Critical behavior of random $n \geq 4$ vector models studied by the renormalization-group technique in $4-\epsilon$ dimensions: Crossover from first-order to smeared transitions*

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The phase transitions in a large class of physical systems are described by $n \geq 4$ component order parameters. Here, the critical behavior of quenched random $n \geq 4$ vector models is studied by means of renormalization-group theory in $4-\epsilon$ dimensions. Recursion relations for average potentials are constructed following the methods derived by Lubensky. For several Hamiltonians describing homogeneous $n \geq 4$ systems there exist no stable fixed points in $4-\epsilon$ dimensions which explains the first-order transitions actually observed in these systems. It is shown that the recursion relations for the corresponding quenched random systems are also unstable. However, the runaway in this case is of a fundamentally different nature. The fluctuations of the local mean-field transition temperature diverge, and this behavior is interpreted as a "smeared" transition. This interpretation is consistent with existing experiments. On the other hand, in the cases where the homogeneous Hamiltonian possesses a stable fixed point in $4-\epsilon$ dimensions, this fixed point remains stable against random perturbations, so no change in the critical behavior is expected. For most of the models studied there is at least one fixed point of order $\epsilon^{1/2}$. These fixed points are all unstable. It is suggested that experiments should be performed to determine the critical behavior of random $n \geq 4$ systems. Of particular interest are the systems with no stable fixed points, such as Cr, Eu, MnO, and UO_2, where crossover from first-order to a smeared transition is predicted.

I. INTRODUCTION

The nature of the phase transitions in quenched random systems,[1] or systems with "frozen" impurities, has recently drawn more and more attention. Whereas the phase transitions in annealed disordered systems where impurities can diffuse freely to reach thermal equilibrium are rather well understood,[2] only little is known about critical properties of quenched systems. Very few rigorous results exist. McCoy and Wu[3] have solved exactly a disordered two-dimensional model in which all the vertical bonds in any horizontal row are identical. They find a "smeared" or rounded phase transition. An interesting question is whether this behavior is specific for systems with long-range correlation of impurities or whether it is a general feature of disordered systems. No rigorous results exist for more realistic models, and no conclusive experiments have been performed.

A convenient approximate method of calculating critical properties of real three-dimensional systems is the renormalization-group theory, or the ϵ expansion. For pure systems the ϵ expansion has been very successful in calculating critical exponents,[4] and also in predicting the order of the phase transition.[5] Generally, the ϵ expansion assumes translational invariance, which is not fulfilled for quenched disordered systems. Lubensky[6] and Grinstein and Luther[7] have independently extended the ϵ-expansion formalism to quenched random systems. Grinstein and Luther derived an effective translational invariant Hamiltonian which leads to the same free energy as the original Hamiltonian,[8] and performed the ϵ expansion on this effective Hamiltonian, whereas Lubensky directly constructed the recursion relations for the probability distributions for random potentials. These theories can be shown to be equivalent. The *isotropic* n-vector model was studied by both groups. The main result was, that if there are no long-range correlations of the random potential there is a *sharp* transition with pure-system exponents if the specific-heat exponent α of the pure system is negative, and if α is positive there is a sharp transition with *new* exponents. For $n \leqslant 3$ it can be shown that anisotropy does not affect the critical behavior of pure systems,[9] since the isotropic fixed point is always stable to first order in ϵ. It can be shown (see Sec. II) that for $n \leqslant 3$ the corresponding *random* isotropic fixed point is always stable. Hence for $n \leqslant 3$ one should always expect sharp transitions. (A special case is the random Ising model where a fixed point of order $\epsilon^{1/2}$ has been found.[10]) The effects of randomness, if any, are difficult to observe experimentally for two reasons. First, the exponents of the pure and random fixed points differ only insignificantly, and second, one cannot avoid macroscopic gradients of impurity concentration through the crystal. Such unintended inhomogeneities will cause a rounding of the transition temperature and obscure the intrinsic critical behavior.[11]

Recently, it was pointed out by Mukamel[12] that the phase transitions in certain physical systems in which the magnetic unit cell is doubled should be described by $n \geq 4$ component order parameters. This is a very important observation, since it turns out that many, if not most, magnetic phase transitions are $n \geq 4$ transitions. Mukamel, Krinsky, and Bak[5,13] constructed Ginzburg-Landau-Wilson Hamiltonians corresponding to several of these systems, and performed a renormalization-group analysis in $4-\epsilon$ dimensions. The first-order transitions in several types of antiferromagnets could be explained by noting that the corresponding Hamiltonians possess *no* stable fixed point. Examples of physical systems corresponding to these Hamiltonians are Cr ($n=12$), Eu ($n=12$), UO_2 ($n=6$), and MnO ($n=8$). The question now naturally arises what kind of critical behavior should be expected for the corresponding quenched random systems. There is no reason that a gradient of dilute impurities or imperfections should automatically cause a rounding of a first-order transition. Hence, if dilute impurities do in fact change the nature of the phase transition, this effect should be interpreted as a genuine intrinsic effect.

Let us briefly review the experimental situation. It turns out, surprisingly, that a rather extensive amount of neutron scattering experiments has been performed on Cr with various impurities.[14-18] Cr is one of the systems with a first-order transition generated by a lack of stable fixed point. A very interesting experiment was performed by Lebech and Mikke.[14] They found that very small amounts of Re (0.18 at.%) in Cr make the phase transition continuous. This observation seems to be consistent with measurements on Cr with impurities of Mn (0.05 at.%),[15] and Fe (0.37–0.5 at.%).[16] With larger concentrations of impurities, the magnetic structure usually becomes commensurate and, in agreement with a simple Landau argument,[5] the phase transition again becomes of first order.[14,17] However, in this context, we are only interested in the dilute case. Another interesting feature is that for[18] Eu early experiments indicated a *continuous* transition. The purity of the samples used in these experiments was reportedly lower than that of the sample used in the experiment of Cohen *et al.*[19] which revealed the first-order transition. Also for MnO it is important to have a sample of good quality in order to observe the first-order transition.[20] These experiments clearly indicate that impurities and other imperfections are important in determining the critical behavior of random $n \geq 4$ systems. Whether the "continuous" phase transition observed in these experiments should be interpreted as a sharp second-order phase transition with temperature-independent exponents or a "smeared" transition in the McCoy-Wu sense is not yet clear. The experiments were certainly not set up to study critical properties of random systems. In any case, these experiments provide sufficient motivation to study these transitions from a theoretical point of view.

In this paper the critical properties of random $n \geq 4$ models are studied by means of renormalization-group theory in $4-\epsilon$ dimensions, using Lubensky's formalism. This paper is organized as follows. In Sec. II the general structure of the recursion relations for the average potentials and higher cumulants is derived. The disorder is characterized by a single variable Δ which behaves like a quartic potential. This variable may or may not be relevant in the Wilson sense[4] according to the symmetry of the Hamiltonian. It is shown that in general the number of fixed points is doubled when randomness is included, and the new fixed points and their exponents are related to the pure fixed points in a simple way. We also show that for $n \leq 3$ there is always a stable fixed point independent of anisotropy. In Sec. III the random $n \geq 4$ models corresponding to the homogeneous Hamiltonians with no stable fixed points are studied. It turns out the randomness does not create stable fixed points for *any* of the models studied by Mukamel, Krinsky, and Bak. Superficially one might be tempted to conclude that the transitions should be of first order. However, it has previously been pointed out by Aharony[21] and Lubensky[6] that the "runaway" for random systems is of a fundamentally different nature than the runaway for homogeneous Hamiltonians. Therefore, one should rather expect a *smeared* transition. Once the "randomness" is turned on, the runaway is of the same nature whether or not the original Hamiltonian had a stable fixed point. On the other hand, in the cases where the pure system has a stable $n \geq 4$ fixed point, this fixed point remains stable with respect to random perturbations. This reflects the fact that the critical exponent α is negative. Hence, random impurities should not influence the critical properties of these systems. For some of the Hamiltonians, namely, those corresponding to $n=4$, the stability of the fixed point is *marginal* to first order in ϵ. The conclusions above are not affected by this fact. In Sec. IV it is shown that most of the random Hamiltonians have at least one fixed point of order $\epsilon^{1/2}$. For the $n=4$ Hamiltonian describing type-II antiferromagnets with $\vec{m} \parallel \vec{k}$ we determine explicitly *three* such fixed points. However, it turns out that these fixed points are always unstable and hence probably without physical significance. In addition, there are always unphysical fixed points which can never be reached. Finally, the results are summarized and discussed in Sec. V, and specific experiments are proposed.

II. RENORMALIZATION-GROUP THEORY FOR RANDOM SYSTEMS

In this section we shall briefly review the renormalization-group theory for quenched random systems and extend the formalism to include all possible fourth-order anisotropy terms of the Ginzburg-Landau-Wilson (GLW) Hamiltonian corresponding to the pure system. This fourth-order anisotropy plays a crucial role for the critical behavior of both homogeneous and random $n \geq 4$ systems.

The most general GLW Hamiltonian for the pure system, including terms of up to fourth order in the order parameter may be written

$$\mathcal{H}_p = \int d^d x \left(\sum_{i=1}^{n} -\tfrac{1}{2}\{r\phi_i^2(x) + [\nabla\phi_i(x)]^2\} - \sum_p u^p \sum_{ijkl} \beta^p_{ijkl}\, \phi_i(x)\phi_j(x)\phi_k(x)\phi_l(x) \right), \tag{1}$$

where ϕ_i are the n components of the order parameter, and the sum $\sum_{p=1}^{l}$ is over all the possible fourth-order invariants O_p of the space group which can be formed by the components of the order parameter. However, in order to describe the corresponding *random* system, we rewrite this Hamiltonian on the more general form

$$\begin{aligned}\mathcal{H}_r = &\int d^d x_1\, d^d x_2 \sum_{i=1}^{n} -\tfrac{1}{2}[r'(x_1,x_2)\phi_i(x_1)\phi_i(x_2)] \\ &+ \int d^d x_1 \cdots d^d x_4 \sum_p -u^p(x_1,x_2,x_3,x_4) \sum_{ijkl} \beta^p_{ijkl}\, \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4) \\ &+ \int d^d x_1 \cdots d^d x_4 \sum_{p'} -w^{p'}(x_1,x_2,x_3,x_4) \sum_{ijkl} \beta^{p'}_{ijkl}\, \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\,.\end{aligned} \tag{2}$$

The coupling constants $r'(x_1, x_2)$ and u^p are now position dependent,[22] since the random system is not translationally invariant. Moreover, since all the symmetry elements of the homogeneous system are absent, there are additional fourth-order terms not invariant under the space group of the pure system. The random system is defined in terms of a probability distribution of the potentials $r'(x_1, x_2)$, u^p, and $w^{p'}$. Mathematically, the free energy is obtained by averaging the logarithm of the partition function over this probability distribution function.

We now transform the reduced Hamiltonian into momentum space:

$$\begin{aligned}\mathcal{H} = &-\frac{1}{2}\int_{\vec{q}_1,\vec{q}_2} \sum_{i=1}^{n} v_2(\vec{q}_1,\vec{q}_2)\phi_i(\vec{q}_1)\phi_i(\vec{q}_2) \\ &- \int_{\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4} \left(\sum_p v_4^p(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4) \sum_{ijkl} \beta^p_{ijkl}\, \phi_i(\vec{q}_1)\phi_j(\vec{q}_2)\phi_k(\vec{q}_3)\phi_l(\vec{q}_4) \right. \\ &\left. + \sum_{p'} v_4^{p'}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4) \sum_{ijkl} \beta^{p'}_{ijkl}\phi_i(\vec{q}_1)\phi_j(\vec{q}_2)\phi_k(\vec{q}_3)\phi_l(\vec{q}_4) \right).\end{aligned} \tag{3}$$

The randomness is now characterized by a probability distribution function of the potentials in $\vec{q}$ space, $P(\{v_l^p(\vec{q}_1, \ldots, \vec{q}_l)\})$. The renormalization group operates on the potentials $\{v_l^p(\vec{q}_1, \ldots, \vec{q}_l)\}$ of a particular member of the ensemble and transform it to a new $\vec{q}$-dependent set of potentials, $\{v_l'^p(\vec{q}_1, \ldots, \vec{q}_l)\}$, where because of the change of scale, the $\vec{q}$ spaces of the initial and the transformed potentials are the same. In this way the renormalization group operation transforms $P(\{v_l^p\})$ to a *new* probability distribution $P'(\{v_l^p\})$. The probability distribution function can alternatively be described in terms of its cumulants C_k. The lowest-order cumulants are $\langle v_l \rangle$ and $\langle v_l v_m \rangle$, where the angular brackets denote averages over $P(\{v_k\})$. The recursion relations for the *potentials* can thus be converted to recursion relations for the *cumulants*. Following Lubensky,[6] recursion relations for $\langle v_2 \rangle$ and $\langle v_4^p \rangle$ can be constructed by averaging the recursion relations for the general inhomogeneous potentials. Since the *average* potentials transform according to the full space group of the pure system we see immediately

$$\langle v_2(\vec{q}_1,\vec{q}_2)\rangle = (r + q_1^2)\delta(\vec{q}_1 + \vec{q}_2)\,, \tag{4}$$

$$\langle v_4^p(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\rangle = u_p\,\delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4)\,, \tag{5}$$

$$\langle v_4^{p'}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\rangle = 0\,, \tag{6}$$

where we have expanded the second-order potential in the long-wavelength limit and, as usual in renormalization-group theory, suppressed the $\vec{q}_i$ dependence of the fourth-order potentials.

In addition to the recursion relations for the average potentials, we also construct recursion relations for the quantity $\langle \delta v_2 \delta v_2 \rangle$, where $\delta v_2(\vec{q}_1, \vec{q}_2) = v_2(\vec{q}_1, \vec{q}_2) - \langle v_2(\vec{q}_1, \vec{q}_2)\rangle$. In the long-wavelength limit we have

$$\langle \delta v_2(\vec{q}_1, \vec{q}_2) \delta v_2(\vec{q}_3, \vec{q}_4)\rangle = \Delta \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4), \quad (7)$$

where Δ is proportional to $\langle \delta v_2(x,x) \delta v_2(x,x)\rangle$. Since $v_2(x,x) \sim T - T_c(x)$, where $T_c(x)$ is the *local* mean-field transition temperature, Δ is simply proportional to the variance of the fluctuations in the transition temperature. We note that Δ by definition is a positive quantity. Therefore, we must reject any solution of the recursion relations with Δ negative as unphysical. It turns out that Δ enters the recursion relations on an equal footing with the quartic potentials, but it should be emphasized that Δ is *not* a coefficient of a term in an effective Hamiltonian.

In general the recursion relations for the *homogeneous* system to second order in $\epsilon = 4 - d$ may be written[4]

$$r' = b^2 \left[r \left(\sum_p c_p u_p \right) A(r) \right],$$
$$u_p' = b^\epsilon \left[u_p - \left(\sum_{p'p''} d^p_{p'p''} u_{p'} u_{p''} \right) K_4 \ln b \right]. \quad (8)$$

Here b is the cutoff ratio in momentum space

$$K_d = 2^{(1-d)} \pi^{-d/2} [\Gamma(\tfrac{1}{2}d)]^{-1} \quad (9)$$

and

$$A(r) = \int_{b^{-1} \leq |q| \leq 1} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + r}. \quad (10)$$

Using the diagram technique, we construct the corresponding equations for the random system. They are, again to second order in the coupling constants,

$$r' = b^2 \left[r + \left(\sum_p c_p u_p - \Delta \right) A(r) \right],$$
$$u_p' = b^\epsilon \left[u_p - \left(\sum_{p'p''} d^p_{p'p''} u_{p'} u_{p''} - 6\Delta u_p \right) K_4 \ln b \right], \quad (11)$$
$$\Delta' = b^\epsilon \left[\Delta - \left(\sum_p 2c_p u_p \Delta - 4\Delta^2 \right) K_4 \ln b \right].$$

We note that the exponent ν, describing the divergence of the correlation length at the critical point is $\frac{1}{2} + \sum_p c_p / 4u_p^* K_4$ to first order in ϵ, where u_p^* are the fixed-point potentials. Using scaling we get

$$\alpha = -\nu d + 2 = \frac{1}{2} \left(\epsilon - \sum_p 2c_p u_p^* K_4 \right). \quad (12)$$

By inspection of the recursion relation for Δ, we note that the stability of the pure fixed point with respect to perturbations in Δ is determined by the sign of α. When α is positive the homogeneous fixed point is unstable with respect to randomness, in agreement with an argument of Harris.[23] This relation can be extended to higher order in ϵ.

Let us first determine the fixed points of (11) to order $O(\epsilon)$. Defining $x_i = (K_4/\epsilon) u_i$ and $y = (K_4/\epsilon)\Delta$, the equations for the fixed points become

$$x_p = \sum_{nm} d^p_{nm} x_n x_m - 6 y x_p, \quad (13)$$

$$y = \sum_p 2c_p x_p y - 4y^2. \quad (14)$$

Suppose the homogeneous system has the fixed point given by $x_p = x_p^*$, $p = 1, \ldots, l$. The random system obviously has a fixed point given by $x_p = x_p^*$, $y = 0$. Now, let us consider the case $y \neq 0$ ($\Delta \neq 0$). We define *new* parameters x_p' by means of the equations

$$x_p = (1 + 6y) x_p', \quad y \neq -\tfrac{1}{6}, 0 \quad (15)$$

and the fixed-point equations now read

$$x_p' = \sum_{nm} d^p_{nm} x_n' x_m', \quad (16)$$

$$y = \sum_p 2c_p (1 + 6y) x_p' y - 4y^2. \quad (17)$$

Equation (16) is identical to the equation determining the fixed point of the pure system, for which the solutions are known. Hence, if the pure system has a fixed point $x_p = x_p^*$, $y = 0$, then the random system has the same fixed point and, in addition, a new *random* fixed point

$$x_p^{R*} = (1 + 6y^*) x_p^*, \quad (18)$$

$$y^* = -\frac{1}{4} + \sum_p \frac{c_p}{2} x_p^{R*}. \quad (19)$$

Clearly, if Δ^* determined by Eq. (17) turns out to be 0, no new fixed point has been obtained. Of more interest is the fact that if $\sum_p c_p x_p^* = \frac{1}{3}$, then Eq. (17) has no solution for Δ. According to Eq. (12) this case occurs whenever $\nu = \frac{1}{2} + \frac{1}{12}\epsilon$ or $\alpha = \frac{1}{6}\epsilon$ for the pure system. In Sec. IV we shall see that in this case there is a fixed point of order $\epsilon^{1/2}$ similar to the one obtained by Khmel'nitsky.[10]

We conclude that it is a rather trivial matter to solve *completely* for the random fixed points, once the fixed points for the corresponding homogeneous problem are known. However, following this procedure, Δ^* may turn out to be negative and the fixed point is unphysical. For example, the random Gaussian fixed point corresponding to $x_p^* = 0$ for all p has $y = -\frac{1}{4}$ and should be rejected.

The interesting question now arises as to what determines the *stability* of the new fixed points. Before considering the general case with arbitrary anisotropic fixed points, let us prove that when $n \leqslant 3$, the isotropic random fixed point is always stable. Brezin *et al.*[9] showed that when $n \leqslant 3$, the isotropic fixed point is always stable for the *pure* system. Let u_1 (or x_1) be the coefficient of the isotropic four-spin term of the Hamiltonian. The recursion relations are now

$$u_1' = u_1 + \left[\epsilon u_1 - \left(4(n+8)u_1^2 + \sum_{p'\neq 1} d^1_{p'1} u_{p'} u_1 + \sum_{p'p''\neq 1} d^1_{p'p''} u_{p'} u_{p''} - 6u_1\Delta \right) K_4 \right] \ln b ,$$

$$\Delta' = \Delta + \left[\epsilon\Delta - \left(8(n+2)u_1\Delta - 4\Delta^2 + \sum_{p\neq 1} 2c_p u_p \Delta \right) K_4 \right] \ln b , \qquad (20)$$

and

$$u_p' = u_p + \left[\epsilon u_p - \left(d^p_{p1} u_p u_1 + \sum_{p'p''} d^p_{p'p''} u_{p'} u_{p''} - 6u_p\Delta \right) K_4 \right] \ln b ,$$

when $p \neq 1$. The recursion relations for u_p, $p\neq 1$, do not contain terms proportional to u_1^2 or $u_1 u_{p'}$, where $p \neq p'$. The isotropic fixed point is

$$x_1^* = 1/4(n+8), \quad x^*_{p,p\neq 1} = 0, \quad y^* = 0 , \qquad (21)$$

where $y^* = \Delta K_4/\epsilon$, and the corresponding random fixed point is

$$x_1^{R*} = \frac{1}{16(n-1)}, \quad x^{R*}_{p,p\neq 1} = 0, \quad y^* = \frac{4-n}{8(n-1)} . \qquad (22)$$

To determine the stability of this fixed point, we linearize the recursion relations around the fixed point. Stability requires the following matrix to have negative eigenvalues only:

$$\begin{pmatrix} 1-8(n+8)x_1^{R*}+6y^* & -d^1_{p1}x_1^{R*} & 6x_1^{R*} \\ 0 & 1-d^p_{p1}x_1^{R*}+6y^* & 0 \\ -8(n+2)y^* & -2c_p y^* & 1+8y^*-8(n+2)x_1^{R*} \end{pmatrix} . \qquad (23)$$

However, since we know that the isotropic fixed point is stable for $n \leq 3$ for the pure system,

$$1 - d^p_{p1} x_1^* < 0 .$$

Using Eqs. (21) and (22) we find that this equation implies that the diagonal elements, $1 - d^p_{p1} x_1^{R*} + 6y^* < 0$. The stability is thus determined by the eigenvalues of the 2×2 submatrix

$$\begin{pmatrix} 1-8(n+8)x_1^{R*}+6y^* & 6x_1^{R*} \\ -8(n+2)x_1^* & 1+8y^*-8(n+2)x_1^{R*} \end{pmatrix} , \qquad (24)$$

and the problem is reduced to the case considered by Lubensky[6] and by Grinstein and Luther.[7] One finds that for $n<4$ the eigenvalues are all negative. Therefore, the isotropic random fixed point is always stable when $n<4$. For *real* systems in three dimensions, it turns out the exponent α seems to be negative, and one should expect the pure fixed point to be stable. In any case, for $n \leq 3$ one should always expect sharp transitions with concentration independent exponents, which do not differ significantly from those of the pure system, and no observable effects of randomness should be expected.

III. $n \geq 4$ HAMILTONIANS WITH NO STABLE FIXED POINTS

In Sec. II, the general structure of the recursion relations for random systems was outlined, and it was shown that the fixed points were closely related to those of the corresponding pure system. We saw, that when $n \leq 3$ one should, in general, not expect any change in critical behavior. In this section, the formalism is applied to random systems where the corresponding $n \geq 4$ Hamiltonians describing the homogeneous system do not possess stable fixed points. In particular, we shall consider the cases studied by Mukamel, Krinsky, and Bak.[5,13] For a list of physical systems corresponding to these Hamiltonians, see Ref. 5. The lack of stable fixed points explains the first-order phase transitions in several magnetic substances, such as Cr, Eu, MnO, and UO_2. However, as discussed in Sec. I, there is clear experimental evidence that the phase transitions in the corresponding quenched random systems may be continuous.[14-16,18]

Most of these Hamiltonians were unstable to first order in ϵ. Characteristic for the recursion relations[13] is that the fixed points are all unstable with respect to perturbations of a simple linear combination of the four-spin potentials, and that the fixed point value of this linear combination is zero. As an example, let us consider the $n=6$ Hamiltonian representing type-I antiferromagnets with $\vec{m}\perp\vec{k}$, such as UO_2 or $NdSn_3$. The space group has five fourth-order invariants, which can be formed by the six components of the order parameter. Mukamel and Krinsky[13] constructed the corresponding GLW Hamiltonian:

$$\mathcal{H}_4 = -\frac{1}{2}\sum_{i=1}^{3}[r(\phi_i^2+\overline{\phi}_i^2)+(\nabla\phi_i)^2+(\nabla\overline{\phi}_i)^2] - u_1\sum_{i=1}^{3}(\phi_i^4+\overline{\phi}_i^4) - u_2\sum_{i=1}^{3}\phi_i^2\overline{\phi}_i^2 - u_3\sum_{i<j}\phi_i^2\phi_j^2+\overline{\phi}_i^2\overline{\phi}_j^2$$
$$-u_4(\overline{\phi}_1^2\phi_2^2+\overline{\phi}_2^2\phi_3^2+\overline{\phi}_3^2\phi_1^2) - u_5(\phi_1^2\overline{\phi}_2^2+\phi_2^2\overline{\phi}_3^2+\phi_3^2\overline{\phi}_1^2)\,. \tag{25}$$

ϕ_i, $\overline{\phi}_i$, $i = 1, 2, 3$ are the six components of the order parameter. For several fixed points, $u_4^* = u_5^* = 0$. The remaining fixed points are related to these fixed points through symmetry relations. The recursion relations for $u_4 - u_5$ can be written

$$u_5' - u_4' = (1 + \lambda\epsilon \ln b)(u_5 - u_4)\,, \tag{26}$$

where

$$\lambda = 1 + 4x_3^* - 24x_1^* - 8x_4^* - 8x_5^* \tag{27}$$

and $\lambda > 0$ for all fixed points (with $x_4^* - x_5^* = 0$). Relations similar to (26) and (27) also hold for the $n = 12$ Hamiltonians describing Eu and Cr,[13] and for the $n = 8$ Hamiltonian describing phase transitions in type-II antiferromagnets of the MnO-type with $\vec{m} \perp \vec{k}$.

Now, suppose that the homogeneous Hamiltonian has a fixed point $u_1^*, u_2^*, \ldots, u_l^*$ where $\sum_p A_p u_p^* = 0$, and the recursion relation for this linear combination is

$$\sum_p A_p u_p' = (1 + \lambda\epsilon \ln b)\sum_p A_p u_p\,, \tag{28}$$

with

$$\lambda = 1 - \sum_p a_p x_p^* > 0\,. \tag{29}$$

The corresponding recursion relation for the random system is, according to Eq. (11),

$$\sum_p A_p u_p' = (1 + \lambda'\epsilon \ln b)\sum_p A_p u_p\,, \tag{30}$$

with

$$\lambda' = 1 - \sum_p a_p x_p^{R*} + 6y^*\,, \tag{31}$$

where x_p^{R*} and y^* are related to x_p^* through Eqs. (17) and (18). Inserting $x_p^{R*} = x_p^*(1+6y^*)$, we find

$$\lambda' = 1 - \sum_p a_p x_p^*(1+6y^*) + 6y$$
$$= \left(1 - \sum_p a_p x_p^*\right)(1+6y^*) = \lambda(1+6y^*)\,. \tag{32}$$

Since Δ (and y) are positive definite, and λ itself is positive, λ' can never be negative. Therefore, *there can be no stable fixed point for the random system*. Mukamel and Krinsky also studied an $n = 4$ model which is marginally stable to first order in ϵ, but unstable to second order in ϵ. In the Appendix it is shown that if randomness is included, the Hamiltonian remains unstable. We may, therefore, conclude that for all the $n \geq 4$ models with no stable fixed points, the corresponding random "Hamiltonian" *also* diverges. Since the lack of stable fixed point in pure systems indicates a first order transition, one might be tempted to draw the conclusion that the transition should remain first order. However, one must remember, that although Δ is treated in a way perfectly similar to the fourth order potentials the physical significance of these quantities is different. For the homogeneous system, the runaway reflects a "blow-up" of the fourth-order terms of the free energy as a function of the order parameters. For the random system not only the corresponding "averaged" potentials but *also* Δ goes off to infinity. From the structure of the recursion relations [Eq. (11)] it follows that a divergence of the potentials u_p necessarily drives a divergence of Δ. Recalling that Δ essentially is proportional to the fluctuations of the local mean-field transition temperature, it seem very likely that the "runaway" should be interpreted as a "smearing" of the transition. Lubensky[6] and Aharony[21] have presented arguments supporting this interpretation. However, it must be stressed that the runaway carries the Hamiltonian out of the range where our approximations are valid, and independent calculations are needed to identify the nature of the transition.

The ultimate test of the validity of the predictions, of course, is experiment. Existing experiments clearly indicate that the transition *does* indeed become continuous, which in itself is remarkable for first-order systems. Since the random Hamiltonian does not have a stable fixed point, as it should whenever the transition is second order,[4] we predict that this transition is *not* a sharp second-order transition. The present calculation suggests that the transition should be rounded or smeared. It would be very interesting to analyze existing experiments on this basis, and also one should perform similar experiments on the other systems mentioned in Ref. 5. In particular, one should test whether the transition becomes "smeared" as predicted in this work, or rather should be interpreted as second order with concentration-independent exponents, as seen to be the case for $n \leq 3$ systems.[11]

Finally, we note that for some of the $n \geq 4$ models studied by Mukamel, Krinsky, and Bak,[13] the

Hamiltonian does indeed possess one stable fixed point. It turns out that the specific-heat exponent α corresponding to this fixed point is always *negative*. Hence, one should expect *sharp* second-order transitions for the corresponding randomized models, just as when $n \leq 3$. Of course, this transition might be smeared by a macroscopic gradient of impurities.[11]

IV. FIXED POINTS OF ORDER $\epsilon^{1/2}$

It has been shown by Khmel'nitsky[10] that for the random Ising model ($n=1$) there exists a fixed point of order $\epsilon^{1/2}$. In this section, it will be demonstrated that this is not an isolated phenomenon, but a very common feature for anisotropic random systems.

Let us consider the case that the critical exponent ν for one of the fixed points of the homogeneous system is $\frac{1}{2}+\frac{1}{12}\epsilon$ ($\alpha=\frac{1}{6}\epsilon$). In Sec. II we found that in this particular case there is no random fixed point of order ϵ corresponding to the "nonrandom" fixed point. Assume that the fixed point of the pure system is $x_1^*, \ldots, x_l^*$ and $\sum_p c_p x_p^* = \frac{1}{3}$ so that the condition $\alpha=\frac{1}{6}\epsilon$ is fulfilled. Now let us insert $x_p = x_p^*$ and $y=\frac{1}{6}A$ into the second-order terms of the fixed-point equations (13) and (14):

$$\sum_{nm} d_{nm}^p x_n x_m - 6 y x_p = A^2 \left(\sum_{nm} d_{nm}^p x_n^* x_m^* - x_p^* \right) = 0 , \tag{33a}$$

$$\sum_p 2 c_p x_p y - 4 y^2 = A^2 \left(\sum_p \frac{1}{3} c_p x_p^* - \frac{1}{9} \right) = 0 . \tag{33b}$$

Equation (33a) follows from the fact that x_p^* is the fixed point of the homogeneous system. Since all the second-order terms vanish, there are clearly no solutions of order ϵ, but, in general, there are solutions of order $\epsilon^{1/2}$ with $u_p^* = (A/K_4) x_p^* \epsilon^{1/2}$ and $\Delta^* = (A/6K_4)\epsilon^{1/2}$. The coefficient A is determined by inserting $u_p = (A/K_4) x_p^* \epsilon^{1/2} + B\epsilon + \cdots$ and $\Delta^* = (A/6K_4)\epsilon^{1/2} + C\epsilon + \cdots$ into the third-order terms of the recursion relations.

Several of the Hamiltonians that we have considered have "Ising-like" fixed points, where $\sum_p c_p x_p^* = \frac{1}{3}$. For the $n=6$ Hamiltonian describing type-I antiferromagnets with $\vec{m} \perp \vec{k}$ (UO_2), there are three such fixed points, ($x_1^* = \frac{1}{54}$, $x_3^* = \frac{1}{18}$), ($x_1^* = \frac{1}{36}$), and ($x_1^* = \frac{1}{72}$, $x_2^* = \frac{1}{12}$), and the $n=12$ Hamiltonian[13] representing phase transitions in Cr and Eu has a simple Ising fixed point. On the other hand, the $n=8$ Hamiltonian describing phase transitions in type-II antiferromagnets with $\vec{m} \perp \vec{k}$ (MnO) does *not* possess fixed points with $\alpha = \frac{1}{6}\epsilon$, so fixed points of order $\epsilon^{1/2}$ are *not* possible for the corresponding random system.

An interesting case is the $n=4$ Hamiltonian which corresponds to the phase transitions in type-II $\vec{m} \parallel \vec{k}$ antiferromagnets (TbP, TbAs, NdSe, etc.).[13] The recursion relations have *three* fixed points with $\alpha = \frac{1}{6}\epsilon$, namely, the Ising fixed point, $v^* = \epsilon/36K_4$, and two more complicated fixed points, $u^* = \epsilon/48K_4$, $v = -\epsilon/72K_4$, $w^* = \pm\epsilon/6K_4$. Since we have already constructed the recursion relations to third order in the quartic potentials (see the Appendix), we can explicitly determine all the $\epsilon^{1/2}$ fixed points. For the random Ising fixed point we get

$$v^* = \frac{1}{K_4}\left(\frac{\epsilon}{330}\right)^{1/2} \quad \text{and} \quad \Delta^* = \frac{1}{K_4}\left(\frac{6\epsilon}{55}\right)^{1/2}, \quad \eta = -\frac{\epsilon}{110};$$

and for the two *new* random fixed points we get

$$u^* = \frac{1}{48K_4}\left(\frac{\epsilon}{6}\right)^{1/2}, \quad v^* = -\frac{1}{72K_4}\left(\frac{\epsilon}{6}\right)^{1/2},$$

$$w^* = \pm\frac{1}{6K_4}\left(\frac{\epsilon}{6}\right)^{1/2} = \pm\Delta^* .$$

The exponent η corresponding to this fixed point is $-\epsilon/(216 \times 12)$.

It now remains to study the stability of these new fixed points. We can prove that they are all unstable using an argument similar to the one used in II to show that the random fixed points of order ϵ are unstable.

Suppose again that one of the recursion relations of the pure system is

$$\sum_p A_p u_p' = (1 + \lambda\epsilon \ln b) \sum_p A_p u_p , \tag{34}$$

with

$$\lambda = 1 - \sum_p a_p x_p^* > 0, \quad x_p^* = \frac{u_p^* K_4}{\epsilon},$$

and

$$\sum_p A_p u_p^* = 0 .$$

Then the corresponding equation for the random system, with fixed point $u_p^{R*} = (A/\epsilon^{1/2}) u_p^*$, is

$$\sum_p A_p u_p' = (1 + \lambda' K_4 \ln b) \sum_p A_p u_p , \tag{35}$$

with

$$\lambda' = -\sum_p a_p u_p^{R*} + 6\Delta^*$$

$$= \frac{A\epsilon^{1/2}}{K_4}\left(-\sum_p a_p x_p^* + 1 \right) = \frac{A\epsilon^{1/2}}{K_4}\lambda > 0 .$$

Therefore, condition (34), which we used to show that there is no stable fixed point of order ϵ of the pure system and no stable fixed point of order

ϵ for the random system, also implies that there is no stable fixed point of order $\epsilon^{1/2}$ for the random system. Clearly, the argument can be extended to show that there are no fixed points of order $\epsilon^{1/n}$, since the stability condition in this case is essentially the same.

A special case is the Hamiltonian studied in the Appendix, corresponding to type-II antiferromagnets with $\vec{m} \parallel \vec{k}$, where a condition like (34) has not been found.[13] The stability of the $\epsilon^{1/2}$ fixed points is determined by the eigenvalues of the 4×4 matrix

$$\begin{bmatrix} 1-96x_1^*-24x_2^*-\frac{1}{2}x_3^* & -24x_1^* & -\frac{1}{2}x_3^* & 6x_1^* \\ -48x_2^* & 1-48x_1^*-72x_2^* & \frac{1}{2}x_3^* & 6x_2^* \\ -48x_3^* & 0 & 1-48x_1^* & 6x_3^* \\ -48y^* & -24y^* & 0 & \frac{2}{3} \end{bmatrix}, \qquad (36)$$

where we have defined $x_1^* = u^*K_4/\epsilon^{1/2}A$, $x_2^* = v^*K_4/\epsilon^{1/2}A$, $x_3^* = w^*K_4/\epsilon^{1/2}A$, and $y^* = \Delta^*K_4/\epsilon^{1/2}A$. By inserting the fixed point values of the homogeneous fixed point we find that there is always at least one positive eigenvalue (the sum of the diagonal elements is positive). Therefore the three fixed points of order $\epsilon^{1/2}$ for this system are all unstable.

V. CONCLUSIONS

The influence of random perturbations on general anisotropic n-vector models has been studied by means of renormalization group theory in $4-\epsilon$ dimensions. The randomness can be associated with an extra term, with coefficient Δ, in the GLW Hamiltonian which may or may not be relevant in the Wilson sense,[4] depending upon the symmetry of the system and the dimensionality of the order parameter. Whereas for $n \leq 3$ (except, maybe, for the random Ising model, and for models with long-range correlation in the random potential) one should always expect sharp second-order phase transitions with concentration-independent exponents, it turns out that when $n \geq 4$ qualitatively different critical behavior may occur.

Recently, Mukamel, Krinsky, and Bak[13] studied several physical realizable $n \geq 4$ vector models using the ϵ-expansion technique. For some of these models there was a stable fixed point. The critical exponent α for this fixed point is *negative* and it is predicted that the critical properties should not be affected by randomness, and the transition should remain sharp, just as when $n \leq 3$. Experiments should be performed to determine the critical behavior. A good example is Ho, where the $n = 4$ exponents for the pure system have been confirmed experimentally.[24]

For most of the pure $n \geq 4$ models, however, it turned out that the corresponding recursion relations do *not* possess a stable fixed point.[5] It has been demonstrated, that for all these systems, the recursion relations for the corresponding dilute random quenched systems are *also* unstable. The renormalized fluctuations of the local mean-field transition temperature *diverge*, and this behavior is interpreted as an indication of a "smeared" transition. *We therefore predict crossover from first-order transitions to "smeared" or rounded transitions for the real physical systems associated with these models.* Existing experiments do indeed indicate that the transition becomes continuous when small quantities of impurities are added, for example in Cr. It should be emphasized that this is a very unusual behavior for a first-order transition. It is suggested that experiments be performed to test this prediction. For a list of systems, where this crossover should be expected, see Ref. 5. In particular, one should investigate whether the continuous transitions actually observed in Cr and Eu,[14-16,18] are sharp second order with concentration-independent exponents, or rather smeared as predicted in this paper.

For most of the models considered in this paper, there exist fixed points of order $\epsilon^{1/2}$. The fixed point found by Khmel'nitsky[10] for the random Ising model is thus not an isolated case but occurs widely for anisotropic random n-vector models. I have explicitly found three such fixed points for the Hamiltonian describing type-II antiferromagnets. These fixed points are in general not stable.

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APPENDIX

Mukamel and Krinsky[13] have studied an $n = 4$ model describing phase transitions in type-II antiferromagnets with $\vec{m}\parallel\vec{k}$ (TbP, TbSb, TbAs, CeS, TbSe, NdSe, and NdTe). They found a fixed point which is *marginally* stable to first order in ϵ, but unstable to second order in ϵ. Clearly, the argument concerning the stability of random fixed points presented in III is not valid in this case, and we have to analyze this Hamiltonian separately.

The Hamiltonian is

$$\mathcal{H}_3 = -\frac{1}{2}\sum_{i=1}^{4}\left[r\phi_i^2 + (\nabla\phi_i)^2\right] - u\left(\sum_{i=1}^{4}\phi_i^2\right)^2 - v\sum_{i=1}^{4}\phi_i^4 - w\phi_1\phi_2\phi_3\phi_4 . \quad \text{(A1)}$$

The recursion relations for the random system should be constructed to second order in ϵ in order to determine the random fixed points, and the stability of such fixed points. The recursion relations are

$$\begin{aligned}
r' &= r + (24u + 12v - \Delta)A(r) - (192u^2 + 96v^2 + 192uv - 48u\Delta - 24v\Delta + \Delta^2)B(r),\\
u' &= u + [\epsilon u - (48u^2 + 24uv + \tfrac{1}{4}w^2 - 6u\Delta)K_d + (1344u^3 + 1152u^2v + 288uv^2 + 15uw^2\\
&\qquad + 3vw^2 - 432u^2\Delta - 360uv\Delta - \tfrac{3}{2}w^2\Delta + 21u\Delta^2)K_4^2 - 2\eta u]\ln b,\\
v' &= v + [\epsilon v - (36v^2 + 48uv - \tfrac{1}{4}w^2 - 6v\Delta)K_d + (864v^2 + 2304uv^2 + 1728u^2v - 12uw^2\\
&\qquad - 3vw^2 - 288v^2\Delta - 288uv\Delta + \tfrac{3}{2}w^2\Delta + 21v\Delta^2)K_4^2 - 2\eta v]\ln b,\\
w' &= w + [\epsilon w - (48uw - 6w\Delta)K_d + (1728u^2w + 576uvw + 3w^3 + 21w\Delta^2)K_4^2 - 2\eta w]\ln b,\\
\Delta' &= \Delta + [\epsilon\Delta - (48u\Delta + 24v\Delta - 4\Delta^2)K_d + (11\Delta^3 - 288u\Delta^3 - 144v\Delta^2 + 576u^2\Delta\\
&\qquad + 288v^2\Delta + 576vu\Delta)K_4^2 - 2\eta\Delta]\ln b,\\
\eta &= \tfrac{1}{4}K_d^2(192u^2 + 96v^2 + 192uv - 48u\Delta - 24v\Delta + \Delta^2).
\end{aligned} \quad \text{(A2)}$$

To first order in ϵ there are several *unstable* fixed points and one marginally stable fixed point, namely the isotropic fixed point $u=\epsilon/48K_4$, $v=0$, $w=0, \Delta=0$. To proceed we write

$$u=\epsilon/48K_4 + u_2\epsilon^2, \quad v=v_2\epsilon^2,\qquad w=w_2\epsilon^2, \quad \Delta + \Delta_2\epsilon^2, \quad \text{(A3)}$$

and determine u_2, v_2, w_2, and Δ_2 from the fixed-point equations. By considering terms of order $O(\epsilon^3)$ the equation for u gives one additional constraint, whereas one has to compare terms of $O(\epsilon^4)$ in the equations for v, w, and Δ to get three additional constraints. The solutions for which $\Delta^*=0$ are all unstable.[13] There exist two random fixed points with $\Delta_2^*\neq 0$, namely,

$$u^* = \frac{\epsilon}{48K_d} + \frac{5\epsilon^2}{24\times 48K_4},\qquad v^*=0, \quad w^*=0, \quad \Delta^* = -\epsilon^2/18K_4 \quad \text{(A4)}$$

and

$$u^* = \frac{\epsilon}{48K_d} + \frac{5\epsilon^2}{24\times 48K_4},\qquad w^*=0, \quad v^* = -\frac{13\epsilon^2}{24\times 12K_4},\qquad \Delta_2^* = \epsilon^2/8K_4 . \quad \text{(A5)}$$

The first random fixed point is unphysical ($\Delta^*<0$), but both fixed points are unstable with respect to perturbations in w, since one of the eigenvalues, $\lambda_w = \frac{1}{2}\epsilon^2$ in both cases.

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