33] [If the assumed value of $\gamma(\infty)$ is changed to 1.250, the listed estimates for $\nu(\infty)$

TABLE III. Results of Padé analysis of the Baker-Hunter transformed second-moment series for $s=\frac{1}{2}$. The critical point is taken to be $v_c(\frac{1}{2})^{-1}=9.7944$. The $[L/M]$ approximants for ν, Δ_1, $\tilde{A}_2$, and $\tilde{A}_2B_2$ are given sequentially. Most higher-order approximants have very weak (|residues| $<10^{-11}$) defects.

L \ M	2	3	4	5	6	7	8	9
2	0.641	0.620	0.638	0.631	0.639	0.638	0.635	0.636
	0.95	⋯	1.10	1.38	0.95	1.30	⋯	⋯
	10.96	12.60	11.13	11.25	11.05	11.14	11.27	11.25
	1.06	0	0.46	0.88	0.88	1.60	⋯	⋯
3	0.637	0.638	0.639	0.638	0.638	0.638	0.638	
	1.21	1.10	0.98	1.07	1.08	1.02	1.11	
	11.19	11.13	11.05	11.10	11.11	11.09	11.11	
	0.93	0.92	0.95	0.93	0.93	0.89	1.02	
4	0.638	0.632	0.638	0.638	0.638	0.638		
	1.00	1.36	1.07	1.08	1.07	1.06		
	11.09	11.26	11.10	11.11	11.10	11.10		
	0.87	0.87	0.93	0.94	0.93	0.93		
5	0.641	0.639	0.638	0.638	0.638			
	0.74	0.95	1.08	1.07	1.06			
	10.81	11.05	11.11	11.10	11.10			
	0.86	0.87	0.93	0.93	0.93			
6	0.637	0.638	0.638	0.638				
	⋯	1.27	1.02	1.06				
	11.20	11.14	11.09	11.10				
	⋯	1.36	0.89	0.93				
7	0.636	0.636	0.638					
	⋯	⋯	1.11					
	11.21	11.25	11.11					
	⋯	⋯	1.01					
8	0.636	0.637						
	⋯	⋯						
	11.26	11.21						
	⋯	⋯						
9	0.638							
	⋯							
	11.13							
	⋯							

TABLE IV. As in Table III, but for $s=\frac{5}{2}$; the critical point is taken to be $v_c(\frac{5}{2})^{-1}=10.451$.

$L \backslash M$	2	3	4	5	6	7	8	9
2	0.652 0.579 6.51 5.53	0.633 0.644 7.64 4.48	0.637 0.618 7.38 4.70	0.637 0.621 7.41 4.66	0.636 0.620 7.44 4.65	0.641 0.581 7.09 4.86	0.644 0.547 6.81 4.99	0.522 ... 7.77 ...
3	0.637 0.621 7.41 4.67	0.637 0.618 7.38 4.70	0.635 0.631 7.50 4.60	0.636 0.623 7.44 4.64	0.637 0.623 7.42 4.66	0.630 0.651 7.81 4.31	0.637 0.614 7.40 4.65	
4	0.637 0.618 7.37 4.70	0.637 0.621 7.41 4.67	0.636 0.623 7.44 4.64	0.635 0.636 7.56 4.55	0.685 0.325 3.46 6.98	0.635 0.630 7.53 4.55		
5	0.634 0.648 7.63 4.52	0.636 0.623 7.43 4.64	0.637 0.621 7.41 4.66	0.686 0.321 3.36 7.01	0.633 0.645 7.68 4.44			
6	0.647 0.525 6.58 5.14	0.641 0.581 7.09 4.86	0.630 0.650 7.80 4.32	0.635 0.630 7.53 4.56				
7	0.638 0.631 7.35 4.88	0.644 0.545 6.80 4.99	0.637 0.615 7.40 4.64					
8	0.701 0.345 2.53 7.77	0.710 0.345 2.33 8.12						
9	0.710 0.346 2.36 8.11							

would be lowered by 0.001.] For $s=\frac{5}{2}$, we estimate $\nu(\frac{5}{2})\simeq 0.635\pm 0.002$, and $\Delta_1 \simeq 0.63\pm 0.02$; and for $s=\infty$, our best estimates are $\nu(\infty)\simeq 0.633\pm 0.002$ and $\Delta_1\simeq 0.62\pm 0.02$. The Padé tables for $s=\frac{5}{2}$ and $s=\infty$ are noticeably noisier than those for $s=\frac{1}{2}$. Again, the values of $v_c(s)$ employed in the transformation were the best values determined from susceptibility analysis in Ref. 2. Now, however, the results, for $s=\infty$ especially, are notably more sensitive to the assumed value of $v_c(s)$. For example, if the best value $v_c(\infty)^{-1}=10.524$ determined in Ref. 1 is substituted for $v_c(\infty)^{-1}=10.522$ in the spin-infinity analysis, the central estimate for $\nu(\infty)$ drops to $\nu(\infty)=0.625\pm 0.005$, consistent with hyperscaling! However, the apparent convergence of the Padé table noticeably worsens under such a change; and no consistent secondary singularity is estimated. Furthermore, the Padé table becomes highly defective in a manner which suggests[2,15] that the critical point has been misestimated. Thus, we do not place much credence in the lower estimate for ν. In this regard it is satisfying that the best apparent convergence of the Baker-Hunter second-moment analysis for $s=\infty$ and $s=\frac{5}{2}$ is obtained using the same critical points that optimized the convergence of the susceptibility analysis.

The striking aspect of Tables IV and V is the large amplitude of the correction terms $B_2(\frac{5}{2})\approx 0.60$ and $B_2(\infty)\approx 0.66$. This is in strong contrast to spin-$\frac{1}{2}$ where no evidence was found for the leading nonanalytic correction to scaling. In the case of $s=\frac{1}{2}$ we assume that the Baker-Hunter analysis indicates a zero amplitude for the leading correction to scaling, and that the correction terms found are due to analytic background—although the possibility that they represent a term $C_2 t^{-(\gamma+2\nu)+2\Delta_1}$ cannot be dismissed (such a term is always expected to be present, but would normally be masked in our analysis by stronger leading cor-

TABLE V. As in Table III, but for $s=\infty$; the critical point is taken to be $v_c(\infty)^{-1}=10.522$.

L \ M	2	3	4	5	6	7	8	9
2	0.652	0.634	0.633	0.638	0.633	0.633	0.634	0.643
	0.571	0.614	0.615	0.598	0.621	0.615	0.600	0.499
	6.19	7.18	7.19	6.91	7.27	7.22	7.12	6.41
	5.58	4.92	4.91	5.15	4.84	4.87	4.90	5.07
3	0.637	0.634	0.634	0.631	0.633	0.632	0.633	
	0.601	0.615	0.614	0.634	0.617	0.624	0.620	
	7.00	7.19	7.17	7.35	7.23	7.30	7.26	
	5.07	4.91	4.92	4.82	4.86	4.81	4.84	
4	0.634	0.639	0.631	0.633	0.636	0.633		
	0.611	0.597	0.634	0.622	0.585	0.621		
	7.15	6.90	7.35	7.27	7.00	7.27		
	4.94	5.17	4.82	4.85	4.98	4.84		
5	0.630	0.633	0.633	0.636	0.632			
	0.647	0.621	0.617	0.584	0.627			
	7.48	7.27	7.23	7.00	7.31			
	4.71	4.84	4.86	4.99	4.82			
6	0.639	0.633	0.632	0.633				
	0.560	0.615	0.624	0.621				
	6.78	7.22	7.30	7.27				
	5.07	4.87	4.82	4.84				
7	0.629	0.634	0.633					
	0.718	0.600	0.621					
	7.66	7.12	7.26					
	5.12	4.90	4.84					
8	0.673	0.643						
	0.349	0.492						
	3.95	6.37						
	6.62	5.04						
9	0.635							
	0.646							
	7.08							
	10.85							

rections).

As a summary, in Table VI we list the best estimates for the critical parameters for $s=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and ∞.

One can, of course, perform Baker-Hunter analysis on the series for $\mu_2/\chi \sim t^{-2\nu}$, which provides estimates for ν independent of γ. We have performed such analysis for all values of spin studied. The resulting analysis is slightly noisier than that of μ_2 itself. However, it fully confirms the results in Table VI. Indeed the range of best values for ν is somewhat smaller than that listed in Table VI, varying from around 0.634–0.635 at $s=\infty$ to 0.638–0.639 at $s=\frac{1}{2}$. The value of Δ_1 is again in the range $\Delta_1 \sim 0.6$; and the amplitude of the leading correction diminishes rapidly with decreasing spin value. Since the $s=\frac{1}{2}$ analysis is especially clean, we would quote the spin-$\frac{1}{2}$ value $\nu=0.638$ as our best universal estimate. The best value for the correction exponent, $\Delta_1 \approx 0.6$ is slightly higher than that found in susceptibility analysis ($\Delta_1 \approx 0.5 \pm 0.1$).[1,2] Finally, we feel that the higher-spin analyses are sufficiently noisy that the somewhat lower estimates $\nu \simeq 0.633$–0.635 for high spins should not be taken too seriously.

TABLE VI. Best estimates for the critical parameters of the spin-s second moment as deduced from Baker-Hunter analysis.

s	$[v_c(s)]^{-1}$	ν	Δ_1	$\tilde{A}_2(s)$	$B_2(s)$
$\frac{1}{2}$	9.7944	0.638	1.1	11.10	0.08
1	10.229	0.638	0.68	8.8	0.4
$\frac{3}{2}$	10.362	0.637	0.66	8.2	0.5
2	10.421	0.635	0.64	7.8	0.5
$\frac{5}{2}$	10.451	0.635	0.63	7.5	0.6
∞	10.522	0.633	0.62	7.3	0.7

A final point concerning the Baker-Hunter analysis: As discussed by Baker and Hunter[15] and Camp and Van Dyke,[2] for series composed only of confluent singularities the sequence of $[N-1/N]$ Padé approximants converges most rapidly. Nevertheless, we have used the full Padé table in arriving at our estimates for critical parameters. This was done for two reasons. First, as a practical matter, for the series considered herein, the two convergence criteria differ little or not at all. Second, we have found that on "dirty" test series, i.e., test series with weak nonconfluent corrections in addition to the stronger confluent corrections, the full-table convergence criterion provides a more faithful guide to the true critical behavior.

B. Four-fit analysis

As noted above, in four-fit analysis, either $v_c(s)$ or $\gamma+2\nu$ is chosen as an input parameter and the remaining four parameters, $\gamma+2\nu$ or $v_c(s)$, $\tilde{A}_2(s)$, $B_2(s)$, and Δ_1 obtained as functions of the series coefficients and the (fifth) input parameter. We may then vary the value of $v_c(s)$ or $\gamma+2\nu$ to obtain the smoothest possible sequence of estimates for the four undetermined parameters.

Since $v_c(s)$ is employed as the input parameter in the Baker-Hunter analysis, we choose to discuss four fits with $\gamma+2\nu$ as the input parameter. Setting γ equal to 1.250 we thus vary ν to obtain a smooth sequence. [We have also taken $v_c(s)$ as the input parameter, and varied it with γ set equal to 1.250 to obtain smooth sequences of estimates for the remaining parameters. There are no consequential differences between the results of the two types of four fits.]

We thus vary ν between 0.625 and 0.650. For all values of s the convergence was greatly degraded if ν was chosen outside the range $0.633<\nu<0.643$. We therefore varied ν in steps of 0.001 through this narrower range. In this range, the estimates for v_c, Δ_1, $\tilde{A}_2(s)$, and $B_2(s)$ exhibited relatively good apparent convergence for all ν. Nevertheless the variation in apparent convergence was sufficient so that we could, in all cases, distinguish a "best" value for ν. Another, independent criterion by which to choose ν is how well the four-fit analysis reproduces the critical point, as determined from our prior analyses of the susceptibility. The critical point $v_c(s)$ determined by choosing ν so as to optimize apparent convergence agreed with that determined from the susceptibility analysis to at least one part in 10^5 for all values of s; so there is no apparent conflict between the two methods of choosing ν.

We show in Table VII the ν variation of the sequence of estimates in the case $s=\frac{1}{2}$. Except for the values of Δ_1 and $\tilde{A}_2$ this is typical of the results for larger spin values also. The apparent convergence of the estimates for $v_c(\frac{1}{2})$ is good for all three ν sequences shown. However, the $\nu=0.638$ sequence shows much better apparent convergence in the estimates for Δ_1 and $\tilde{A}_2$. Further, the estimate for $v_c(\frac{1}{2})$ obtained from the $\nu=0.638$ sequence is within one part in 10^5 of that from susceptibility analysis,[2] while those obtained from the $\nu=0.643$ and $\nu=0.633$ sequences are, respectively, 20 and 30 parts in 10^5 different from $v_c(\frac{1}{2})$ as determined from analysis of the susceptibility.[2] The apparent convergence of the $\nu=0.637$ and $\nu=0.639$ sequences is nearly as good as that of the $\nu=0.638$ sequence, while the $\nu=0.636$ and $\nu=0.640$ sequences are noticeably less well converged than the $\nu=0.638$ sequence. Thus we would quote $\nu(S=\frac{1}{2})=0.638\pm0.002$ with reasonable confidence.

TABLE VII. Variation of the sequence of four-fit estimates for v_c, Δ_1, $\tilde{A}_2$, and B_2 as a function of the value of ν assumed in the analysis, for the case $s=\frac{1}{2}$. [See Eq. (13).]

	$v_c(\frac{1}{2})$	Δ_1	$\tilde{A}_2$	B_2
	$\nu=0.633$			
4	0.102 05	1.221	11.258	0.0723
5	0.102 06	1.262	11.270	0.0731
6	0.102 06	1.282	11.275	0.0739
7	0.102 07	1.336	11.284	0.0772
8	0.102 07	1.403	11.293	0.0830
9	0.102 07	1.471	11.300	0.0910
10	No root with $\Delta_1<1.50$ was found			
	$\nu=0.638$			
4	0.102 11	1.098	11.117	0.0839
5	0.102 10	1.090	11.114	0.0839
6	0.102 10	1.063	11.103	0.0835
7	0.102 10	1.062	11.103	0.0835
8	0.102 10	1.066	11.104	0.0836
9	0.102 10	1.066	11.104	0.0836
10	0.102 10	1.066	11.104	0.0836
	$\nu=0.643$			
4	0.102 16	0.980	10.925	0.1013
5	0.102 15	0.938	10.895	0.1027
6	0.102 14	0.884	10.857	0.1038
7	0.102 13	0.853	10.837	0.1042
8	0.102 13	0.829	10.821	0.1042
9	0.102 12	0.802	10.805	0.1041
10	0.102 12	0.778	10.790	0.1039

In Table VIII we list our best estimates for the critical parameters of μ_2—as determined from this kind of four-fit analysis—for all values of spin studied. We list the value of $v_c(s)$ determined by the apparent-convergence criterion. These estimates for $v_c(s)$ are, as noted above, in com-

TABLE VIII. Best values for ν, Δ_1, $v_c(s)$, $\tilde{A}_2$, and B_2 from four-fit analysis. [See Eq. (13).] The uncertainties listed are based on smoothness of the sequence of four fits.

s	$v_c(s)$	ν	Δ_1	$\tilde{A}_2(s)$	$B_2(s)$
$\frac{1}{2}$	0.102 10 ± 1	0.638 ± 2	1.07 ± 8	11.104 ± 60	0.084 ± 2
1	0.097 77 ± 1	0.637 ± 3	0.68 ± 2	8.727 ± 60	0.381 ± 17
$\frac{3}{2}$	0.096 52 ± 1	0.638 ± 2	0.63 ± 1	7.853 ± 60	0.534 ± 15
2	0.095 97 ± 1	0.637 ± 3	0.62 ± 2	7.568 ± 90	0.594 ± 21
$\frac{5}{2}$	0.095 68 ± 1	0.635 ± 4	0.62 ± 2	7.488 ± 80	0.612 ± 19
∞	0.095 06 ± 2	0.635 ± 4	0.61 ± 2	6.948 ± 60	0.751 ± 25

plete agreement with those estimated from susceptibility analysis.[2] Except for $s=\frac{5}{2}$ and ∞ all estimates for ν are clustered at $\nu \simeq 0.637-0.638$. For $s=\frac{5}{2}$ and ∞ the apparent convergence was best for $\nu = 0.635$. However, it was very good for a fairly wide range of values of ν, and we would *not* say that these estimates were significantly out of line with the higher estimates for other spins. Rather, we take this difference to be an indication of the minimum uncertainties of the four-fit analysis.

As we have seen above in the Baker-Hunter analysis the magnitude of the leading corrections to scaling decreases as s decreases, and we find a changeover from $\Delta_1 \simeq 0.6$ for large s to $\Delta_1 \simeq 1$ at $s=\frac{1}{2}$. This is consistent with the Baker-Hunter analysis, and with our interpretation that the leading correction to scaling disappears at $s=\frac{1}{2}$, where it is replaced by the analytic correction with exponent equal to unity.

In summary, four-fit analysis (i) confirms Baker-Hunter analysis quantitatively, (ii) is in agreement with susceptibility analysis[1,2] (with the minor exception that Δ_1 is estimated here to be ~0.6, rather than 0.5), (iii) finds a universal value $\nu = 0.638^{+0.002}_{-0.008}$ for the correlation-length exponent, and (iv) is consistent with corrections to scaling, with exponent $\Delta_1 \sim 0.6$, whose amplitude vanishes at $s=\frac{1}{2}$.

IV. SPECIFIC-HEAT ANALYSIS

The series for the internal energy and specific heat have traditionally been the least well behaved of the standard high-temperature series.[21] This poor behavior is manifested in our analysis by the failure of Baker-Hunter and four-fit analyses to provide reasonable results for the spin-s specific-heat series for $s > \frac{1}{2}$. For spin-$\frac{1}{2}$, the four-fit analysis again fails to produce convergent estimates for the critical parameters of the specific heat. The Baker-Hunter analysis of the spin-$\frac{1}{2}$ specific heat is not terribly good. However, using the 14-term spin-$\frac{1}{2}$ specific-heat series of Sykes *et al.*,[21] we estimate that $\alpha \simeq 0.12 \pm 0.01$ if the critical point is forced to be that estimated from susceptibility and moment-series analyses. No secondary singularity was seen in the analysis. (However, a large additive constant would not be found by Baker-Hunter analysis, but could seriously deteriorate convergence of the analysis.)

In the absence of definitive results from Baker-Hunter and four-fit analyses, we turned to analysis based on ratio methods. One of the major difficulties encountered in studying the internal energy and specific-heat series is that there are expected to be large additive constant terms in both functions at the critical point.[10,15] Ratio analyses are somewhat less sensitive to such terms than are, for example, Padé methods.[11] Thus, while the analysis reported below is far from definitive, it cannot be dismissed as evidence on the question of hyperscaling.

As noted above, Sykes *et al.*[21] concluded on the basis of rather direct ratio methods that $\alpha = 0.125$ for the spin-$\frac{1}{2}$ Ising model. We have performed ratio analyses of the energy density $\epsilon(t)$, the specific heat c_H, and the temperature derivative of the specific heat, as well as Padé analysis of the specific heat for $s=\frac{1}{2}$, $s=\frac{5}{2}$, and $s=\infty$. The methods of end shifts[8] and Neville tables[18] were used in the ratio analyses. In both types of analysis the critical point was specified as an input parameter of the analysis and chosen to agree with the "best" critical point from analysis of susceptibility and moments. Similarly, estimates for the critical exponent were obtained by evaluating Padé approximants to $t\, d \ln c_H/dt$ at $t=0$, where $t = 1 - T_c/T$ and T_c is obtained from susceptibility analysis.

The most singular part of the internal energy, the specific heat, and the temperature derivative of the specific heat are, respectively, expected to behave as $t^{1-\alpha}$, $t^{-\alpha}$, and $t^{-(1+\alpha)}$ in the critical region ($t \approx 0$). Based on our analyses of these functions we would quote an *apparent* value $\alpha \simeq 0.12 \pm 0.02$ for $s=\frac{1}{2}$, $s=\frac{5}{2}$, and $s=\infty$. Rather than show our analysis for all cases we display in Table IX the end-shift analysis of c_H for $s=\frac{1}{2}$, and in Table X the Neville analysis of $\epsilon(t)$ for $s=\frac{5}{2}$ and $s=\infty$. The results are entirely typical of the other analyses performed. The end-shifted ratio analysis of $c_H(s=\frac{1}{2})$ displayed in Table IX is quite well converged to $\alpha \simeq 0.11$. The small value of the end shift $\Delta \sim 0.2$ indicates at most small corrections to scaling.[2] The Neville analysis (Table X) of $\epsilon(s=\frac{5}{2})$ and $\epsilon(s=\infty)$ is similarly indicative that $\alpha \simeq 0.12$. For $s=\infty$ the second-order estimates, $\alpha_{l,2}$ are particularly well behaved and indicate that $\alpha \simeq 0.125$.

To see whether the apparent convergence to $\alpha \simeq 0.11-0.12$ is masking hidden (weaker) singularities we have formed the functions $f_1(t)=t^{\alpha-1}\epsilon(t)$, $f_2(t)=t^{\alpha}c_H(t)$, and $f_3(t)=t^{\alpha+1}dc_H(t)/dt$. Two choices were made for α: $\alpha \simeq \frac{1}{8}$, to agree with series estimates, and $\alpha=\frac{1}{12}$, which is consistent with the hyperscaling relation $d\nu = 2-\alpha$. The series for f_1, f_2, and f_3 were then analyzed by ratio methods as well as by the Baker-Hunter transformation.

In Table XI we display the end-shift analysis of the series for f_2 for the spin-$\frac{1}{2}$ case. The first analysis listed is that obtained by assuming that $\alpha=\frac{1}{12}$; the second is that obtained by assuming $\alpha=\frac{1}{8}$, and the third is that obtained by assuming $\alpha=0.11$—the Table IX series estimate for $s=\frac{1}{2}$. We interpret these results as follows. Assume, following Sykes *et al.*,[21] that $c_H \approx At^{-\alpha}-B$. Then $t^{\alpha}c_H \approx A - Bt^{\alpha}$. That is, the reduced series should have a branch-cut singularity with critical exponent, $-\alpha$. If the value of α is underestimated (overestimated) in forming the reduced series f_2, the apparent exponent from end-shift analysis will be smaller (larger) than the inputted estimate for α. *With this interpretation,* a scan through the results in Table XI indicates that the estimate $\alpha=\frac{1}{12}$ is badly inconsistent with the series, while $\alpha=\frac{1}{8}$ is closer to consistency and $\alpha=0.11$ is fully consistent with the series. To see whether there

TABLE IX. End-shifted ratio analysis of c_H for $s=\frac{1}{2}$. $v_c(\frac{1}{2})$ is chosen to equal 0.102 10 in agreement with moment analysis. The estimates α_n obtained by using nth-order series for $n=7,8,\ldots,14$ are shown. The end shift Δ_n is also shown. (Series taken from Ref. 21.)

n	α_n	Δ_n
7	0.052	0.28
8	0.242	−1.18
9	0.210	−0.89
10	0.158	−0.35
11	0.126	0.01
12	0.114	0.16
13	0.112	0.19
14	0.113	0.17

TABLE X. Neville analysis of the internal energy $\epsilon(t)$ for $s=\frac{5}{2}$ and $s=\infty$. The $s=\frac{5}{2}$ and $s=\infty$ critical points are forced, respectively, to equal $v_c(\frac{5}{2})=0.095\,66$ and $v_c(\infty)=0.095\,05$. Assuming $\epsilon(t)\sim t^{1-\alpha}$, mth-order Neville estimates using coefficients c_n up to order l are shown for $l=6,\ldots,12$ and $m=1$, 2, and 3. (See Ref. 18 for the definition of the Neville estimates $\alpha_{l,m}$ in terms of the series coefficients $\{c_n\}$.)

	l	$\alpha_{l,1}$	$\alpha_{l,2}$	$\alpha_{l,3}$
$s=\frac{5}{2}$	6	0.34	0.02	−0.55
	7	0.25	0.09	0.16
	8	0.24	0.19	0.33
	9	0.20	0.11	−0.02
	10	0.19	0.13	0.17
	11	0.17	0.12	0.11
	12	0.16	0.12	0.11
$s=\infty$	6	0.34	−0.09	−0.85
	7	0.28	0.17	0.42
	8	0.25	0.18	0.20
	9	0.22	0.11	−0.01
	10	0.20	0.14	0.18
	11	0.19	0.12	0.10
	12	0.17	0.12	0.12

TABLE XI. End-shift ratio analysis of $t^{\alpha}c_H(t)$ for the spin-$\frac{1}{2}$ Ising model. Again, the critical point is set equal to $v_c(\frac{1}{2})=0.102\,10$. Results for the exponent $\overline{\alpha}$ of the singular part of $t^{\alpha}c_H(t)$, as well as for the end shift (see Ref. 8) are shown for the three assumed values $\alpha=\frac{1}{12}$, $\alpha=\frac{1}{8}$, and $\alpha=0.11$.

	n	$\overline{\alpha}$	Δ_n
$\alpha=\frac{1}{12}$	7	−0.071	−0.23
	8	0.133	−1.52
	9	0.082	−1.14
	10	0.010	−0.52
	11	−0.032	−0.12
	12	−0.046	0.03
	13	−0.047	0.04
	14	−0.045	0.01
$\alpha=\frac{1}{8}$	7	−0.147	−0.40
	8	0.072	−1.66
	9	0.007	−1.21
	10	−0.079	−0.53
	11	−0.129	−0.10
	12	−0.145	0.05
	13	−0.146	0.06
	14	−0.143	0.03
$\alpha=0.11$	7	−0.122	−0.35
	8	0.092	−1.62
	9	0.032	−1.20
	10	−0.050	−0.53
	11	−0.097	−0.11
	12	−0.112	0.04
	13	−0.113	0.05
	14	−0.110	0.02

are other weaker corrections in c_H which could cause the apparent inconsistency of $\alpha = \frac{1}{12}$, we applied the Baker-Hunter transformation to $t^{1/12}c_H$ and $t^{1/8}c_H$. Both transformed series were poorly behaved and no weaker confluent singularity was detected in either case.

The analysis of f_2 for $s=\infty$ is fully consistent with that for $s=\frac{1}{2}$ shown in Table XI. Namely, if we assume that $c_H \approx At^{-\alpha}+B+\cdots$, then $\alpha=0.12$ is consistent with the $s=\infty$ series and $\alpha\simeq 0.08$ is badly inconsistent. Furthermore, the analysis of f_1 indicates that $\epsilon(t)\approx E_0 t^{1-\alpha}+E_1+E_2t+\cdots$, in accord with the analysis of f_2. Again, the choice $\alpha\approx\frac{1}{8}$ is consistent with the assumed form, and the choice $\alpha\simeq\frac{1}{12}$ is inconsistent with it.

The most striking fact about the specific-heat analysis is that the corrections to scaling found in the susceptibility and moments are apparently completely masked by the presence of a large additive constant in both the specific heat and internal energy. In fact, we find no evidence for corrections to scaling of the form predicted by renormalization-group theory. This may be due to the rather poor convergence of the series (as compared to the series for χ and μ_2). Alternately, it may provide a hint as to why the hyperscaling relation fails (if it indeed does fail).

In summary, we estimate that $\alpha\simeq 0.12\pm 0.02$ for $s=\frac{1}{2}$, $s=\frac{5}{2}$, and $s=\infty$ in agreement with previous $s=\frac{1}{2}$ estimates.[21] No evidence for confluent singularities is found—although such singularities, if weak, would be effectively masked by large additive constant contributions in the energy and specific heat.

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[2]W. J. Camp and J. P. Van Dyke, Phys. Rev. B 11, 2579 (1975). See W. J. Camp and J. P. Van Dyke [J. Phys. A 9, 731 (1976)] for a similar analysis of the spin-s Heisenberg susceptibility, moments, and crossover function.

[3]F. J. Wegner, Phys. Rev. B 11, 4529 (1972).

[4]Failure to incorporate the possibility of confluent corrections into the series analysis leads to an apparent weak spin dependence of, for example, the exponent γ in violation of universality. See D. Jasnow and M. Wortis, Phys. Rev. 176, 739 (1968).

[5]K. G. Wilson and J. Kogut, Phys. Rep. 12, 75 (1974). See also the succinct review of M. E. Fisher, Rev. Mod. Phys. 46, 597 (1974). Both the above contain extensive additional references.

[6]There is a distinction between the "true" correlation length (defined by the long-distance correlations) and the "effective" correlation length (defined from the behavior of spherical moments). A careful study of $s=\frac{1}{2}$ Ising correlations in $d=2$ and 3 by M. E. Fisher and R. J. Burford [Phys. Rev. 156, 583 (1967)] suggests that the corresponding exponents are, in fact, identical. We assume this identification in discussion, though, of course, our numerical work, based on μ_2, is limited to the exponent of the effective correlation length.

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[8]J. P. Van Dyke and W. J. Camp, Phys. Rev. B 9, 3121 (1974).

[9]See Fisher and Burford, Ref. 6.

[10]W. J. Camp, Phys. Rev. B 6, 960 (1972), and further references cited therein.

[11]See, e.g., M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967); or H. E. Stanley, *Phase Transitions and Critical Phenomena* (Oxford, New York, 1971).

[12]Fisher, Ref. 5. We follow Fisher in using the notation λ_k for the eigenvalues of the linearized recursion relations. Reference 3 denotes the same quantity y_k.

[13]We neglect here the complication of possible "marginal" fields with corresponding $\lambda_k=0$.

[14]Except, of course, for those special cases in which the correction amplitudes like $B_n(s)$ happen to vanish. References 1 and 2 suggest that this occurs for the $s=\frac{1}{2}$ Ising model.

[15]G. A. Baker, Jr. and D. L. Hunter, Phys. Rev. B 7, 3377 (1973).

[16]M. A. Moore, D. M. Saul, M. Wortis, J. Phys. C 7, 162 (1974); D. M. Saul, M. Wortis, and D. Stauffer, Phys. Rev. B 9, 4964 (1974).

[17]Direct analyses by conventional methods which do *not* explicitly allow for confluent singularities of the form (6) are plagued by strong monotonic trends in high-order approximants. If such difficulties are ignored, these analyses suggest a pronounced s dependence, $\nu(s=\frac{1}{2})=0.638$ and $\nu(\infty)\lesssim 0.625$, in strong violation of universality. In another context (that of crossover scaling) the apparent nonuniversality of series-determined exponents has been discussed by E. K. Riedel and F. J. Wegner [Phys. Rev. B 9, 294 (1974)].

[18]M. A. Moore, D. Jasnow, and M. Wortis, Phys. Rev. Lett. 22, 940 (1969). An earlier study of series for the $s=\frac{1}{2}$ simple-cubic Ising ferromagnet [M. E. Fisher and R. J. Burford, Phys. Rev. 156, 583 (1967)] found the somewhat higher estimate $\nu=0.6430\pm 0.0025$; however, the analysis is complicated by the presence of the large even-odd alternations associated with the existence of antiferromagnetic singularities.

[19]We remark here that although Refs. 1 and 2 conclude $\Delta_1 = 0.50 \pm 0.08$, there was one method of analysis (Baker-Hunter, Ref. 2) which consistently led to somewhat higher estimates in the range $\Delta_1 = 0.55$–0.60.

[20]H. B. Tarko and M. E. Fisher, Phys. Rev. Lett. 31, 926 (1973). The argument [M. E. Fisher (private communication)] runs as follows: If leading corrections to scaling are absent for the $d=3$ Ising model, then it is possible that the first corrections which *do* come in are of the form $t^{2\Delta_1} f(h/t^{\Delta})$ [a higher-order term in the expansion (5)]. On the critical isotherm ($t=0$), this leads to corrections varying with magnetic field as $h^{-\zeta}$ with $\zeta = 2\Delta_1/\Delta \simeq 0.77$, in precise agreement with the actual behavior found by Tarko and Fisher for the correlation length.

[21]M. F. Sykes, D. L. Hunter, D. S. McKenzie and B. R. Heap, J. Phys. A 5, 667 (1972). This work refers to $s=\frac{1}{2}$ only.

[22]Such corrections, expected phenomenologically, are generated within the renormalization-group framework by nonlinear corrections to the scaling fields.

[23]C. Domb, in ***Phase Transitions and Critical Phenomenon***, edited by C. Domb and M. S. Green (Academic, London, 1974), Vol. 3. See especially pp. 434 and 435 for a discussion and earlier references.

[24]A somewhat similar set of "compromise" exponents obeying both thermodynamics scaling and hyperscaling has been suggested by P. C. Hohenberg (private communication).

[25]M. Wortis, D. Jasnow, and M. A. Moore, Phys. Rev. 185, 805 (1969); M. Wortis, in Ref. 23, p. 114.

[26]W. J. Camp and J. P. Van Dyke (unpublished) have derived spin-s series through order K^{10} in which the coefficient of K^n is explicitly given as a finite polynominal in the variables $X = s(s+1)$.

[27]It is conventional to take the nearest-neighbor distance $|\vec{\delta}|$ equal to unity.

[28]Expansions in Ref. 2 were in terms of the variable K (the variable v denoted tanh$K$). Thus, critical points given herein differ from those of Ref. 2 by the scale factor $(s+1)/3s$.

[29]The coefficients $c_n(s)$ for $s=1$, 2, and ∞ appear in a modified form in Table VI of Ref. 23, p. 391.

[30]W. J. Camp and J. P. Van Dyke (unpublished); D. M. Saul and M. Wortis (unpublished).

[31]We note that — as with the susceptibility—attempts to use five-fit analysis in which both $\gamma + 2\nu$ and $v_c(s)$ are unknowns [or to replace $\gamma + 2\nu$ or $v_c(s)$ by Δ_1 as the input parameter in the method of four-fits] are notably unsuccessful.

[32]This is the best estimate for $v_c(\frac{1}{2})^{-1}$ determined from susceptibility analysis in Ref. 2. The best estimate obtained in Ref. 1 was about 0.001% higher. The results of our analysis for $s=\frac{1}{2}$, particularly as regards $\gamma + 2\nu$ and Δ_1 are insensitive to such a difference.

[33]In the Baker-Hunter analysis of μ_2, we obtain ν consistently for a given value of $v_c(s)$ by using the Baker-Hunter estimate for γ obtained from susceptibility analysis using the same value of $v_c(s)$. For $s=\frac{1}{2}$ and $s=\frac{5}{2}$ this is 1.25, but for $s=\infty$ exponents in the range $\gamma \simeq 1.245$–1.247 were consistently produced by Baker-Hunter analysis of χ.