Specific heat of interacting Bose systems at low temperatures

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The constant-volume specific heat C_V of Bose liquids at low temperature T is investigated using a microscopic generalization of the dynamical quasiparticle model introduced by Landau. The utility of the dynamical quasiparticle model, which is to be distinguished from the statistical quasiparticle model, is based on the outstanding feature of Bose liquids that the single-particle spectrum and the density spectrum coincide. Contributions from collective modes not included in the dynamical quasiparticle model are estimated to be negligible at low T. An exact expression is derived for the leading lifetime contribution δC_V due to the damping of the phonon and δC_V is found to be second order in Γ, the imaginary part of the matrix self-energy. An explicit microscopic calculation of δC_V is performed for a one-parameter Bose gas in a perturbation approximation that includes three-phonon processes and satisfies the general symmetry requirements for Bose systems. The surprising result is that $\delta C_V \sim T^5 \ln T$. The application of this result to superfluid ^{4}He is discussed. It is concluded that the leading phonon dispersion cannot be rigorously determined from a measurement of the $O(T^5)$ term of C_V.

I. INTRODUCTION

The usual approach[1] to understanding the low-temperature thermodynamic properties of Bose liquids, e.g., superfluid ^{4}He, is based on the assumption that the system can be regarded as an ideal gas of infinitely long-lived "simple quasiparticles" with temperature-independent energies. As an example let us assume[2] that the excitation spectrum E_k at small wave vector k for superfluid ^{4}He has the form

$$E_k = c\hbar k[1 + e_2(\hbar k/mc)^2 + \cdots], \tag{1.1}$$

where e_2 is dimensionless and independent of k, m is the mass of the helium atom, and c is the zero-temperature sound speed. We find that the constant-volume specific heat C_V for low temperature T takes the form

$$C_V = \frac{2\pi^2 k_B^4}{15\hbar^3 c^3} T^3 \left[1 - \tfrac{100}{7}\pi^2 e_2 \left(\frac{k_B T}{mc^2}\right)^2 + \cdots\right]. \tag{1.2}$$

If this simple approach gives correctly the $O(T^5)$ term in C_V, then we see from Eq. (1.2) that a measurement of the T^5 term in C_V would determine the coefficient e_2 in the phonon dispersion [Eq. (1.1)]. Because of the interest in the nature of the deviation of the phonon dispersion (1.1) from linearity, several workers[3,4] have used Eqs. (1.1) and (1.2) to analyze the data from thermodynamic measurements of superfluid ^{4}He. Their conclusion is that at saturated vapor pressure the phonon spectrum curves upward, i.e., e_2 is positive, which is in agreement with the results of other investigations.[5]

At finite T phonon-phonon interactions[2] lead to quasiparticles with finite lifetimes, and the simple quasiparticle calculation of C_V breaks down. It is instructive to estimate the leading T dependence of the lifetime correction δC_V to the simple quasiparticle contribution. For example in a Fermi liquid the quasiparticle damping constant γ_k is order T^2, and since γ_k is related to the width of the one-particle spectral function, it might be expected that the leading T dependence of $\delta C_V \sim T(T^2) \sim T^3$. This estimate can be placed on a firmer basis by anticipating the formal result of Sec. III B that δC_V is proportional to $(\gamma_k/E_k)^2$ times the leading T dependence of the simple quasiparticle contribution. Since k is measured from the Fermi momentum, $E_k \sim T(T_F)$, where T_F is the Fermi temperature. Thus we obtain $\delta C_V \sim T(T/T_F)^2 \sim T^3$, which has been confirmed in a microscopic analysis.[6] If we apply similar arguments to a Bose liquid and use the quantum hydrodynamic result[7] $\gamma_k/E_k \sim T^4$, we find $\delta C_V \sim T^3(T^4)^2 \sim T^{11}$. An additional factor of T^{-2} arises from the singular behavior of the phonon coherence factor, and our estimate becomes $\delta C_V \sim T^9$. If this estimate $\delta C_V \sim T^9$ is correct, then the above interpretation of the $O(T^5)$ term in C_V in terms of the phonon dispersion is valid. However, to the best of our knowledge, there does not exist in the literature a microscopic analysis of the leading T dependence of the lifetime corrections to C_V of a Bose liquid. To fill this void we present here a microscopic generalization of the simple quasiparticle model taking into account lifetime effects, a derivation of an exact expression for the leading lifetime contribution δC_V for a Bose liquid, and an explicit microscopic calculation of δC_V for a one-parameter Bose gas in a perturbation approximation that includes three-phonon processes and satisfies the general symmetry requirements for Bose systems. Although the details of the model calculation are not directly applicable to superfluid ^{4}He, unexpected qualitative features will be found and will be helpful in improving our physical insight into the thermodynamic properties of Bose systems at low T.

Previous work[6-8] on microscopic calculations of the thermodynamic properties of quantum systems can be divided into two general classes. The simple quasiparticle model discussed above was originally formulated by Landau[1] and can be considered as a special case of the first class. Landau introduced the simple quasiparticles dynamically as "particles" of a fictitious ideal gas. The natural generalization of Landau's formulation is to consider "dynamical quasiparticles" with finite lifetimes. The energy of the dynamical quasiparticle is complex in general and is determined microscopically by the pole of the single-particle propagator. Thermodynamic properties can be calculated from a formal expression for the grand potential as a functional of the single-particle propagator. From this exact expression it is possible to derive a convenient form for the entropy which is given solely in terms of the single-particle propagator. This form of the entropy, the "dynamical quasiparticle contribution," is approximate since modes not associated with the single-particle propagator are omitted. The dynamical quasiparticle approach is particularly useful for a Bose liquid because of its unique property[9] that the single-particle spectrum and the density spectrum coincide. Thus one dynamical mode, i.e., the dynamical quasiparticle, dominates the low-T thermodynamic properties of a Bose liquid and the nondynamical quasiparticle contribution to the entropy is negligible. The dynamical quasiparticle approach is not as convenient for a Fermi liquid[6,7] for which several modes, such as the spin-fluctuation mode, must be included.

In the second class[8,10] of calculations the thermodynamic properties are given as functionals of a distribution function for the excitations, and the equilibrium distribution function is determined by minimizing the grand potential. The "statistical quasiparticle energies," which are real and can not be identified with the pole of a (causal) propagator, are determined statistically as functional derivatives of the total energy of the system with respect to the quasiparticle distribution function. The form of the entropy is identical to the simple quasiparticle (Fermi or Bose ideal gas) form evaluated using the statistical quasiparticle energies and is exact for all T. Since there is no single dynamical mode that dominates the thermodynamics of a normal Fermi liquid, the statistical quasiparticle approach is useful, and in fact the quasiparticle energies that enter Landau's theory[11] of normal Fermi liquids are essentially statistical quasiparticle energies.

Because of the unique feature of Bose systems that one dynamical mode dominates the thermodynamics, we adopt here the dynamical quasiparticle approach to the study of low-T thermodynamic properties of a Bose liquid. In the following we shall be concerned with dynamical quasiparticles exclusively unless otherwise noted. By "quasiparticle" we mean a dynamical quasiparticle with a complex excitation energy. By "simple quasiparticle" we are referring to a dynamical quasiparticle with a purely real excitation energy.

In Sec. II after introducing some basic definitions, we show, from the formal expression for the grand potential in terms of the one-particle propagator, that the dynamical quasiparticle contribution to the entropy can be written in a transparent form in terms of an entropy spectral function. We estimate the T dependence of the contributions from collective modes not included in the dynamical quasiparticle model and find that they are negligible at low T as expected. The relation of the present formulation to other work on Bose systems is also discussed. In Sec. III we separate the entropy spectral function into a simple quasiparticle part and a lifetime part by formally expanding the single-particle propagator in powers of Γ, the imaginary part of the matrix self-energy. The simple quasiparticle part of the entropy spectral function leads directly to the Landau form for the entropy. An exact expression for the leading lifetime correction δC_V, which turns out to be second order in Γ, is derived.

To gain further understanding of the low-T thermodynamics of Bose systems, we consider in Sec. IV a one-parameter model of a Bose gas for which explicit calculations can be performed. We show how the dielectric formulation[9] can be extended to $T > 0$ and be used to generate perturbation approximations in T and the dimensionless coupling constant g that simultaneously satisfy the general symmetry requirements[9,12] of Bose systems and yield consistent thermodymamics.[13] To first order in g which includes three-phonon processes, the simple quasiparticle contribution to C_V is found to be analytic in T through $O(T^7)$. Our main result[14] is that the leading order of the lifetime contribution δC_V is $O(g^2)$ and in dimensionless units is given by

$$\delta C_V = -0.035 g^2 T^5 \ln(1/gT^4) . \tag{1.3}$$

In Sec. V we discuss the interpretation and implications of the model calculation and the limitations of quantum hydrodynamics. We also give a crude estimate of the magnitude of the coefficient of the $T^5 \ln T$ term in C_V for superfluid ^{4}He.

The notation of this paper follows closely to that in Ref. 9. The reader is assumed to be familiar with the elementary features of the Bose gas and the well-known Bogoliubov approximation.

II. DYNAMICAL QUASIPARTICLE MODEL

To emphasize the nature of the dynamic quasiparticle (DQ) model we present a derivation of a transparent expression for the DQ contribution to the entropy S of a Bose liquid. We introduce some basic definitions in Sec. II A, derive the entropy expression in Sec. II B, and discuss the DQ model and its relation to other work in Sec. II C.

A. Basic definitions

We construct a grand canonical ensemble at temperature T and chemical potential μ with the grand potential Ω

$$\Omega = -T \ln \langle e^{-(H-\mu N)/T} \rangle , \tag{2.1}$$

where H is the Hamiltonian, N is the total number of spinless bosons, and the bracket $\langle \cdots \rangle$ denotes the average over the grand canonical ensemble. The volume of the system, Boltzmann's constant, and $\hbar$ are taken to be unity. To describe the condensate we let $a_0 = a_0^\dagger = n_0^{1/2}$, where n_0, a c number, is the condensate density and a_k and $a_k^\dagger$ are the usual Bose annihilation and creation operators. Bosons having nonzero momenta are described by the grand potential Ω',

$$\Omega' \equiv \Omega - \mu N' = -T \ln \langle e^{-(H-\mu N')/T} \rangle \tag{2.2}$$

and N' is the number of noncondensate bosons. The two parameters, n_0 and μ, can be eliminated by requiring that the stationary property, $\partial\Omega/\partial n_0 = 0$, be satisfied at the correct n_0, i.e.,

$$\frac{\partial \Omega'}{\partial n_0} = \mu \tag{2.3}$$

and with the condition

$$n = n_0 + n' . \tag{2.4}$$

Equations (2.1)–(2.4) constitute the standard description of a Bose liquid with a uniform condensate at rest.

B. Derivation of the entropy expression

The outstanding feature of a Bose liquid is that the discrete spectra of the amplitude fluctuations, density fluctuations, and longitudinal-current fluctuations coincide.[9] This coincidence is a general consequence of Bose condensation, which invalidates the number of excited particles as a good quantum number, and of rotation-translation invariance, which leads to the classification of the states of the system in terms of the momentum $\vec{k}$ and the helicity m (angular momentum along $\hat{k}$). Thus at a particular $\vec{k}$, all of the zero-helicity ($m=0$) excitations, which include amplitude, density, and longitudinal-current excitations, are degenerate; these excitations are referred to as the elementary excitation or quasiparticle of the Bose liquid. We assume that all of the $m \neq 0$ excitations at small k can be ignored in comparison with the elementary excitation. For example, the transverse ($m = \pm 1$) excitations of a Bose gas have been shown by Ma[15] to be nonpropagating at long wavelengths. Thus it is convenient to express the thermodynamic properties of Bose liquids at low T solely in terms of the elementary excitation, which can be chosen to be represented by the one-particle Green's function $\mathcal{G}_{\mu\nu}$. The choice of $\mathcal{G}_{\mu\nu}$ rather than a density response function to represent the elementary excitation is dictated by the existence of a simple closed functional for the grand potential Ω' in terms of $\mathcal{G}_{\mu\nu}$.

It is convenient to define the one-particle thermodynamic Green's function $\mathcal{G}_{\mu\nu}(k, \omega_n)$ as a 2×2 matrix

$$\mathcal{G}_{\mu\nu}(k, \omega_n) = -\int_0^\beta dt\, e^{\omega_n t} \langle \hat{T} a_{k\mu}(t) a_{k\nu}^\dagger \rangle , \tag{2.5}$$

where $a_{k\mu} = a_k$ if $\mu = +$, and $a_{k\mu} = a_{-k}^\dagger$ if $\mu = -$; $\omega_n \equiv 2\pi n i T$, n is an integer; $\hat{T}$ is the time-ordering operator for $\beta \equiv 1/T \leq t \leq 0$, and the bracket $\langle \cdots \rangle$ denotes the average over the grand canonical ensemble. The grand potential Ω' can be written as a functional[16] of $\mathcal{G}$:

$$2\Omega'[\mathcal{G}] = \Phi[\mathcal{G}] - \mathrm{Tr}_n \mathfrak{M}[\mathcal{G}]\mathcal{G} - \mathrm{Tr}_n \ln(-\mathcal{G}^{-1}) , \tag{2.6}$$

where

$$\mathrm{Tr}_n = -T \sum_n \int d^3k (2\pi)^{-3}\, \mathrm{tr},$$

tr is the matrix trace, matrix multiplication is implied between the matrices $\mathcal{G}$ and $\mathfrak{M}$, and the logarithm is defined in terms of its series expansion. The matrix self-energy $\mathfrak{M}$ is defined by the Dyson equation

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} - \mathfrak{M} ,$$
$$\mathcal{G}_0^{-1}(k, \omega_n) = \omega_n \tau_3 - (\epsilon_k + \mu)\tau_0 , \tag{2.7}$$

where $\epsilon_k \equiv k^2/2m$, μ is the chemical potential, τ_0 is the unit 2×2 matrix, and τ_1, τ_2, and τ_3 are the three Pauli matrices. The functional $\Phi[\mathcal{G}]$ is given by the sum of all distinct, connected vacuum diagrams with no self-energy insertions. A very useful property of the functional $\Omega'[\mathcal{G}]$ is the stationarity condition under variations of $\mathcal{G}$, i.e.,

$$\frac{\delta\Omega'[\mathcal{G}]}{\delta\mathcal{G}} = 0\,, \tag{2.8}$$

at a fixed value of the condensate density n_0 and at a particular function $\mathcal{G}$ that satisfies the following relation involving the first functional derivative of $\Phi[\mathcal{G}]$ at constant n_0

$$\mathfrak{M}[\mathcal{G}] = \frac{\delta\Phi[\mathcal{G}]}{\delta\mathcal{G}}\,. \tag{2.9}$$

The second functional derivative of $\Phi[\mathcal{G}]$ with respect to $\mathcal{G}$ gives the kernel of the Bethe-Salpeter equation.

Since the entropy S is related to the partial derivative of Ω' with respect to T, we analyze the T dependences of Ω'. First, there is an explicit T dependence that arises from the discrete frequencies and the frequency sums. Second, there is the implicit T dependence of $\mathcal{G}$ itself. These two different T dependences of $\Omega'[\mathcal{G}]$ can be conveniently denoted by the obvious notation $\Omega'_T[\mathcal{G}_T]$. Because of the stationarity property [Eq. (2.8)] of $\Omega'_T[\mathcal{G}_T]$ with respect to $\mathcal{G}_T$, the implicit T dependence of $\mathcal{G}_T$ can be ignored in evaluating S. Thus we can write

$$S(T_0) = -\lim_{T\to T_0}\left(\frac{\partial}{\partial T}\,\Omega'_T[\mathcal{G}_{T_0}]\right)_\mu\,, \tag{2.10}$$

where only the explicit T dependence Ω'_T due to the discrete frequency sums is differentiated.

The frequency sums may be converted by standard techniques[16] into frequency integrals over the real axis, and the explicit T dependence due to the frequency sums is then reflected in the T dependence of the Bose statistical function $f(\omega) \equiv (e^{\omega/T} - 1)^{-1}$. In this manner, the second term of Eq. (2.6) becomes

$$\mathrm{Tr}_n(\mathfrak{M}[\mathcal{G}]\,\mathcal{G})_{T_0} = \mathrm{Tr}\{[\mathrm{Re}\mathfrak{M}(k,\omega)\rho(k,\omega) + \Gamma(k,\omega)\mathrm{Re}\mathcal{G}(k,\omega)]_{T_0} f(\omega)\} \tag{2.11}$$

and the third term of Eq. (2.6) yields

$$\mathrm{Tr}_n[\ln(-\mathcal{G}^{-1})]_{T_0} = \mathrm{Tr}\{2\,\mathrm{Im}\ln[-\mathcal{G}^{-1}(k,\omega - i0+)]_{T_0} f(\omega)\}\,, \tag{2.12}$$

where

$$\mathrm{Tr} \equiv (2\pi)^{-4}\int d^3k \int d\omega\,\mathrm{tr}$$

and the spectral functions, $\rho(k,\omega)$ and $\Gamma(k,\omega)$, are defined by

$$\mathcal{G}(k,\omega_n) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{\rho(k,\omega)}{\omega_n - \omega}\,, \tag{2.13a}$$

$$\mathfrak{M}(k,\omega_n) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{\Gamma(k,\omega)}{\omega_n - \omega} \tag{2.14a}$$

or in terms of response functions

$$\rho(k,\omega) = -2\,\mathrm{Im}\mathcal{G}(k,\omega + i0+)\,, \tag{2.13b}$$

$$\Gamma(k,\omega) = -2\,\mathrm{Im}\mathfrak{M}(k,\omega + i0+)\,. \tag{2.14b}$$

The functions $\mathcal{G}(k,\omega \pm i0+)$ are analytic continuations of $\mathcal{G}(k,\omega_n)$ into the upper (lower) half plane of ω.

The explicit T dependence of the first term of Eq. (2.6) deserves a more detailed analysis. An "integral" version of Eq. (2.9) can be written in the form:

$$\Phi[\mathcal{G}] = \sum_{\nu=1}^{\infty}\frac{1}{2\nu}\,\mathrm{Tr}_n(\mathfrak{M}^{(\nu)}[\mathcal{G}]\,\mathcal{G}) + \mathrm{const.}\,, \tag{2.15}$$

where $\mathfrak{M}^{(\nu)}[\mathcal{G}]$ is the sum of all self-energy diagrams with $(2\nu - 1)$ $\mathcal{G}$ lines and the constant consists of contributions from the condensate. To extract the explicit T dependence, we first consider only the $f(\omega)$ that arises from the $\mathcal{G}$ line that has been singled out in Eq. (2.15). We find for the νth term

$$\mathrm{Tr}_n(\mathfrak{M}^{(\nu)}[\mathcal{G}]\mathcal{G}) = \mathrm{Tr}\{[\mathrm{Re}\mathfrak{M}^{(\nu)}(k,\omega)\rho(k,\omega)]_{T_0} f(\omega)\}\,. \tag{2.16}$$

To obtain the rest of the T dependence, we separate successively a different $\mathcal{G}$ line in the νth term of Eq. (2.15). The contribution from each explicit $\mathcal{G}$ line is the same as Eq. (2.16) and thus the factor $1/2\nu$ in Eq. (2.15) is cancelled. Summing over ν we obtain

$$\Phi_T[\mathcal{G}_{T_0}] = \mathrm{Tr}\{[\mathrm{Re}\mathfrak{M}(k,\omega)\rho(k,\omega)]_{T_0} f(\omega)\}\,. \tag{2.17}$$

This analysis of $\Phi_T[\mathcal{G}_{T_0}]$, resulting in Eq. (2.17), does not leave out the contribution from the $m=0$ collective excitations in contrast to the application of the same treatment of Φ in a Fermi system. The reason lies in the unique property of a Bose system that the $m=0$ collective excitations and the single-particle excitations coincide.[9]

Equations (2.11), (2.12), and (2.17) can now be collected in Eq. (2.6) to get

$$\Omega'_T[\mathcal{G}_{T_0}] = -\tfrac{1}{2}\,\mathrm{Tr}\{[X(k,\omega)]_{T_0} f(\omega)\}\,, \tag{2.18}$$

$$X(k,\omega) = 2\,\mathrm{Im}\ln[-\tau_3\mathcal{G}^{-1}(k,\omega - i0+)] + \Gamma(k,\omega)\mathrm{Re}\mathcal{G}(k,\omega)\,, \tag{2.19}$$

with τ_3 introduced to ensure the vanishing of the logarithm at the limits $\omega \to \pm\infty$. The entropy S can be found from Eqs. (2.10) and (2.18) to be

$$S(T)=\tfrac{1}{2}\,\mathrm{Tr}\left(X(k,\omega)\frac{\partial f(\omega)}{\partial T}\right). \tag{2.20}$$

A more transparent form of S may be obtained by using the identity

$$\frac{\partial f(\omega)}{\partial T}=-\frac{\partial n(\omega)}{\partial \omega}, \tag{2.21}$$

$$n(\omega)\equiv[1+f(\omega)]\ln[1+f(\omega)]-f(\omega)\ln f(\omega) \tag{2.22}$$

and partially integrating Eq. (2.20) with respect to ω to obtain

$$S(T)=\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,n(\omega)\,\sigma(k,\omega), \tag{2.23a}$$

$$\sigma(k,\omega)\equiv\frac{1}{2}\,\frac{\partial}{\partial\omega}\,\mathrm{tr}X(k,\omega). \tag{2.23b}$$

Since

$$2\left(\mathrm{Im}\frac{\partial}{\partial\omega}\ln[-\tau_3\mathcal{G}^{-1}(k,\omega-i0+)]\right)=\frac{\partial\,\mathrm{Re}(\mathcal{G}^{-1})}{\partial\omega}\rho-\frac{\partial\Gamma}{\partial\omega}\mathrm{Re}\mathcal{G}, \tag{2.24}$$

it follows from Eqs. (2.19), (2.23), and (2.24) that σ can be written compactly as

$$\sigma(k,\omega)=\tfrac{1}{2}\,\mathrm{tr}\left(\frac{\partial\,\mathrm{Re}(\mathcal{G}^{-1})}{\partial\omega}\rho+\Gamma\frac{\partial\,\mathrm{Re}\mathcal{G}}{\partial\omega}\right). \tag{2.25}$$

Equations (2.22), (2.23), and (2.25) constitute the basic formulas for the DQ contribution to the entropy of a Bose liquid. The function $\sigma(k,\omega)$ can be interpreted as the DQ entropy spectral function, and is expressed solely in terms of functions related to the one-particle Green's function. From the symmetry properties of $\mathcal{G}$, e.g., $\mathcal{G}_{\mu\nu}(\vec{k},\omega)=\mathcal{G}_{-\nu-\mu}(-\vec{k},-\omega)$, it is easy to see that $\sigma(\vec{k},\omega)=\sigma(-\vec{k},-\omega)$.

The constant volume specific heat C_V is given by

$$C_V=T\left(\frac{\partial S}{\partial T}\right)_{VN}. \tag{2.26}$$

After a differentiation of Eqs. (2.23) and (2.25) and a partial integration, we find the following symmetrical form for C_V:

$$C_V=\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega\left(\frac{\partial f(\omega)}{\partial T}\sigma(k,\omega)-\frac{\partial f(\omega)}{\partial\omega}\lambda(k,\omega)\right), \tag{2.27a}$$

$$\lambda(k,\omega)\equiv\frac{1}{2}\,\frac{\partial}{\partial T}\,\mathrm{tr}X(k,\omega)=\tfrac{1}{2}\,\mathrm{tr}\left(\frac{\partial\,\mathrm{Re}(\mathcal{G}^{-1})}{\partial T}\rho+\Gamma\frac{\partial\,\mathrm{Re}\mathcal{G}}{\partial T}\right), \tag{2.27b}$$

where $\lambda(k,\omega)$ may also be considered a spectral function. The partial derivatives in Eq. (2.27) are to be understood to be at constant N and V and at a constant value of the arguments of f and $\mathcal{G}$. Although C_V is the physical quantity of interest, we shall find it more convenient to calculate the entropy using Eq. (2.23) and the entropy spectral function (2.25) and then to obtain C_V via Eq. (2.26).

C. Discussion

The simplicity of the DQ model is manifest in the entropy expression (2.23). All the complications due to lifetime effects are included in the DQ entropy spectral function σ. The crucial point in our derivation of Eq. (2.23) is our assumption, in the steps leading from Eqs. (2.15) to (2.17), that it is possible to separate successively one different $\mathcal{G}$ line at a time in the νth term of Eq. (2.15). This procedure implicitly assumes that there is no new contribution to $\delta\Phi/\delta T$ arising from two $\mathcal{G}$ lines grouped together. Upon closer examination it is seen[6] that an additional contribution to $\delta\Phi/\delta T$ does indeed arise if there exists at least a pair of intermediate states on the energy shell. In other words, corrections to the DQ model can arise from real scatterings among the quasiparticles. The dominant effect of real scatterings is to produce collective modes of quasiparticles, which can of course be classified by $\vec{k}$ and m. Since the zero-helicity excitations in a Bose liquid coincide, the $m=0$ collective mode, which is second sound, is already included in the poles of the one-particle propagator. Therefore only the $m\neq 0$ collective modes can give corrections to the DQ model of a Bose liquid. A crude estimate of the contribution from the $m\neq 0$ collective modes can be obtained from the observation that the $m\neq 0$ modes are oscillations in the normal component of the Bose liquid; hence we expect that their contribution to C_V might be $C_V'\sim T^3(\rho_n/\rho_s)\sim T^7$, where $\rho_{n(s)}$ is the normal (super) fluid density. Such an estimate assumes that the $m\neq 0$ modes are propagating modes like the phonon; since they are not propagating, $C_V'\sim T^7$ is certainly an overestimate. To obtain a more reasonable estimate of the contribution of a nonpropagating mode, we use the expression derived by Riedel[17]

$$S'\propto\int d\omega\frac{\partial f(\omega)}{\partial T}\int dk\,k^2\left[\left(\frac{D_I}{D_R}\right)^3+\cdots\right], \tag{2.28}$$

where $D(k,\omega)$ is the propagator for the mode in question, the subscripts R and I denote the real and imaginary parts, respectively, and the prime on the S' is to remind us that this is a correction to the DQ model. The transverse ($m=\pm1$) mode at small k can be given by a propagator of the form[11,15]

$$D(k,\omega)=\frac{i\eta k^2}{\omega+i(\eta/\rho_n)k^2}\,,\quad k\ll\bar{k}\,, \tag{2.29}$$

where η is the first viscosity and $\bar{k}$ is some cutoff momentum. Substituting Eq. (2.29) into Eq. (2.28), we find

$$S'\sim(T\rho_n/\eta)^{3/2}\sim T^{15}\,, \tag{2.30}$$

where we have used[2,15] $\rho_n\sim T^4$ and $\eta\sim T^{-5}$. Thus we conclude that the corrections to the DQ model of a Bose liquid at low T are negligible.

In contrast, in a Fermi liquid the relevant collective mode is the nonpropagating spin fluctuations, which can be described by a propagator of the form

$$D_s(k,\omega)=\frac{ir_s k}{\omega+ic_s k}\,,\quad k\ll\bar{k}\,, \tag{2.31}$$

where c_s and r_s are independent of k and T. The corresponding correction to the DQ model of a Fermi liquid can be obtained from Eqs. (2.28) and (2.31) and is non-negligible[6,17]:

$$S'\sim T^3\ln T\,. \tag{2.32}$$

It follows that at low T the DQ model is not as useful for a Fermi liquid as for a Bose liquid.

Another feature of the present derivation of the entropy expression (2.23) for a Bose liquid is the expansion of the grand potential Ω' and the Green's function $\mathcal{G}_{\mu\nu}$ about an arbitrary finite temperature $T_0>0$ rather than about $T_0=0$. Such a $T_0>0$ expansion has been utilized by Fulde and Wagner,[18] who wrote down Eq. (2.23) for the entropy of phonons in an amorphous solid. The importance of the $T_0>0$ expansion can be seen from an inspection of the third-order diagrams for $\Phi[\mathcal{G}]$, which reveal that the functional dependence of $\Phi[\mathcal{G}]$ on $\mathcal{G}_{T_0}$ is not the same for $T_0=0$ as for $T_0>0$. Hence the entropy expression derived by an expansion about $T_0=0$, as done by Götze and Wagner,[19] is valid only to leading order in T in contrast to the general expression (2.23) valid for all T. We discuss the $T_0=0$ derivation further in Sec. III A.

The entropy form (2.23) has also been derived by Kane and Kadanoff,[20] who were interested in the nonequilibrium states of a Bose system. Their derivation depends on several "nonequilibrium identities," which have been shown[21] to be correct at $T=0$ but only approximate at $T>0$. The terms that violate the "nonequilibrium identities" are those that involve real scattering processes at $T>0$. Thus the nonequilibrium derivation of Eq. (2.23) has been shown to be valid only to leading order in T.

III. LOW-TEMPERATURE OR QUASIPARTICLE EXPANSION

In the previous section we derived a general expression for the DQ entropy in terms of the entropy spectral function σ. We consider in this section the separation of both σ and ρ, the one-particle spectral function, into a simple quasiparticle part, σ^{QP} and ρ^{QP}, and a lifetime part, $\delta\sigma$ and $\delta\rho$, by formally expanding σ and ρ in powers of Γ, the imaginary part of the self-energy. Such a separation is expected to be useful at long wavelengths or equivalently at low T for which the quasiparticle is well defined.

We show in Sec. III A that σ^{QP} assumes a simple δ-function form with no normalization factors in contrast to the form of ρ^{QP}. This form of σ^{QP} leads directly to the Landau simple quasiparticle form for the entropy and the specific heat. In Sec. III B we derive an exact expression for the leading correction to σ^{QP} and find that it is proportional to Γ^2.

A. Simple quasiparticle limit

We first investigate the form of $\rho(k,\omega)$, the spectral function (2.13) of the one-particle propagator, in the simple quasiparticle limit, which is defined as the limit in which $\Gamma(k,\omega)$ vanishes. We write Dyson's equation (2.7) for the response function $\mathcal{G}$ [see Eq. (2.13b)] in the form

$$\mathcal{G}^{-1}(k,\omega)=\mathcal{G}_{QP}^{-1}(k,\omega)+\tfrac{1}{2}i\Gamma(k,\omega)\,, \tag{3.1}$$

where the response function $\mathcal{G}^{QP}\equiv(\mathcal{G}_{QP}^{-1})^{-1}$ is given by

$$\begin{aligned}\mathcal{G}_{QP}^{-1}(k,\omega)&=\mathcal{G}_0^{-1}(k,\omega)-\mathrm{Re}\mathfrak{M}(k,\omega)\\&=(\omega-\mathrm{Re}\mathcal{A})\tau_3-(\epsilon_k+\mathrm{Re}\mathcal{S}-\mu)\tau_0\\&\quad-\mathrm{Re}\mathfrak{M}_2\tau_1+i0+,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathcal{A}&=\tfrac{1}{2}(\mathfrak{M}_{++}-\mathfrak{M}_{--})\,,\\ \mathcal{S}&=\tfrac{1}{2}(\mathfrak{M}_{++}+\mathfrak{M}_{--})\,,\\ \mathfrak{M}_2&=\mathfrak{M}_{+-}=\mathfrak{M}_{-+}\,.\end{aligned} \tag{3.3}$$

We take the inverse of the matrix $\mathcal{G}_{QP}^{-1}$ in Eq. (3.2) to obtain the following form for $\mathcal{G}^{QP}$:

$$\mathcal{G}^{QP}=\mathfrak{N}^{QP}/\mathfrak{D}^{QP}\,, \tag{3.4a}$$

$$\mathfrak{N}^{QP}=Z(\omega\tau_3+\mathcal{E}_0\tau_0-\mathcal{E}_1\tau_1)\,, \tag{3.4b}$$

$$\mathfrak{D}^{QP}=Z^2(\omega^2-\mathcal{E}_0^2+\mathcal{E}_1^2), \tag{3.4c}$$

where $\omega Z=\omega-\mathrm{Re}\mathcal{A}$, $\mathcal{E}_0 Z=\epsilon_k+\mathrm{Re}\mathcal{S}-\mu$, and $\mathcal{E}_1 Z=\mathrm{Re}\mathfrak{M}_2$. If we write

$$E^2(k,\omega)=\mathcal{E}_0^2-\mathcal{E}_1^2\,,\quad E(k,\omega)\equiv[E^2(k,\omega)]^{1/2},$$

and

$$\mathfrak{D}^{QP} = Z^2[\omega - E(k,\omega)][\omega + E(k,\omega)]$$

and use Eq. (2.13b), we can write ρ^{QP} as

$$\rho^{QP} = \rho_0^{QP}\tau_0 + \rho_1^{QP}\tau_1 + \rho_3^{QP}\tau_3\,, \tag{3.5}$$

where

$$\rho_0^{QP} = (\pi\mathcal{E}_0/ZE)[\delta(\omega - E) - \delta(\omega + E)]\,, \tag{3.6a}$$

$$\rho_1^{QP} = (-\pi\mathcal{E}_1/ZE)[\delta(\omega - E) - \delta(\omega + E)]\,, \tag{3.6b}$$

$$\rho_3^{QP} = (\pi/Z)[\delta(\omega - E) + \delta(\omega + E)]\,. \tag{3.6c}$$

Although ρ^{QP} has formally the familiar δ-function form, it contains normalization factors. The argument of the δ functions in Eqs. (3.6) can be simplified by expanding $E(k,\omega)$ about a zero of $\omega \mp E(k,\omega)$, namely

$$\pm E_k \equiv E(k, \omega = \pm E_k)\,, \tag{3.7}$$

$$\delta[\omega \mp E(k,\omega)] = \delta\left(\omega \mp E_k \mp (\omega \mp E_k)\frac{\partial E}{\partial\omega}\bigg|_{\omega = \pm E_k} - \cdots\right)$$

$$= \left[1 - \left(\frac{\partial E}{\partial\omega}\right)_k\right]^{-1}\delta(\omega \mp E_k)\,, \tag{3.8}$$

where $(\partial E/\partial\omega)_k = (\partial E/\partial\omega)_{\omega = E_k}$ and $E(k,\omega) = E(k,-\omega)$ have been used. Note that there could be another zero of $[\omega \mp E(k,\omega)]$.

The form of σ^{QP} can be found using Eq. (2.25) with $\Gamma = 0$, Eqs. (3.2) and (3.5) to be

$$\sigma^{QP} = \tfrac{1}{2}\,\mathrm{tr}\left(\frac{\partial \mathcal{G}_{QP}^{-1}}{\partial\omega}\rho^{QP}\right)$$

$$= Z\left(\rho_3 - \frac{\partial\mathcal{E}_0}{\partial\omega}\rho_0 - \frac{\partial\mathcal{E}_1}{\partial\omega}\rho_1\right)$$

$$+ \frac{\partial Z}{\partial\omega}(\omega\rho_3 - \mathcal{E}_0\rho_0 - \mathcal{E}_1\rho_1)\,. \tag{3.9}$$

The second term on the right-hand side of Eq. (3.9) can be seen to vanish using Eq. (3.6) and the relation $E^2 = \mathcal{E}_0^2 - \mathcal{E}_1^2$. If we again use the latter relation, we obtain after some algebra the simple form

$$\sigma^{QP}(k,\omega) = \pi[\delta(\omega - E_k) + \delta(\omega + E_k)]\,. \tag{3.10}$$

Note that $\sigma^{QP}(k,\omega)$ is an even function of ω in agreement with the general property of $\sigma(k,\omega)$ and that in contrast to ρ^{QP}, σ^{QP} has no normalization factors. Thus E_k can be interpreted as the simple quasiparticle energy, and is defined by $\mathcal{G}_{QP}^{-1}(k, E_k) = 0$. Similarly the simple quasiparticle limit of $\lambda(k,\omega)$ [Eq. (2.27b)] is found to be

$$\lambda^{QP}(k,\omega) = \pi\frac{dE_k}{dT}[\delta(\omega - E_k) + \delta(\omega + E_k)]\,. \tag{3.11}$$

If we substitute Eq. (3.10) into Eq. (2.23a) and Eq. (3.11) into Eq. (2.27a), we obtain the simple quasiparticle form generalized to a T-dependent excitation energy $E_k(T)$ for the entropy and specific heat, respectively:

$$S^{QP} = \int \frac{d^3k}{(2\pi)^3}[(1+f_k)\ln(1+f_k) - f_k\ln f_k]\,, \tag{3.12}$$

$$C_V^{QP} = \int \frac{d^3k}{(2\pi)^3}E_k(T)\frac{d}{dT}f_k\,, \tag{3.13}$$

where $f_k = (e^{E_k(T)/T} - 1)^{-1}$ and the total temperature derivative is at a constant N. Equations (3.12) and (3.13) have been widely used to describe superfluid ^{4}He at low temperatures.[22]

Inspection of Eq. (3.13) shows that the leading T dependence of C_V^{QP} depends on the behavior of $E_k \sim T \ll mc^2$ (c is the $T = 0$ macroscopic sound speed). Since the form of E_k in the limit $E_k \sim T \ll mc^2$ has not been established to all orders of perturbation theory, we assume that in this limit E_k has the form $E_k(T) = c(T)k$ with $c(T) = c + O(T^n)$, $n > 0$, i.e., the temperature corrections to c vanish as $T \to 0$, which is substantiated by quantum hydrodynamics.[2] With this assumption we easily obtain, using Eq. (3.13), the well-known T^3 law of Landau[1]:

$$C_V^{QP} = \tfrac{2}{15}\pi^2(T/c)^3 + \cdots\,. \tag{3.14}$$

Other workers[19] have derived Eq. (3.14) by taking $T_0 = 0$ in the derivation of the entropy. This procedure leads to an expression for S similar to Eq. (2.23a) but with σ evaluated at $T = 0$. The simple quasiparticle limit of σ would then have the same form as Eq. (3.10), but with E_k evaluated at $T = 0$. Since it is known[23] to all orders of perturbation theory that at $T = 0$ and in the long-wavelength limit $E_k = ck + \cdots$, the result (3.14) follows simply. However, this derivation involves the tacit assumption that the temperature corrections to σ at $T = 0$ vanish as $T \to 0$, an assumption which is similar in spirit to the one used above for $c(T)$.

In deriving Eq. (3.8), which is an intermediate step to Eq. (3.14), we have assumed the existence of only one zero in $\omega = E(k,\omega)$. There exists a second zero in $\omega = E(k,\omega)$, *viz.*, second sound. However, since second sound is strongly damped ($\sim T^{-5}$) as $T \to 0$, its contribution to C_V at low T is negligible ($\sim T^{19}$) and Eq. (3.14) is unchanged.

The derivation of Eq. (3.14) is not complete until the lifetime contribution to C_V, considered in the following section, is shown to be higher order in T. Also since the deviation of E_k from linearity in k at $T = 0$ and the T dependence of E_k have not been established for a Bose liquid, the nature of the deviation from T^3 of the specific heat of a Bose liquid is not known.

B. Expansion of the lifetime part

The usefulness of the simple quasiparticle limit of $\sigma(k,\omega)$ suggests an attempt to separate $\sigma(k,\omega)$

into a simple quasiparticle part σ^{QP} and a lifetime part $\delta\sigma$. Such a separation can be constructed for the propagating mode in $\mathcal{G}$, i.e., the elementary excitation, by expanding σ as a functional of Γ formally in powers of Γ. We write

$$\mathcal{G}=\mathcal{G}^{QP}+\delta\mathcal{G}, \quad \rho=\rho^{QP}+\delta\rho, \quad \sigma=\sigma^{QP}+\delta\sigma. \qquad (3.15)$$

The expansion of the response function $\mathcal{G}$ is found by writing Eq. (3.1) in the form:

$$\mathcal{G}=\mathcal{G}^{QP}-\tfrac{1}{2}i\mathcal{G}^{QP}\Gamma\mathcal{G}, \qquad (3.16)$$

where $\mathcal{G}^{QP}$ is defined in Eq. (3.2). The first iteration of Eq. (3.16) in powers of Γ gives

$$\delta\mathcal{G}^{(1)}=-\tfrac{1}{2}i\mathcal{G}^{QP}\Gamma\mathcal{G}^{QP} \qquad (3.17)$$

and can be interpreted as the first-order (in Γ) correction to $\mathcal{G}^{QP}$. To find the corresponding contribution to $\delta\rho$, we substitute Eq. (3.16) into Eq. (2.13b) and use Eq. (3.1) to write ρ in the form

$$\rho=(\mathrm{Re}\,\mathcal{G}^{QP})\Gamma\,\mathrm{Re}\,\mathcal{G}, \qquad (3.18)$$

which is exact and reduces properly to ρ^{QP} as $\Gamma\to 0$. If we let $\mathcal{G}\to\mathcal{G}^{QP}$ and keep $\Gamma\neq 0$ in Eq. (3.18), we obtain the first-order contribution

$$\delta\rho=(\mathrm{Re}\,\mathcal{G}^{QP})\Gamma\,\mathrm{Re}\,\mathcal{G}^{QP}. \qquad (3.19)$$

Substituting Eqs. (3.1) and (3.19) into Eq. (2.25) to find the corresponding contribution to $\delta\sigma$, we obtain the unexpecting result that to first order (in Γ) $\delta\sigma$ *vanishes*. This result for $\delta\sigma$ can be made conspicuous by using the exact relation (3.18) to rewrite σ [Eq. (2.25)] in the form:

$$\sigma=\tfrac{1}{2}\,\mathrm{tr}\left[\left(\frac{\partial\,\mathrm{Re}(\mathcal{G}^{-1})}{\partial\omega}\mathrm{Re}\,\mathcal{G}^{QP}-\frac{\partial(\mathrm{Re}\,\mathcal{G})^{-1}}{\partial\omega}\mathrm{Re}\,\mathcal{G}\right)\Gamma\,\mathrm{Re}\,\mathcal{G}\right], \qquad (3.20)$$

which can be shown to reduce to σ^{QP} [see Eq. (3.10)] as $\Gamma\to 0$. If we substitute $\mathcal{G}^{QP}$ for $\mathcal{G}$ and keep $\Gamma\neq 0$ in Eq. (3.20), it is easy to see that to $O(\Gamma)$, $\delta\sigma$ vanishes. Thus although the lifetime effects enter in the one-particle spectral function to first order in Γ [see Eq. (3.19)], these lifetime effects do not enter in this order to the thermodynamics of the system. The second iteration of Eq. (3.16) gives

$$\delta\mathcal{G}^{(2)}=-\tfrac{1}{4}\mathcal{G}^{QP}\Gamma\mathcal{G}^{QP}\Gamma\mathcal{G}^{QP}. \qquad (3.21)$$

Using the second-order contribution [Eq. (3.21) in Eq. (3.20)] we find that the leading correction to σ^{QP} is second order in Γ:

$$\delta\sigma=-\tfrac{1}{2}\,\mathrm{tr}\left(\frac{\partial\mathcal{G}_{QP}^{-1}}{\partial\omega}\,\mathrm{Im}(\mathcal{G}^{QP}\Gamma\mathcal{G}^{QP}\Gamma\mathcal{G}^{QP})\right). \qquad (3.22)$$

This new relation (3.22) is an exact result for the leading lifetime correction to the simple quasiparticle entropy spectral function and is convenient for subsequent analysis as it depends only on functions related to the one-particle propagator.

We are interested in the low-T limit of Eq. (3.22) for a Bose liquid. In a Bose liquid at $T=0$, $\mathcal{G}$ approaches $\mathcal{G}^{QP}$ for small k and ω so that ignoring the matrix sum, we expect Eq. (3.22) to be proportional to $\delta''(\omega-E_k)$. Inspection of Eq. (2.23a) shows that the important values of ck are $O(T)$, and thus an estimate of the T dependence of Eq. (3.22) depends on our knowledge of the behavior of $\Gamma(k,\omega)$ for $ck\sim\omega\sim T$. Since the behavior of $\Gamma(k,\omega)$ in this limit has not been established to all orders in perturbation theory and since the matrix Γ is not directly related to the experimentally observed quantity γ_k (the width of the one-phonon peak), we cannot make an estimate of the leading T dependence of $\delta\sigma$ for a Bose liquid. We can gain further understanding of the low-T thermodynamics of Bose systems by considering a simple model of a Bose gas for which we can evaluate E_k, $\Gamma(k,\omega)$, $\delta\sigma$, and C_V explicitly.

IV. MODEL CALCULATION OF C_V

We consider now the evaluation of the simple quasiparticle and leading lifetime contribution to the low-T specific heat C_V of a model Bose gas. We present the small parameters of the model in Sec. IV A and discuss in Sec. IV B the nature of a perturbation calculation in the dielectric-function approach. We give an outline of the model calculation of the simple quasiparticle and leading lifetime contribution to C_V in Secs. IV C and IV D, respectively, and reserve a presentation of some of the details of the calculation to Appendixes A–C. The reader who is not interested in the calculation may on the first reading skip all of Sec. IV D and go on to Sec. V.

A. One-parameter model of a Bose gas

We consider a dilute gas of spinless bosons of mass m at density n and assume that the short-range two-body interaction can be summarized by the s-wave scattering length a. A detailed discussion of many of the $T=0$ properties of the model is given in Ref. 9, and unless otherwise noted the notation of Ref. 9 is adopted hereafter. [Exception: the one-particle spectral function $\rho(k,\omega)$ was called $\mathcal{A}(k,\omega)$ in Ref. 9.]

The zeroth approximation in the model corresponds to the well-known Bogoliubov approximation in which the natural units for momentum and energy are, respectively,

$$k_0\equiv(4\pi na)^{1/2}\equiv ms_0, \quad T_0\equiv 4\pi na/m\equiv ms_0^2, \qquad (4.1)$$

where s_0 is the phonon speed in the Bogoliubov approximation. The model can be characterized by the small dimensionless parameter

$$g \equiv k_0^3 n^{-1} = (4\pi a)^{3/2} n^{1/2} . \qquad (4.2)$$

The temperature T is assumed to be small compared with T_0, which defines the small dimensionless parameter

$$t \equiv T/T_0 = mT/4\pi n a . \qquad (4.3)$$

In Sec. IV and Appendixes A and B, we shall work in a convenient set of units in which the momentum and energy are measured in terms of k_0 and T_0, respectively; so that

$$k_0 = T_0 = m = s_0 = 1 , \quad g = n^{-1} = 4\pi a, \quad t = T . \qquad (4.4)$$

Our procedure is to calculate the $O(g)$ contribution to $\mathcal{G}$ and Γ and then to expand the $O(g)$ contribution in terms of T. Hence our perturbation expansion is a double expansion in g and T with $g < T$.

B. Approximations in the dielectric formulation

Our calculation to $O(g)$ of $\mathcal{G}$ and Γ will be performed in the framework of the generalized dielectric formulation. We discuss here only its salient features; the extension to $T > 0$ is given in Appendixes A and B.

Our basic philosophy is to incorporate as many exact conditions into the formalism as possible before any perturbation expansions are made. For example in Sec. II we ensured consistent thermodynamics by incorporating the condition of Φ derivability[13] [Eq. (2.9)] into our derivation of the entropy formula (2.23). However, the manner in which Eq. (2.9) is used is important.

If we follow the usual approach[12] of approximating Φ and using Eq. (2.9) to obtain an approximate self-energy, then the approximate two-particle Green's function generated by the second functional derivative of $\Phi[\mathcal{G}]$ is of a higher order than the resulting approximate one-particle Green's function. As a consequence, the one-particle and density spectra do not coincide for a given approximation, which violates a general and basic feature of a Bose system.[9] Furthermore, the gapless condition on the one-particle spectrum and the conservation laws cannot be simultaneously satisfied in a given approximation by this procedure.[12]

These difficulties are all circumvented in the dielectric function approach,[9] in which the response functions are expressed in terms of the corresponding regular functions (those that do not involve an isolated single-interaction line nor an isolated one-particle line) rather than in terms of the self-energy $\mathfrak{M}$. In particular, the denominators of the amplitude, density, and longitudinal-current (i.e., zero-helicity) response functions can all be related to the dielectric function ϵ. Thus the zero-helicity spectra coincide, and the elementary excitation spectrum is given by the zero of $\epsilon(k, \omega)$. By using the generalized Ward identities to express the regular functions involving the density, in terms of regular functions involving the longitudinal current, we see that an approximation for the regular longitudinal-current response function yields automatically an approximate density response function that is consistent with local number conservation and the relevant sum rules.

Since the properties of the Bose liquid are dominated by the elementary excitation, we are led to use Eq. (2.9) in its "integral" form [Eq. (2.15)] to express Ω' solely in terms of $\mathcal{G}$. In this way a given approximation for $\mathcal{G}$ in the resulting form for Ω' automatically satisfies Φ derivability and the approximate theory yields consistent thermodynamics.[13] Hence although the usual approach to Φ derivability is not useful, we can cast Eq. (2.9) into an "integral" form [Eq. (2.15)] for Bose liquids and make approximations on $\mathcal{G}$ rather than directly on Φ. In the dielectric formulation the exact $\mathcal{G}$ can be expressed in terms of regular functions involving the longitudinal current. By developing a perturbation expansion for these regular functions rather than for $\mathcal{G}$ or $\mathfrak{M}$ directly, we can now generate a perturbation theory that in a given approximation yields consistent thermodynamics, coincidence of excitations, a gapless one-particle excitation spectrum, and a density-response function that satisfies the relevant sum rules.

The present calculation is now reduced to an evaluation to $O(g)$ of the regular functions involving the longitudinal current, *viz.*, the regular self-energy $M_{\mu\nu}$, current vertex Λ_μ^z, and current-current-correlation F^{zzr}. These regular functions to $O(g)$ are evaluated explicitly in Appendix A.

C. Simple quasiparticle contribution to $O(g)$

The simple quasiparticle contribution to the entropy and specific heat is given [see Eqs. (3.12) and (3.13)] in terms of the T-dependent quasiparticle spectrum $\omega_k(T)$, which is defined in Sec. III A in terms of the real part of the pole of the one-particle response function $\mathcal{G}$. As discussed in Sec. IV B we can alternatively find the quasiparticle spectrum from the zero of the dielectric function. We shall use the notation ω_k for the quasiparticle spectrum of the Bose gas model and retain the notation E_k for the quasiparticle of a Bose liquid.

To obtain $\omega_k(T)$ to $O(g)$, we first expand ω_k^2 in terms of g,

$$\omega_k^2 = \omega_k^{2(0)} + g\omega_k^{2(1)} + O(g^2),$$

where $\omega_k^{2(0)} = k^2(1+\frac{1}{4}k^2)$ is the Bogoliubov spectrum and the $O(g)$ correction $\omega_k^{2(1)}$ is given in terms of a sum of one-loop integrals [see Eq. (B1)]. Only the real part of $\omega_k(T)$ is of interest in this section. We separate the T-independent and T-dependent parts of $\omega_k^{2(1)}$:

$$\omega_k^{(1)} \equiv \tfrac{1}{2}\omega_k^{2(1)}/\omega_k^{(0)} = \omega_k^{(1)}(0) + \tilde{\omega}_k^{(1)}(T). \tag{4.5}$$

The T-independent part $\omega_k^{(1)}(0)$ was evaluated in Ref. 24 and has a nonanalytic term in the $k \ll 1$ limit:

$$\omega_k^{(1)}(0)/k = \pi^{-2}[1 - \tfrac{17}{90}k^2 - \tfrac{3}{320}k^4 \ln(1/k) + O(k^4)]. \tag{4.6a}$$

We observe from Eq. (3.12) or Eq. (3.13) that for low T we are interested in $\tilde{\omega}_k^{(1)}(T)$ in the limit $k \sim T \ll 1$. In Appendix B $\tilde{\omega}_k^{(1)}(T)$ is evaluated in this limit and the leading T dependence is found [see Eq. (B2)] to be also nonanalytic:

$$\tilde{\omega}_k^{(1)}(T)/k = \tfrac{3}{20}\pi^2 T^4 \ln(1/T) + O(T^4), \tag{4.6b}$$

in agreement with a previous calculation[25] in the $k \ll T \ll 1$ limit. Substituting Eqs. (4.5) and (4.6) into Eq. (3.12) and using the integrals

$$\int_0^\infty dz\, z^7 f(z) = \frac{8\pi^8}{15} \quad \text{and} \quad \int_0^\infty dz\, z^3 f(z) = \frac{\pi^4}{15}$$

with $f(z) = (e^z - 1)^{-1}$, we find that the nonanalytic terms in Eqs. (4.6) give contributions to the entropy of equal magnitude [order $T^7 \ln(1/T)$] but with opposite sign and thus cancel exactly. Our final result for the low-T simple quasiparticle contribution to C_V to $O(g)$ is

$$C_V^{\mathrm{QP}} = \tfrac{2}{15}\pi^3 T^3 (C_0 + C_2 T^2 + C_4 T^4 + \cdots), \tag{4.7a}$$

$$C_0 = 1 - 3\pi^{-2} g, \quad C_2 = -\tfrac{25}{14}\pi^2 + 5g\tfrac{169}{63}. \tag{4.7b}$$

The evaluation of the C_4 coefficient is tedious and has not been attempted. We have not investigated the nature of the higher-T terms. Note that to $O(g)$ the leading correction to the T^3 behavior of C_V^{QP} is of the order of T^5, the coefficient C_2 of the T^5 term is not affected by the T dependence of $\omega_k(T)$, and C_V^{QP} is analytic in T through at least order T^7. The significance of the exact cancellation of the nonanalytic terms in Eq. (4.6) when integrated is not fully understood.

D. Lifetime contribution to $O(g^2)$

The behavior of the leading lifetime part of the entropy spectral function, $\delta\sigma$ [see Eq. (3.20)], depends on the form of $\mathcal{G}^{\mathrm{QP}}$ and Γ. As emphasized in Sec. III B, lifetime effects enter in the one-particle Green's function in $O(g)$, i.e., $\Gamma = g\Gamma^{(1)} + O(g^2)$, and thus the leading order of $\delta\sigma$ is $O(g^2)$. It follows from Eq. (3.20) that to $O(g^2)$ the $\mathcal{G}^{\mathrm{QP}}$ in $\delta\sigma$ needs to be evaluated only to zeroth order, i.e., the well-known Bogoliubov approximation $\mathcal{G}^{(0)}$. Since the singular behavior of the coherence factors associated with $\mathcal{G}^{(0)}$ will be important in determining the T dependence of $\delta\sigma$, it is convenient to isolate the singular coherence factors and write $\mathcal{G}_{\mu\nu}^{(0)}$ ($\mu, \nu = +-$) in the form

$$\mathcal{G}_{\pm\pm}^{(0)} = \mathcal{G}_s \pm \mathcal{G}_r, \quad \mathcal{G}_{+-}^{(0)} = -\overline{\mathcal{G}}_s, \tag{4.8}$$

$$\mathcal{G}_s = \eta_s(a-b), \quad \overline{\mathcal{G}}_s = \overline{\eta}(a-b), \quad \mathcal{G}_r = \tfrac{1}{2}(a+b), \tag{4.9}$$

where the coherence factors, η_s and $\overline{\eta}$,

$$\eta_s - \overline{\eta} = \tfrac{1}{2}\lambda, \quad \eta_s + \overline{\eta} = 1/2\lambda, \tag{4.10}$$

are proportional to $1/k$ for small k. Here

$$\lambda = \frac{\epsilon_k}{\omega_k}, \quad \frac{1}{a} = \omega - \omega_k, \quad \frac{1}{b} = \omega + \omega_k, \quad \omega_k = k(1+\tfrac{1}{4}k^2)^{1/2},$$

and the superscript (0) and the k and ω dependences of various functions have been suppressed whenever it will not cause confusion. In the limit $k \sim \omega \sim T \ll 1$, $\mathcal{G}_s \sim \overline{\mathcal{G}}_s \sim T^{-2}$ and $\mathcal{G}_r \sim T^{-1}$. Note that $\mathcal{G}_s$ and $\overline{\mathcal{G}}_s$ are both odd in ω, whereas $\mathcal{G}_r$ is even in ω.

It is also convenient to divide the imaginary part of the self-energy $\Gamma_{\mu\nu}$ into even and odd pieces [see Eq. (3.3)]:

$$\Gamma_{\pm\pm} = -2(\mathcal{S} \pm \mathcal{A})_I, \quad \Gamma_{+-} = -2\mathfrak{M}_{2I}, \tag{4.11}$$

where we have adopted a shorthand notation $(\mathcal{S} \pm \mathcal{A})_I \equiv \mathrm{Im}(\mathcal{S} \pm \mathcal{A})$, etc. In the dielectric-function approach,[9] the $O(g)$ contribution to Γ can be expressed in terms of one-loop regular functions (see Appendix C):

$$(\mathcal{S} - \mathfrak{M}_2)_I^{(1)} = (S - M_2)_I, \tag{4.12}$$

$$\tfrac{1}{2}(\mathcal{S} + \mathfrak{M}_2)_I^{(1)} = (2/\omega^2)(S - M_2)_I - (2/\omega)L_{3I} + L_{4I} + \tfrac{1}{2}(S + M_2)_I, \tag{4.13}$$

$$\mathcal{A}_I^{(1)} = -(2/\omega)(S - M_2)_I + L_{3I}, \tag{4.14}$$

where

$$L_{3I} \equiv A_I + (k/\omega)\beta_\mu \Lambda_{\mu I}^z, \tag{4.15}$$

$$L_{4I} \equiv (k/\omega)\delta_\mu \Lambda_{\mu I}^z + (k^2/\omega^2)F_I^{zzr} \tag{4.16}$$

and the superscript (1) on the one-loop regular functions or integrals, explicitly given in Appendix A, is suppressed. We note for future reference that the leading T dependences for $k \sim \omega \sim T \ll 1$ are [see Eqs. (A21)–(A26)]:

$$(S - M_2)_I \sim T^2, \quad \beta_\mu \Lambda_{\mu I}^z \sim A_I \sim T^3,$$

$$(S + M_2)_I \sim \delta_\mu \Lambda_{\mu I}^z \sim F_I^{zzr} \sim T^4,$$

so that from Eqs. (4.13) and (4.14):

$$(\mathcal{S}+\mathfrak{M}_2)_I^{(1)} \sim T^0, \quad \mathcal{A}_I^{(1)} \sim T, \quad \text{and} \quad (\mathcal{S}-\mathfrak{M}_2)_I^{(1)} \sim T^2 .$$

The first step in the evaluation of $\delta\sigma$ is to perform the matrix sum in Eq. (3.22). Using the fact that $\partial \mathcal{G}_{QP}^{-1}/\partial\omega = \tau_3$, we write Eq. (3.22) in the form

$$\delta\sigma = -\tfrac{1}{2}g^2 \,\mathrm{Im}(\Delta_{++} - \Delta_{--}), \tag{4.17}$$

$$\Delta_{\mu\nu} = (\mathcal{G}^{(0)}\Gamma^{(1)}\mathcal{G}^{(0)}\Gamma^{(1)}\mathcal{G}^{(0)})_{\mu\nu}, \tag{4.18}$$

where matrix multiplication is implied. The evaluation of Δ directly from Eq. (4.18) is straightforward but tedious. We can separate the evaluation into several steps and write Δ as

$$\Delta = 4HK, \tag{4.19}$$

$$H = -\tfrac{1}{2}\mathcal{G}^{(0)}\Gamma^{(1)}, \tag{4.20}$$

$$K = -\tfrac{1}{2}\mathcal{G}^{(0)}\Gamma^{(1)}\mathcal{G}^{(0)}, \tag{4.21}$$

where the matrix K also enters into the leading correction to $\mathcal{G}^{(0)}$ given in Eq. (3.17), i.e.,

$$\delta\mathcal{G}^{(1)} = igK. \tag{4.22}$$

To be concrete we consider K_{++} in some detail. Using Eqs. (4.8) and (4.11) we can write K_{++} in the form

$$\begin{aligned} K_{++} = {} & (\mathcal{G}_s^2 \mathcal{S}_I - 2\mathcal{G}_s\overline{\mathcal{G}}_s\mathfrak{M}_{2I} + \overline{\mathcal{G}}_s^2\mathfrak{M}_{2I}) \\ & + [\mathcal{G}_s^2\mathcal{A}_I + 2\mathcal{G}_s\mathcal{G}_r(\mathcal{S}-2\mathfrak{M}_2)_I - \overline{\mathcal{G}}_s^2\mathcal{A}_I] \\ & + (\mathcal{G}_r^2\mathcal{S}_I + 2\mathcal{G}_s\mathcal{G}_r\mathcal{A}_I) + \mathcal{G}_r^2\mathcal{A}_I . \end{aligned} \tag{4.23}$$

Since the expansion (4.22) of $\mathcal{G}$ is expected to be valid at low T, we expect that $K_{++}/\mathcal{G}_{++}$ would vanish as $T\to 0$. In the limit $k\sim\omega\sim T\ll 1$, $\mathcal{G}_s^2\mathcal{S}_I \sim (T^{-2})^2 T^0 \sim T^{-4}$, so that $K_{++}/\mathcal{G}_{++}\sim T^{-2}$, which does not vanish as $T\to 0$. However, if we group together terms of the same leading T dependence as noted by the brackets in Eq. (4.23), this singular behavior is cancelled and $K_{++}/\mathcal{G}_{++}\sim T^2$ as expected. To see this we substitute the explicit forms (4.9) and (4.10) of $\mathcal{G}_s$ and $\overline{\mathcal{G}}_s$ into Eq. (4.23), cancel singular terms, collect terms, and obtain

$$\begin{aligned} K_{++} = {} & \left(\frac{(a-b)^2(\mathcal{S}-\mathfrak{M}_2)_I}{8\lambda^2} + \mathcal{G}_r^2\mathcal{S}_I + 2\mathcal{G}_s\mathcal{G}_r\mathcal{A}_I\right) \\ & + \left(\frac{(a-b)^2\mathcal{A}_I}{4} + \frac{\lambda(a-b)\mathcal{G}_r(\mathcal{S}+\mathfrak{M}_2)_I}{2} \right. \\ & \left. + \frac{(a-b)\mathcal{G}_r(\mathcal{S}-\mathfrak{M}_2)_I}{2\lambda} + \mathcal{G}_r^2\mathcal{A}_I\right) \\ & + \frac{\lambda^2(a-b)^2(\mathcal{S}+\mathfrak{M}_2)_I}{8}, \end{aligned} \tag{4.24}$$

where the leading T dependence of the three groups of terms in Eq. (4.24) is T^{-2}, T^{-1}, and T^0, respectively. Further cancellation occurs when we substitute Eqs. (4.12)–(4.16) into Eq. (4.24). The T^{-2} terms combine to give a leading behavior of T^0; similarly the T^{-1} terms combine to give terms proportional to T; thus $K_{++}/\mathcal{G}_{++}\sim T^2$. Similar cancellations or combinations can be found for the other K functions in the same manner. The final result of this algebra is

$$K_{\pm\pm} = K_0 \pm K_1, \quad K_{\pm\mp} = K_0, \tag{4.25}$$

$$K_0 = -p(S-M_2)_I/2D^2, \tag{4.26}$$

$$K_1 = \frac{(\omega^2-k^2)[L_{3I} + k^2(S-M_2)_I/2\omega]}{D^2}, \tag{4.27}$$

where

$$p(k,\omega) \equiv 2k^2 - \omega^2 - k^4\omega^{-2}, \tag{4.28}$$

$$D(k,\omega) \equiv \omega^2 - \omega_k^2 . \tag{4.29}$$

A similar analysis of the H functions yields

$$H_{\pm\pm} = H_0 \pm H_1, \quad H_{+-} - H_{-+} = \overline{H}_1, \tag{4.30}$$

$$H_0 = -(\omega^2-k^2)(S-M_2)_I/\omega^2 D, \tag{4.31}$$

$$H_1 = \frac{(\omega^2-2k^2)(S-M_2)_I/2\omega - L_{3I}}{D^2}, \tag{4.32}$$

$$\overline{H}_1 = -[2L_{3I} + \omega(S-M_2)_I]/D . \tag{4.33}$$

The leading T dependence of these functions are as follows: $K_0\sim H_0\sim T^0$, $K_1\sim H_1\sim \overline{H}_1\sim T$, and higher-order terms have been neglected.

From Eqs. (4.19), (4.25), and (4.30) we see that

$$\Delta_{++} - \Delta_{--} = 8(H_0K_1 + H_1K_0) + 4\overline{H}_1K_0 . \tag{4.34}$$

Substituting Eqs. (4.26), (4.27), and (4.31)–(4.33) into Eqs. (4.34) and (4.17), we find the following simple form for the leading T dependence of $\delta\sigma$:

$$\delta\sigma = -g^2 8p(S-M_2)_I \,\mathrm{Im}(N_{3I}/D^3), \tag{4.35}$$

$$N_{3I} \equiv A_I + k\beta_\mu\Lambda_{\mu I}^z/\omega + k^2(S-M_2)_I/2\omega, \tag{4.36}$$

where $N_{3I}\sim T^3$. Noting that $\mathrm{Im}(\omega-\omega_k+i0+)^{-3}$ can be interpreted as $-\tfrac{1}{2}\pi\delta''(\omega-\omega_k)$, $p(\omega_k)=p'(\omega_k)=0$, $p''(\omega_k)=-8$, and anticipating two integrations by parts, we can express Eq. (4.35) for $\omega>0$ in the the form:

$$\delta\sigma = -g^2 4\pi k^{-3}(S-M_2)_I N_{3I}\delta(\omega-\omega_k). \tag{4.37}$$

From the leading T dependence of the one-loop integrals in Eq. (4.37), we see that $\delta\sigma\sim T^2\delta(\omega-\omega_k)$ and thus the leading lifetime contribution δS to the entropy is order T^5. However, a careful evaluation of $(S-M_2)_I$ shows that $(S-M_2)_I\sim T^2\ln T$ rather than simply T^2, and thus the leading T dependence of δS is $T^5\ln T$. To see explicitly the origin of the logarithm, we use Eq. (A5) for the integral $(S-M_2)$ and consider the part of the integral proportional to $f_{p'}Q^+$:

$$(S-M_2)_I = -\frac{\pi}{2}\int\frac{d^3p}{(2\pi)^3}\left(1+\frac{\lambda_p}{\lambda_{p'}}\right) f_{p'}\,\delta(\omega_k-\omega_{p'}-\omega_p), \tag{4.38}$$

where $\vec{p}' = \vec{p} + \vec{k}$ and the argument of $(S - M_2)_I$ is $k, \omega = \omega_k$. We interchange $\vec{p}$ and $-\vec{p}'$, use the small p expansions $\lambda_p \approx \frac{1}{2}p$ and $\omega_p \approx p$, and write Eq. (4.38) as

$$(S - M_2)_I = -\frac{1}{8\pi k}\int_0^\infty dp\, p f_p \int_{|p-k|}^{p+k} dp'\, p'\left(1 + \frac{p'}{p}\right) \times \delta(p' + p - k). \tag{4.39}$$

The δ-function singularity occurs in Eq. (4.39) at the limit of the range of integration and thus Eq. (4.39) is not well defined. A careful limiting procedure shows that if ω_k/k is an increasing function of k at small k, then one-loop integrals such as Eq. (4.39) are nonzero. However, if ω_k/k is a decreasing function of k at small k, then Eq. (4.39) vanishes. Since in our model $\omega_k/k = (1 + \frac{1}{4}k^2)^{1/2}$ is an increasing function of k, we obtain

$$(S - M_2)_I = -\frac{1}{8\pi}\int_0^k dp(k-p) f_p. \tag{4.40}$$

The logarithmic divergence in the integral $\int_0^k dp f_p$ in Eq. (4.40) arises from the large number of phonons in low momentum states and implies that a straightforward expansion in powers of g does not exist for the imaginary part of the self-energy function in the limit $k \sim \omega \sim T \ll 1$. Because of the nature of our perturbation calculation, quasiparticle damping has not been included in the propagators (4.9). Since we are interested only in the dominant logarithmic term in Eq. (4.40), we can introduce a low-p cutoff, $p_c = gkT^4$, which is a measure of the phonon damping.[25] The contribution to $(S - M_2)_I$ from the R^+ part of the integrand [see Eq. (A5)] is the same as Eq. (4.40) so that the dominant contribution to $(S - M_2)_I$ is

$$(S - M_2)_I = -(1/4\pi)T^2 z \ln(1/gT^4), \tag{4.41}$$

where $z \equiv k/T$. If we use Eqs. (A22), (A23), and (A25) in Eq. (4.37), we find that N_{3I} does not contain a logarithmic singularity and is given by

$$N_{3I} = T^3[\tfrac{1}{8}\pi z - z^3/32\pi - (3/4\pi)F(z)], \tag{4.42a}$$

where $F(z)$ is a positive integral

$$F(z) = \int_0^z dx\, x(z-x) f(x) \tag{4.42b}$$

and $f(x) = (e^x - 1)^{-1}$. Using Eqs. (4.41), (4.42), (4.37), and (2.23), and the evenness of $\sigma(k, \omega)$ with respect to ω, we obtain the following form for the leading low-T lifetime contribution to the entropy:

$$\delta S = \frac{g^2}{2\pi^2} T^5 \ln\left(\frac{1}{gT^4}\right)\int_0^\infty dz\, n(z) \times\left(\frac{z}{8} - \frac{z^3}{32\pi^2} - \frac{3}{4\pi^2}F(z)\right), \tag{4.43}$$

where the statistical factor $n(z)$ is defined in Eq. (2.22). The integral in Eq. (4.43) involving $F(z)$ is evaluated numerically and the other integrals can be done analytically. Our final result for the leading low-T lifetime contribution to the specific heat of a Bose gas is

$$\delta C_V = C_L T^5 \ln(1/gT^4), \tag{4.44a}$$

$$C_L = -0.035g^2, \tag{4.44b}$$

which is quoted in Eq. (1.3). The significance of Eqs. (4.44) and its derivation is discussed in Sec. V.

V. DISCUSSION

In Sec. IV we have performed a microscopic calculation of the simple quasiparticle and leading lifetime contribution to C_V for a one-parameter model of a Bose gas. We discuss in Sec. V A the interpretation and implications of the model calculation and compare them to some qualitative features found from quantum hydrodynamics. The limitations of quantum hydrodynamics are also discussed. In Sec. V B we consider the interpretation of C_V for superfluid ^{4}He and give a crude estimate of the coefficient of the $T^5 \ln T$ term for superfluid ^{4}He.

A. Qualitative features of the model calculation

The main result of our model calculation [Eq. (4.44)] is that the leading lifetime contribution δC_V is $O(g^2)$ and proportional to $T^5 \ln T$. Thus the naive argument given in Sec. I yielding $\delta C_V \sim T^9$ is incorrect. In order to understand the origin of the $T^5 \ln T$ dependence of δC_V, we discuss the microscopic calculation and compare it to the results of a quantum-hydrodynamics (QHD) calculation upon which the argument presented in Sec. I is based.

The significance of the $O(g)$ one-loop diagrams (see Ref. 9, Fig. 6) is that they are the simplest diagrams which contribute to $\mathrm{Im}\,\omega_k$, the quasiparticle damping. The corresponding result[25] for $\mathrm{Im}\,\omega_k$ can be interpreted as the transition rate from a quasiparticle with momentum $\vec{k}$ to two-quasiparticle states, i.e., a three-phonon process. The vertices in the one-loop diagrams result in the λ_p factors that appear in the one-loop integrals Eqs. (A4)–(A9). If we ignore the structure of the vertices, the $O(g)$ contribution to $\mathrm{Im}\,\omega_k$ can be represented as in Fig. 1(a). It was shown in Sec. III B that the total contribution of the one-loop diagrams to the entropy is identically zero. Hence the typical structure of the lowest-order contribution to $\delta\sigma$ is given by the two-loop $O(g^2)$ diagram shown in Fig. 1(b). If we consider Fig. 1(a) as a

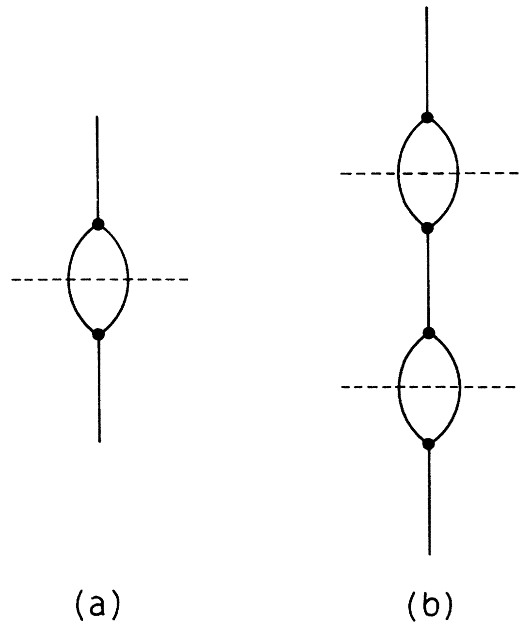

FIG. 1. Structure of (a) one-loop contributions to $\mathrm{Im}\,\omega_k$ and $\delta\rho$, and (b) two-loop contributions to $\delta\sigma$. Dotted line represents the imaginary part, and the detailed structure of the vertices is ignored.

diagram for $\delta\rho$, we can see from Fig. 1(b) that

$$\sigma \sim \frac{1}{k}\,\frac{\Gamma_k^2}{(\omega-\omega_k)^2+\frac{1}{4}\Gamma_k^2}\,\rho, \tag{5.1}$$

$$\rho \sim \frac{1}{k}\,\frac{\Gamma_k}{(\omega-\omega_k)^2+\frac{1}{4}\Gamma_k^2}\,. \tag{5.2}$$

We have assumed that the matrix structure and k and ω dependence of $\Gamma(k,\omega)$ can be effectively represented by a single function of k, Γ_k. The k^{-1} factor arises from the divergent phonon residue. The form (5.1) can also be obtained directly from Eq. (2.25) by making the above assumptions on $\Gamma(k,\omega)$. Note that from Eqs. (5.1) and (5.2), $\sigma(k,\omega)$ falls off much more rapidly in the wings as a function of ω than $\rho(k,\omega)$ and has a stronger peak. Thus, attempts to include the lifetime contribution to the specific heat by introducing a Lorentzian line shape[22] for $\sigma(k,\omega)$ are most likely overestimates. In the limit that Γ_k is small, we can separate the lifetime correction in Eq. (5.1) as

$$\delta\sigma \sim (\Gamma_k^2/k^2\omega_k^2)\delta(\omega-\omega_k). \tag{5.3}$$

If we assume that Γ_k is equal to the damping constant γ_k of the quasiparticle and use the result[26] that $\gamma_k/k \sim g\,T^4$, then we find $\delta\sigma \sim g^2T^6\delta(\omega-\omega_k)$. The complicated structure of the vertices and the matrix structure of Γ, which have been ignored in obtaining this estimate, give rise to an important factor of T^{-4}. Thus $\delta\sigma \sim g^2T^2\delta(\omega-\omega_k)$ as found in Sec. IV C. The reason for this factor, as in the appearance of a T^{-4} factor in the damping of second sound,[27] is the near proportionality of the phonon energy and momentum, which renders the emission and absorption processes anomalous in the sense that only small angle processes are possible. As we have seen from the integrand in Eq. (4.40), the logarithmic factor is due to the large occupation number of phonons in the low momentum states. Thus the origin of the $T^5\ln T$ term in δC_V can be said to be threefold: the large number of phonons in the small-k limit, the small-k divergent phonon residue, and the anomalous three-phonon absorption and emission contribution.

We now compare the microscopic calculation to the calculation of $\delta\sigma$ based on the assumed validity of quantum hydrodynamics[2,7] (QHD). Since there is no need for a matrix notation in QHD, we can write Eq. (3.22) as

$$\delta\sigma = -\frac{1}{2}\,\frac{\partial \bar{\mathcal{G}}_{QP}^{-1}}{\partial\omega}\,\bar{\Gamma}(k,\omega)^2\,\mathrm{Im}[\bar{\mathcal{G}}^{QP}(k,\omega)]^3, \tag{5.4}$$

where $\bar{\mathcal{G}}_{QP}^{-1}=\omega^2-c^2k^2$. Here $\bar{\Gamma}(k,\omega)$ is the imaginary part of the inverse of the phonon Green's function $\bar{\mathcal{G}}$, a two-particle function, and is not simply related to the matrix $\Gamma(k,\omega)$ defined in Eq. (3.1). If we retain only the three-phonon contribution to $\bar{\Gamma}(k,\omega)$, we have[28]

$$\bar{\Gamma}(k,\omega)=4\pi\int\frac{d^3p}{(2\pi)^3}\,[2V^2(\vec{k},\vec{p},\vec{p}+\vec{k})(f_p-f_{\vec{p}+\vec{k}})\delta(\omega+\epsilon_p-\epsilon_{\vec{p}+\vec{k}})+V^2(\vec{k},\vec{p},\vec{k}-\vec{p})(1+f_p+f_{\vec{k}-\vec{p}})\delta(\omega-\epsilon_p-\epsilon_{\vec{k}-\vec{p}})$$
$$-V^2(\vec{k},\vec{p},-\vec{k}-\vec{p})(1+f_p+f_{\vec{k}+\vec{p}})\delta(\omega+\epsilon_p+\epsilon_{\vec{k}+\vec{p}})], \tag{5.5}$$

where

$$V(\vec{k},\vec{p},\vec{p}\,')=(ckpp'/32mn)^{1/2}(2u-1+\hat{k}\cdot\hat{p}+\hat{p}\cdot\hat{p}\,'+\hat{p}\,'\cdot\hat{k}), \tag{5.6}$$

$u=(n/c)(\partial c/\partial n)$ is the Grüneisen constant, and $f_p=(e^{\beta cp}-1)^{-1}$. An inspection of Eq. (5.5) shows that for $k\sim\omega\sim T\ll 1$, $\bar{\Gamma}(k,\omega)/k\sim T^4$, and hence from Eq. (5.4), $\delta\sigma\sim T^6$ and $\delta C_V\sim T^9$, in agreement with the naive argument given in Sec. I but in disagreement with the microscopic calculation.

The missing factor in the QHD calculation is T^{-4}, which arises in the microscopic calculation from the near proportionality of the phonon energy and momentum. At $T=0$ the relationship of QHD to microscopic theory has been investigated by Josephson,[29] who pointed out that in the case of almost collinear phonons the validity of QHD may be in doubt, because singularities other than the pole singularities taken into account in QHD may have important effects. In the microscopic calculation, we see from Fig. 1(a) that ρ has the form [Eq. (5.2)] and $\mathcal{G}$ has a branch singularity due to a phonon decaying into almost collinear phonons. The branch structure plays an important role in the two-loop contributions [Fig. 1(b)] to $\delta\sigma$, but is not important in the one-loop calculation of $\mathrm{Im}\omega_k$. Therefore the leading contribution to $\mathrm{Im}\omega_k$ is amenable to QHD but the leading contribution to $\delta\sigma$ is not. Our conclusion should not be interpreted as denying the validity of QHD in general. The QHD Hamiltonian should properly be regarded as a pseudo-Hamiltonian that generates the asymptotically correct correlation functions in the hydrodynamic limit. When used within its limitations, QHD should illustrate certain correlations among macroscopic quantities. However, if one regards it as a microscopic model and computes higher-order perturbation terms, the results may not be very meaningful.

From the microscopic calculation, we see that the effective expansion parameter $\tilde{g}$ of $\delta\sigma$ (or δC_V) with respect to σ^{QP} (or C_V^{QP}) can be written as $\tilde{g}\sim\gamma_k/kT^2\omega_k\sim gT$ and the leading lifetime correction is $O(\tilde{g}^2)$. Alternatively the leading lifetime correction to the *statistical* quasiparticle energy is $O(g^2k^2)$ and has the form $g^2k^3\ln(1/gk^4)$, in contrast to the nonanalytic dynamical quasiparticle energy[24] $gk^5\ln(1/k)$.

Although C_V^{QP} and δC_V are calculated to $O(g)$ and $O(g^2)$, respectively, both are calculated from the one-loop functions that represent three-phonon processes. The $O(g^2)$ contribution to C_V^{QP}, which arises from four-phonon processes, is not known; in particular it is not whether an $O(g^2T^5\ln T)$ term exists in C_V^{QP}. Nevertheless we can say unambiguously that the $O(g)$ three-phonon process gives rise to a $g^2T^5\ln T$ term in the lifetime contribution δC_V.

The numerical value of C_L in Eq. (4.44) supersedes the value cited in Ref. 14. The $g^2T^5\ln T$ term in δC_V owes its existence to the upward curvature of ω_k, but the numerical value of C_L itself is independent of the magnitude of the (upward) curvature.

As discussed elsewhere[25] the quasiparticle in our model should be interpreted as "zero sound," and thus the form of many of the ω-dependent functions in the present calculation are valid only for $\omega\gg g$. In the region $O<\omega<gT^5$, second sound[27] appears. However, it can be shown that second sound contributes to C_V in $O(g^4)$ and thus can be ignored here.

B. Interpretation of C_V measurements in superfluid ^{4}He

Since the effective expansion parameter of δC_V with respect to C_V^{QP} is $\tilde{g}\sim gT$, it is not unreasonable to assume that the leading low-T behavior of δC_V in superfluid ^{4}He, in which g is not small, is order $T^5\ln T$ and arises from the three-phonon processes. In the model calculation the existence of the $T^5\ln T$ term in δC_V depends on the upward curvature of ω_k. The phonon curvature of superfluid ^{4}He at low pressures[5] is now accepted as upward; however, at high pressures[4] the phonon curvature is thought to be downward and the three-phonon process is forbidden by energy-momentum conservation. In the latter case the simplest conjecture is that the induced three-phonon process would also yield the $T^5\ln T$ dependence of δC_V.

The quantity of physical interest is of course $C_V=C_V^{\mathrm{QP}}+\delta C_V$. Although we have argued that $\delta C_V\sim T^5\ln T$, it is not known whether C_V^{QP} has a $T^5\ln T$ term or not. In the event that $C_V^{\mathrm{QP}}\sim T^5\ln T$, it follows that the excitation spectrum E_k has the form (1.1) but with an additional logarithmic term $\sim e_{2L}k^3\ln k$, which at small k dominates the e_2k^3 term. Hence in principle a measurement of the T^5 term of C_V would not determine the leading phonon dispersion (e_{2L}). On the other hand if C_V^{QP} does not have a $T^5\ln T$ term, then at low T the $T^5\ln T$ term in δC_V would dominate the T^5 term. It follows that an analysis of the low-T specific-heat data with the $T^5\ln T$ term ignored would in principle also not determine the leading phonon dispersion (e_2).

Although these conclusions are mathematically rigorous, the important consideration from the practical point of view is the magnitude of the logarithmic term. Since there appears to be little evidence for a $e_{2L}k^3\ln k$ term in E_k, we restrict ourselves to the case where C_V^{QP} has no $T^5\ln T$ term and attempt to estimate the magnitude of the $T^5\ln T$ term in δC_V for helium.

It is convenient first to reduce the low-T specific-heat data to a dimensionless form. Phillips *et al.*[3] fitted their data to the form

$$C_V=(A/T_0^3)\,T^3+(B/T_0^5)\,T^5+\cdots\,, \tag{5.7}$$

where at saturated vapor pressure ($V=27.58$ cm^3/mole) the coefficients were determined to be $A/T_0^3=81.57$ mJ/mole K^4 and $B/T_0^5=-15.6$ mJ/mole K^6 and $T_0\equiv mc^2(0)/k_B$. If we take the characteristic temperature T_0 to be $T_0\approx 27.7$ K, which corresponds to $c(0)\approx 2.4\times10^4$ cm/sec, we can rewrite Eq. (5.7) in dimensionless form

$$C_V/AT^3 = 1 + QT^2 + \cdots, \tag{5.8a}$$

$$Q = -146.7, \tag{5.8b}$$

where T is now the temperature divided by T_0. The dimensionless coefficient Q may include both simple quasiparticle and lifetime contributions, since it was obtained by a straightforward fit[3] of the data. If we assume that at low T the specific heat has only a simple quasiparticle contribution, then the low-k form of the quasiparticle spectrum leading to Eq. (5.8) is given by Eq. (1.1) with $e_2 = -(7/100\pi^2)Q \approx 1.04$.

The lifetime correction to the T^3 phonon contribution to C_V, from the model calculation, can be written in dimensionless form as

$$C_V/AT^3 = 1 + LT^2\ln(1/gT^4) + \cdots, \tag{5.9a}$$

$$L = -(15/2\pi^2)g^2C_L. \tag{5.9b}$$

Taking this form as valid for helium, we can now estimate the lifetime contribution by inserting an effective value of g into Eqs. (5.9). Such an effective g for helium can be obtained from a comparison of the results of a microscopic calculation with that from QHD.[7,26] For example, the microscopic calculation[25] of the leading T dependence of the phonon speed in a Bose gas gives

$$[c(T) - c(0)]/c(0) = \tfrac{3}{40}\pi^2 gT^4\ln(1/T);$$

the analogous result from quantum hydrodynamics[2] is

$$[c(T) - c(0)]/c(0) = \tfrac{1}{30}\pi^2(u+1)^2T^4\ln(1/T),$$

where u is the Grüneisen constant and T is the temperature divided by T_0. Both results agree if we take $g = [\frac{2}{3}(u+1)]^2$. This identification is corroborated by the results for phonon damping. Since u is approximately 2.75, we find that $g \approx 6.25$. Such a value is not too large, especially if we note that the effective expansion parameter is gT.

If we take $g \approx 6.25$ for helium, then from Eq. (5.9b) we find $L \approx -1$. At 0.4 K, the coefficient of the T^2 term in Eq. (5.9a) is $L\ln(1/gT^4) \approx -16$. Comparing this figure with Q in Eq. (5.8b), we see that δC_V is approximately a 10% correction at saturated vapor pressure. At higher pressures the coefficient B in Eq. (5.7) or Q in Eq. (5.8) decreases in magnitude. At $V = 26.23$ cm^3/mole, Phillips *et al.* found $A/T_0^3 = 52.41$ mJ/mole K^4, $B/T_0^5 = -1$ mJ/mole K^6, $c(0) = 2.732\times10^4$ cm/sec, $T_0 \approx 36$ K, and the dimensionless coefficient is calculated to be $Q \approx -25$. If we assume that the lifetime coefficient L in Eq. (5.9) does not vary strongly with pressure, we find that the magnitude of δC_V is comparable to the alleged quasiparticle contribution.[3,4] At still higher pressure ($V \sim 26$ cm^3/mole), the curvature of the phonon dispersion changes sign. Detailed measurements of C_V and other properties of superfluid ^{4}He in the neighborhood of the change in sign in the phonon dispersion is clearly of interest.

Thus we conclude that the leading phonon dispersion cannot be rigorously determined from a measurement of the $O(T^5)$ term of C_V. If the above estimate is valid, the lifetime contribution might be a 10% (or larger) correction.

APPENDIX A: ONE-LOOP INTEGRALS AT $T \geqslant 0$

In the dielectric-function approach (see Sec. IV B), all functions of interest are expressed in terms of the *regular* self-energy $M_{\mu\nu}$, current-vertex Λ^z_μ, and current-current-correlation F^{zzr}. For ease of reference we collect here the evaluations at $T>0$ of these regular functions to $O(g)$.

The diagrams corresponding to the regular functions ($M_{\mu\nu}$, Λ^z_μ, and F^{zzr}) to $O(g)$ all have a one-loop structure, as shown in Fig. 6 of Ref. 9. The evaluation of the one-loop diagrams at $T>0$ is straightforward[16] and yields

$$\begin{aligned} M^{(1)}_{\mu\nu}(k) = &-\int\frac{d^3p}{(2\pi)^3}T\sum_\epsilon\left[\tfrac{1}{2}\mathcal{G}^{(0)}_{\lambda\lambda}(p)\,\delta_{\mu\nu} + \mathcal{G}^{(0)}_{\mu\nu}(p+k)\right] \\ &-\int\frac{d^3p}{(2\pi)^3}T\sum_\epsilon\{\mathcal{G}^{(0)}_{\mu\nu}(p+k)\mathcal{G}^{(0)}_{\lambda\sigma}(p)\delta_\lambda\delta_\sigma \\ &+\tfrac{1}{2}[\mathcal{G}^{(0)}_{\mu\lambda}(p+k)\mathcal{G}^{(0)}_{-\sigma-\nu}(p) \\ &+\mathcal{G}^{(0)}_{\nu\lambda}(p+k)\mathcal{G}^{(0)}_{-\sigma-\mu}(p)]\delta_\lambda\delta_\sigma\}, \end{aligned} \tag{A1}$$

$$\begin{aligned} \Lambda^{z(1)}_\mu(k) = &-\int\frac{d^3p}{(2\pi)^3}T\sum_\epsilon\hat{k}\cdot(\vec{p}+\tfrac{1}{2}\vec{k})\beta_\nu \\ &\times\mathcal{G}^{(0)}_{\mu\nu}(p+k)\,\mathcal{G}^{(0)}_{\nu\lambda}(p)\,\delta_\lambda, \end{aligned} \tag{A2}$$

$$\begin{aligned} F^{zzr(1)}(k) = &-\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}T\sum_\epsilon[\hat{k}\cdot(\vec{p}+\tfrac{1}{2}\vec{k})]^2 \\ &\times\mathcal{G}^{(0)}_{\mu\nu}(p+k)\,\mathcal{G}^{(0)}_{\mu\nu}(p)\,\beta_\mu\beta_\nu, \end{aligned} \tag{A3}$$

where $\delta_{\mu\nu}$ is the Kronecker δ function, $\delta_\mu = 1$, $\beta_\mu = \mathrm{sgn}(\mu)$, p and k denote $(\vec{p}, \epsilon)$ and $(\vec{k}, \omega)$, respectively, ϵ and ω are discrete frequency variables $2\pi niT$, and $\mathcal{G}^{(0)}_{\mu\nu}$ is the matrix Green's function in the Bogoliubov approximation. The form (A1) for $M^{(1)}_{+-}$ differs in part from that given in Eq. (4.1) of Ref. 9; the latter ($T=0$) form does not satisfy the general symmetry property $M_{\mu\nu}(k) = M_{-\mu\,-\nu}(-k)$. The results of the $T=0$ calculation,[9] however, are not affected by this lack of symmetry. The frequency sums can be performed in the usual way,[16] and we obtain the following one-loop integrals at $T>0$:

$$\tfrac{1}{2}(S^{(1)}+M_2^{(1)}-\mu^{(1)})=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\lambda_{\vec p}\lambda_{\vec p+\vec k}(Q^+-R^+),\tag{A4}$$

$$\begin{aligned}\tfrac{1}{2}(S^{(1)}-M_2^{(1)}-\mu^{(1)})&=\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\frac{1-\lambda_{\vec p}^2}{\lambda_{\vec p}}(1+2f_{\vec p})\\&+\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\left[\left(1+\frac{\lambda_{\vec p}}{\lambda_{\vec p+\vec k}}\right)Q^++\left(1-\frac{\lambda_{\vec p}}{\lambda_{\vec p+\vec k}}\right)R^+\right],\end{aligned}\tag{A5}$$

$$A^{(1)}=\int\frac{d^3p}{(2\pi)^3}\lambda_{\vec p}(Q^-+R^-),\tag{A6}$$

$$\delta_\mu\Lambda_\mu^{z(1)}=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\hat k\cdot(\vec p+\tfrac{1}{2}\vec k)[(\lambda_{\vec p}-\lambda_{\vec p+\vec k})Q^-+(\lambda_{\vec p}+\lambda_{\vec p+\vec k})R^-],\tag{A7}$$

$$\beta_\mu\Lambda_\mu^{z(1)}=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\hat k\cdot(\vec p+\tfrac{1}{2}\vec k)\frac{\lambda_{\vec p}}{\lambda_{\vec p+\vec k}}(Q^+-R^+),\tag{A8}$$

$$F^{zz r(1)}=\frac{1}{8}\int\frac{d^3p}{(2\pi)^3}\frac{[\hat k\cdot(\vec p+\frac{1}{2}\vec k)]^2}{\lambda_{\vec p}\lambda_{\vec p+\vec k}}\times[(\lambda_{\vec p}-\lambda_{\vec p+\vec k})^2Q^+-(\lambda_{\vec p}+\lambda_{\vec p+\vec k})R^+],\tag{A9}$$

where

$$Q^\pm\equiv(1+f_{\vec p}+f_{\vec p+\vec k})\left(\frac{1}{\omega-\omega_{\vec p}-\omega_{\vec p+\vec k}}\mp\frac{1}{\omega+\omega_{\vec p}+\omega_{\vec p+\vec k}}\right),\tag{A10}$$

$$R^\pm\equiv(f_{\vec p}-f_{\vec p+\vec k})\left(\frac{1}{\omega-\omega_{\vec p}+\omega_{\vec p+\vec k}}\mp\frac{1}{\omega+\omega_{\vec p}-\omega_{\vec p+\vec k}}\right).\tag{A11}$$

It is convenient here to express the regular self-energy functions $M_{\mu\nu}$ in terms of $S\equiv\frac{1}{2}(M_{++}+M_{--})$, $A\equiv\frac{1}{2}(M_{++}-M_{--})$, and $M_2\equiv M_{+-}=M_{-+}$, and to omit the zero superscript on $\omega_k^{(0)}$ whenever it does not cause confusion. The one-loop integrals at $T=0$ can be easily recovered from Eqs. (A4)–(A11) by letting all the statistical factors $f_{\vec p}\equiv(e^{\omega_{\vec p}/T}-1)^{-1}$ in $Q^\pm$ and $R^\pm$ go to zero.

To evaluate explicitly Eqs. (A4)–(A9), we reduce the one-loop integrals to functions of k by taking $\omega=\omega_k^{(0)}$, which is sufficient for many applications. We divide the one-loop integrals into T-independent and T-dependent parts. Since the T-independent parts of Eqs. (A4)–(A9) have been evaluated in the $T=0$ limit,[9] we concentrate, in the remainder of this appendix, on the T-dependent parts in the limit $k\sim T\ll 1$.

We first consider the real parts of the T-dependent one-loop integrals at $\omega=\omega_k^{(0)}$. As an example, we evaluate the integral

$$I(k)=\int\frac{d^3p}{(2\pi)^3}\lambda_{\vec p}\lambda_{\vec p+\vec k}f_{\vec p}\times\left(\frac{1}{\omega_{\vec k}-\omega_{\vec p+\vec k}-\omega_{\vec p}}-\frac{1}{\omega_{\vec k}+\omega_{\vec p+\vec k}+\omega_{\vec p}}\right),\tag{A12}$$

where the principal part is understood. If we use the fact that the integrand of $\frac{1}{2}(S+M_2-\mu)$ in Eq. (A4) is symmetric in $\vec p$ and $\vec p+\vec k$, it is easy to see that $I(k)$ is the T-dependent part of $\frac{1}{2}(S+M_2-\mu)$ that arises from Q^+. An inspection of Eq. (A12) shows that the dominant contribution to the p integral is from $p\sim T\ll 1$; similarly from Eq. (2.23) the dominant contribution to the k integral is from $k\sim T\ll 1$. In the limit $k\sim T\sim p\ll 1$, we see from Eq. (A12) that the dominant T dependence arises from the singularity of the denominator $(\omega_{\vec k}-\omega_{\vec p+\vec k}-\omega_{\vec p})^{-1}$. We let $\omega_{\vec p+\vec k}=Ty(1-\frac{1}{8}T^2y^2+\cdots)$, $\lambda_p=\frac{1}{2}\omega_p+\cdots$, $\omega_p=Tx$, $\omega_k=Tz$, and write the singular part of I as

$$I=\frac{T^4}{16\pi^2 z}\int_0^z dx\,x^2f(x)\int_{z-x-T^2\delta}^{z+x}dy\,y^2(z-y-x)^{-1},\tag{A13}$$

where $\delta=\frac{3}{8}xz(z-x)$ and $f(x)=(e^x-1)^{-1}$. Since we are interested in the leading T dependence of Eq. (A13), only the range of integration shown needs to be considered. Performing the angular y integration, we obtain

$$I=-\frac{1}{8\pi^2 z}T^4\ln\left(\frac{1}{T}\right)\int_0^z dx\,x^2(x-z)^2f(x).\tag{A14}$$

The part of $\frac{1}{2}(S+M_2-\mu)$ that arises from R^+ and the remaining integrals in Eqs. (A6)–(A10) are evaluated in the same manner. The T-dependent results for $k\sim T\ll 1$ are

$$\tfrac{1}{2}(S^{(1)}+M_2^{(1)}-\mu^{(1)})=\tfrac{1}{30}\pi^2T^4\ln(1/T)+O(T^4),\tag{A15}$$

$$\tfrac{1}{2}(S^{(1)}-M_2^{(1)}-\mu^{(1)})=O(T^4),\tag{A16}$$

$$A^{(1)}=-\tfrac{1}{12}zT^3\ln(1/T)+O(T^3),\tag{A17}$$

$$\delta_\mu\Lambda_\mu^{z(1)}=(\tfrac{2}{15}\pi^2+\tfrac{1}{24}z^2)T^4\ln(1/T)+O(T^4),\tag{A18}$$

$$\beta_\mu\Lambda_\mu^{z(1)}=-\tfrac{1}{6}zT^3\ln(1/T)+O(T^3),\tag{A19}$$

$$F^{zz r(1)}=(\tfrac{2}{15}\pi^2+\tfrac{1}{12}z^2)T^4\ln(1/T)+O(T^4).\tag{A20}$$

In Sec. IV D the method of calculating the imaginary part of the one-loop integrals has been illustrated by evaluating the integral $(S-M_2)_I\equiv\mathrm{Im}(S-M_2)$ in some detail. We shall not repeat the discussion here and merely list the results.

The leading T dependences of the imaginary part of the one-loop integrals at $\omega=\omega_k^{(0)}$ for $k\sim T\ll 1$ are

$$(S^{(1)}+M_2^{(1)})_I=-T^4(z^4/480\pi+z_3/48\pi+\tfrac{1}{30}\pi^3)+O(T^6), \tag{A21}$$

$$(S^{(1)}-M_2^{(1)})_I=-T^2\left(\frac{z^2}{16\pi}+\frac{z}{4\pi}\int_0^z dx f(x)\right)+\cdots, \tag{A22}$$

$$A_I^{(1)}=-T^3\left(\frac{z^3}{96\pi}-z\frac{\pi}{24}+\frac{1}{4\pi}\int_0^z dx\, x(z-x)f(x)\right)+O(T^5), \tag{A23}$$

$$\delta_\mu\Lambda_{\mu I}^{z(1)}=T^4\left(\frac{z^4}{960\pi}-\frac{z^2\pi}{48}-\frac{\pi^3}{15}+\frac{1}{8\pi z}\int_0^z dx(z-x)(z-2x)^2 f(x)\right)+O(T^6), \tag{A24}$$

$$\beta_\mu\Lambda_\mu^{z(1)}=T^3\left(\frac{z^3}{96\pi}+z\frac{\pi}{12}+\frac{1}{8\pi}\int_0^z dx(2x-z)^2 f(x)\right)+\cdots, \tag{A25}$$

$$F_I^{zzr(1)}=-T^4\left(\frac{z^4}{640\pi}+z^2\frac{\pi}{24}+\frac{\pi^3}{15}+\frac{1}{32\pi z}\int_0^z dx(2x-z)^4 f(x)\right)+\cdots, \tag{A26}$$

where $z=k/T$. Note that Eqs. (A22), (A25), and (A26) contain the logarithmically divergent integral $\int_0^z dx f(x)$, and must be handled with care.

APPENDIX B: QUASIPARTICLE SPECTRUM AT $T\geqslant 0$

A simple application of the one-loop integrals evaluated in Appendix A is to the calculation of the quasiparticle spectrum ω_k to $O(g)$ at $T>0$. Following our earlier calculation[24] of ω_k to $O(g)$ at $T=0$, we develop a perturbation expansion for ω_k by writing $\omega_k^2=\omega_k^{2(0)}+g\omega_k^{2(1)}+O(g^2)$. As shown in Ref. 9 [Eq. (4.18′)], $\omega_k^{2(1)}$ is given by

$$\frac{\omega_k^{2(1)}}{k^2}=\frac{S^{(1)}+M_2^{(1)}-\mu^{(1)}}{2}+\frac{k^4}{4\omega_k^{2(0)}}\frac{S^{(1)}-M_2^{(1)}-\mu^{(1)}}{2}+\frac{k^2}{\omega_k^{2(0)}}\left(k^{-1}\omega_k^{(0)}\delta_\mu\Lambda_\mu^{z(1)}+\frac{k\beta_\mu\Lambda_\mu^{z(1)}}{2}-\frac{k^2n'^{(1)}}{4}+\frac{\omega_k^{(0)}A^{(1)}}{2}+F^{zzr(1)}\right)+v^{(1)}, \tag{B1}$$

where $\beta_\mu=\mathrm{sgn}(\mu)$, $\delta_\mu=1$, $\mu=+$ or $-$, repeated Greek indices are summed, and all the one-loop integrals are evaluated at $\omega=\omega_k^{(0)}$. We can divide $\omega_k^{2(1)}$ into T-independent and T-dependent parts: $\omega_k^{2(1)}(T)=\omega_k^{2(1)}(0)+\tilde\omega_k^{2(1)}(T)$. We evaluate here only the T-dependent part $\tilde\omega_k^{2(1)}(T)$. The quantity $v^{(1)}$ is T independent and can therefore be ignored.

The real part of $\tilde\omega_k^{2(1)}(T)$ is obtained by substituting Eqs. (A15)–(A20) into the real part of Eq. (B1). We find for $k\sim T\ll 1$ the leading T dependence

$$\tilde\omega_k^{2(1)}(T)/k^2=\tfrac{3}{10}\pi^2T^4\ln(1/T)+O(T^4,k^2T^2,\ldots), \tag{B2}$$

which is quoted in Eq. (4.6). It is easy to see from Eq. (B1) that $n'^{(1)}=(3\pi^2)^{-1}+\frac{1}{12}T^2+\cdots$ contributes only to the $O(k^2T^2)$ term. In earlier papers[9,25] the real part of $\omega_k(T)$ was calculated in the $k\ll 1$, $T=0$ limit and in the $k\ll T\ll 1$ limit. Comparing these results, we see that the leading k dependence in all three limits agree and that the leading T dependence of the $k\sim T\ll 1$ and $k\ll T\ll 1$ limits is also consistent. The leading T dependence in the $T\ll k\ll 1$ limit has yet to be calculated.

To evaluate the imaginary part $\mathrm{Im}\omega_k^{(1)}$, we substitute Eqs. (A21)–(A26) into the imaginary part of (B1). Upon collecting terms, we find that the logarithmic divergent terms in Eqs. (A21), (A25), and (A26) cancel, and we obtain for $k\sim T\ll 1$:

$$\mathrm{Im}\omega_k^{(1)}=-\frac{T^5}{8\pi}\left(\frac{3}{80}z^5+\frac{z^4}{24}+\frac{3\pi^4}{5}z+\frac{1}{2}\int_0^z dx\,x^2(7z^2-16zx+8x^2)f(x)\right)+O(T^7), \tag{B3}$$

where $z=k/T$. Previously we have calculated $\mathrm{Im}\omega_k^{(1)}$ in the $k\ll 1$, $T=0$ limit[9]

$$\mathrm{Im}\omega_k^{(1)}=-(3/640\pi)k^5[1-\tfrac{4}{21}k^2+O(k^4)] \tag{B4}$$

and in the $k\ll T\ll 1$ limit[25]

$$\mathrm{Im}\omega_k^{(1)}=-\tfrac{3}{40}\pi^3kT^4[1-\tfrac{110}{21}\pi^2T^2+O(T^4)]. \tag{B5}$$

Comparing Eqs. (B3)–(B5) we see that the leading k dependence in all three limits agree and the leading T dependence of the $k\sim T\ll 1$ and $k\ll T\ll 1$ limits is also consistent.

APPENDIX C: SOME SELF-ENERGY IDENTITIES AT $T\geqslant 0$

We derive in this appendix some relations for the self-energy which follow from the application of the generalized Ward identities.[9] We first establish the $O(g)$ results quoted in Eqs. (4.12)–(4.16). The second group of identities is exact and related to certain $T=0$ long-wavelength ($k=0$) identities for the self-energy required by Götze and Wagner[19] in their investigation of the leading

T dependence of C_V. We derive these identities at arbitrary T and give some additional static ($\omega=0$) identities, which can be considered as generalizations of the Hugenholtz-Pines[12] relation. To $O(g)$ we show that the $k=0$ identities contain a logarithmic singularity ignored in Ref. 23.

In Ref. 9 [see Eq. (2.41)] we expressed the self-energy $\mathfrak{M}_{\mu\nu}$ in terms of the regular self-energy $M_{\mu\nu}$, density vertex Λ_μ, and density-density-correlation F^r:

$$\mathfrak{M}_{\mu\nu}=M_{\mu\nu}+(v/\epsilon^r)\Lambda_\mu\Lambda_\nu\,, \tag{C1}$$

$$\epsilon^r=1-vF^r\,. \tag{C2}$$

The generalized Ward identities[9] of interest are

$$\omega\Lambda_\mu=k\Lambda_\mu^z+n_0^{1/2}\beta_\nu G_{\nu\mu}^{-1}\,, \tag{C3}$$

$$\omega^2F^r=k^2(F^{zzr}+n/m)-n_0^{1/2}\beta_\mu(k\Lambda_\mu^z+\omega\Lambda_\mu)\,, \tag{C4}$$

where Λ_μ^z is the (regular) current vertex, F^{zzr} is the (regular) current-current correlation, the inverse of the irreducible Green's function $G_{\mu\nu}^{-1}$ satisfies $G_{\mu\nu}^{-1}=(G_{\mu\nu}^0)^{-1}-M_{\mu\nu}$, and $G_{\mu\nu}^0$ is the Green's function for the noninteracting system. These expressions (C1)–(C4) are exact and valid at any T. Our approach is to use the Ward identities (C3) and (C4) to express $\mathfrak{M}_{\mu\nu}$ in terms of the regular functions $M_{\mu\nu}$, Λ_μ^z, and F^{zzr}, and to expand formally both sides of the resulting expression to first order in g. For example, we write

$$\mathfrak{M}_{\mu\nu}=\mathfrak{M}_{\mu\nu}^{(0)}+g\,\mathfrak{M}_{\mu\nu}^{(1)}+O(g^2)\,,$$

$$g^{1/2}\Lambda_\mu^z=\tfrac{1}{2}k\beta_\mu(1-\tfrac{1}{2}g\,n'^{(1)})+g\Lambda_\mu^{z(1)}+O(g^2),$$

$$gF^{zzr}=gF^{zzr(1)}+O(g^2),$$

etc., where the zeroth order is the Bogoliubov approximation. It is convenient to express $\mathfrak{M}_{\mu\nu}$ in terms of $\mathcal{S}$, $\mathfrak{A}$, and $\mathfrak{M}_2$ defined in Eq. (3.3). After some straightforward algebra, we obtain the $O(g)$ identities:

$$\mathcal{S}^{(1)}=S^{(1)}+L^{(1)}\,, \tag{C5}$$

$$\mathfrak{A}^{(1)}=-(2/\omega)(S^{(1)}-M_2^{(1)}-\mu^{(1)})+A^{(1)}+(k/\omega)\beta_\mu\Lambda_\mu^{z(1)}\,, \tag{C6}$$

$$\mathfrak{M}_2^{(1)}=M_2^{(1)}+L^{(1)}\,, \tag{C7}$$

where

$$L^{(1)}=v^{(1)}-n'^{(1)}\left(1-\frac{k^2}{\omega^2}\right)+\frac{2}{\omega^2}(S^{(1)}-M_2^{(1)}-\mu^{(1)})-\frac{2}{\omega}A^{(1)}-\frac{2k}{\omega^2}\beta_\mu\Lambda_\mu^{z(1)}+\frac{k}{\omega}\delta_\mu\Lambda_\mu^{z(1)}+\frac{k^2}{\omega^2}F^{zzr(1)}\,. \tag{C8}$$

If we take the imaginary part of Eqs. (C5)–(C8), we obtain the results quoted in Eqs. (4.12)–(4.16).

To derive the exact $T=0$ identities, we use the generalized Ward identities to express the self-energy in terms of regular functions multiplied by factors of k or ω. The limit of small k or ω can then be taken easily. As an illustration consider the self-energy $\mathcal{S}-\mathfrak{M}_2-\mu$, which using Eq. (C1) can be written

$$\mathcal{S}-\mathfrak{M}_2-\mu=S-M_2-\mu+(v/2\epsilon^r)(\beta_\mu\Lambda_\mu)^2. \tag{C9}$$

By the use of the generalized Ward identities[9]

$$S-M_2-\mu=\tfrac{1}{2}n_0^{-1/2}(k\beta_\mu\Lambda_\mu^z-\omega\beta_\mu\Lambda_\mu)-\epsilon_k\,, \tag{C10}$$

$$\beta_\mu\Lambda_\mu=n_0^{-1/2}(kF^{zr}-\omega F^r), \tag{C11}$$

$$\beta_\mu\Lambda_\mu^z=n_0^{-1/2}(kF^{zzr}-\omega F^{zr}+nk/m), \tag{C12}$$

we can rewrite Eq. (C9) as

$$\mathcal{S}-\mathfrak{M}_2-\mu=\frac{1}{2n_0}\left(k^2F^{zzr}-2\omega kF^{zr}+\omega^2F^r+\frac{v}{\epsilon^r}(kF^{zr}-\omega F^r)^2\right)+\frac{n'}{n_0}\epsilon_k\,. \tag{C13}$$

Hence in the $k=0$ limit we find the long-wavelength identity

$$(\mathcal{S}-\mathfrak{M}_2-\mu)(k=0,\omega)=(1/2n_0)\omega^2(F^r/\epsilon^r)(k=0,\omega). \tag{C14}$$

Similarly we can establish

$$\mathfrak{A}(k=0,\omega)=\omega\left[1-\frac{1}{2n_0^{1/2}}\left(\delta_\mu\Lambda_\mu\frac{1}{\epsilon^r}\right)(k=0,\omega)\right]. \tag{C15}$$

The identities (C14) and (C15) are valid for arbitrary T and to all orders of the perturbation expansion. By a careful counting of diagrams at $T=0$, Gavoret and Nozières[23] derived the ($k=0$, $\omega\to 0$) identities [(4.7) and (4.8) in Ref. 23]:

$$(\mathcal{S}-\mathfrak{M}_2-\mu)(k=0,\omega\to 0)=-\omega^2\frac{1}{2n_0}\frac{\partial n'}{\partial\mu}\,,$$
$$\mathfrak{A}(k=0,\ \omega\to 0)=-\omega\frac{\partial n'}{\partial n_0}\,. \tag{C16}$$

If we compare Eq. (C16) with Eqs. (C14) and (C15) we can identify at $T=0$

$$\frac{\partial n'}{\partial\mu}=-\frac{F^r}{\epsilon^r}(k=0,\ \omega\to 0),$$
$$\frac{\partial n'}{\partial n_0}=\frac{1}{2n_0^{1/2}}\left(\delta_\mu\Lambda_\mu\frac{1}{\epsilon^r}\right)(k=0,\ \omega\to 0). \tag{C17}$$

To $O(g)$ we can explicitly evaluate the right-hand side of Eqs. (C17) (at $T=0$)

$$F^{r(1)}(k,\omega)=\frac{1}{8}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\lambda_{\vec{p}}^{-}\lambda_{\vec{p}+\vec{k}}^{-}}\,(1-\lambda_{\vec{p}}^{-}\lambda_{\vec{p}+\vec{k}}^{-})\,Q^{+}, \tag{C18}$$

$$\delta_\mu\Lambda_\mu^{(1)}(k,\omega)=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,(\lambda_{\vec{p}}^{-}\lambda_{\vec{p}+\vec{k}}^{-}-1)\,Q^{+},$$

where the statistical factors in Q^+ [see Eq. (A10)] are set equal to zero. It is easy to see that $F^{r(1)}(k=0,\ \omega\to 0)\sim \ln\omega$ and that $\delta_\mu\Lambda_\mu^{(1)}(k=0,\ \omega\to 0)$ is finite. Since $\epsilon^r=1-vF^r$, we find from Eq. (C17) that $\partial n'/\partial\mu$ is well behaved but $\partial n'/\partial n_0$ has a logarithmic singularity that was ignored in Ref. 23. However, the logarithmic singularity cancels out in the final results of Gavoret and Nozières.[23]

There are trivial static ($\omega=0$) identities that follow from symmetry in ω, e.g.,

$$\mathcal{A}=\beta_\mu\Lambda_\mu=\delta_\mu\Lambda_\mu^{z}=F^{zr}=0, \tag{C19}$$

where the argument is $(k,\ \omega=0)$. The nontrivial $\omega=0$ identities can be derived from the generalized Ward identities [Eqs. (C10)–(C12)]. By taking the $\omega=0$ limit of Eqs. (C10)–(C12) and using the fact that $\beta_\mu\Lambda_\mu(k,\ \omega=0)=0$ and $F^r(k,\ \omega=0)$ does not diverge faster that ω^{-2}, we find in the $\omega=0$ limit

$$\epsilon_k(mF^{zzr}+n')=n_0(S-M_2-\mu)=n_0^{1/2}\tfrac{1}{2}\,k\beta_\mu\Lambda_\mu^{z}-n_0\epsilon_k\,, \tag{C20}$$

$$n_0\frac{d}{d\omega}(S-M_2-\mu)=n_0^{1/2}k\frac{d}{d\omega}(\beta_\mu\Lambda_\mu^{z})-\tfrac{1}{2}k^2\frac{d}{d\omega}(F^{zzr}), \tag{C21}$$

where all functions are evaluated at $(k,\ \omega=0)$. Using Eqs. (C9) and (C11), we can rewrite one of the $\omega=0$ identities (C20) as

$$\mathcal{S}-\mathfrak{M}_2-\mu=n_0^{-1}\epsilon_k[n'+mF^{zzr}+m(v/\epsilon^r)(F^{zr})^2]. \tag{C22}$$

The identities (C20)–(C22) are valid for arbitrary k and T. In the $k\to 0$ limit, F^{zzr} and F^{zr} vanish, and Eq. (C22) reduces to the identity derived by Hugenholtz and Pines.[12]

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