Parametric excitation of collective modes in an electron layer on a liquid surface

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Parametric excitations of electrostatic oscillations in a two-dimensional electron gas on a liquid surface and a surface wave (ripplon) of the liquid are studied. The excitations are achieved by an oscillating (pump) electric field along the liquid surface at a frequency near the characteristic frequency of the two-dimensional electron gas. The threshold value of the pump field depends, among other parameters, on the damping rate of the oscillation mode of the electrons. If the damping is controlled by the electron to atomic helium scattering, then the threshold value for the pump electric field can become as low as 100 mV/cm.

I. INTRODUCTION

A sheet of electrons bound to the surface of liquid helium or neon has many interesting properties. It can be a strongly correlated system.[1-5] Since the Coulomb interaction energy becomes greater than the thermal as well as zero-point energies, a two-dimensional Wigner lattice may be formed[1-3]; however, such a lattice has not been observed yet.

On the other hand, the electrodynamical properties are recognizable more easily than the structure of electron gas. The electron density wave which is similar to the usual electron plasma wave exists and its dispersion relation has been investigated.[3,6] We simply call this a plasma wave here. The damping of the plasma wave which gives information on microscopic processes such as self-collisions, collisions with the vapor of the liquid or the ripples on the surface,[3,4] has not been studied over a wide range of wave numbers. The surface wave of the liquid "ripplon" is another normal mode in the present system. A vertical electric field which confines electrons on the liquid surface modifies the dispersion relation of the surface wave by localizing electrons in the troughs of surface waves and causing a downward frequency shift,[7] at the ripplon.

In this paper, we present a method whereby both the plasma wave and the surface wave may be excited by means of a parametric process using a high-frequency "pump" electric field along the surface. Our stability analysis is analogous to those well known in the nonlinear theory of plasmas.[8,9] The result indicates that the threshold of the pump rf field for this instability is determined only by the damping rate of the plasma wave, since the frequency of the surface wave is always small compared with the damping rate of the plasma wave.[8] Therefore, one can infer the damping rate of the plasma wave from the observed threshold value of the pump. We also expect that the excited surface wave decreases the electron mobility.

We present the theory in the following order. Section II aims to illustrate the basic coupling mechanism involved in the parametric process. The result obtained there is applicable to a situation where the wavelength of the excited waves is much shorter than the depth of the liquid, but much longer than the microscopic (kinetic) scale lengths. The effects due to finite depth of the liquid are taken into account in Sec. III. In Sec. IV, the coupled equation is solved and the growth rates of the instability are presented for various range of parameters. The threshold value of the pump field is obtained there as a function of the damping rate and the wavelengths of the excited plasma wave. A brief discussion is given in Sec. V.

II. COUPLED EQUATIONS FOR SHORT-WAVELENGTH EXCITATION

To illustrate the basic coupling mechanism between the electron plasma wave and the liquid surface wave through the externally applied pump field we consider a situation in which the depth of the liquid can be regarded as infinite.

We assume the coordinate z to be perpendicular to the equilibrium surface as shown in Fig. 1, and consider small perturbations in the electron density and liquid surface in the x-z plane. We use a fluid approximation to describe the electron dynamics. The liquid is assumed to be incompressible and to have a unit index of refraction, and its macroscopic electric properties are ignored. The microscopic properties such as scattering of electrons by the ripplons are included in the effective friction term in the descriptions of the electron dynamics. The forces acting upon the fluid are the gravitational force, surface tension, pressure, and the electric force applied to the electron layer which pushes electrons down on the liquid surface. The pump field is an oscillating uniform electric field, $E_0 \cos\omega_0 t$, applied in the horizontal (x) direction.

To derive the coupled equation between the plasma wave and the surface wave, we must analyze three types of dynamics, i.e., the high-frequency oscillations associated with the plasma wave, the low-frequency motion of the electrons associated with the surface wave, and the low-frequency dynamics of the liquid.

Let us first study the high-frequency electron dynamics. The necessary equations are the equation of continuity,

$$\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial x}(\sigma v_x) = 0 , \tag{1}$$

the equation of motion,

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = \frac{e}{m}\frac{\partial \varphi}{\partial x} - \frac{e}{m} E_0 \cos\omega_0 t , \tag{2}$$

and Poisson's equation,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 4\pi e \sigma \delta(z - z_0) . \tag{3}$$

In these expressions, z_0 is the equilibrium position of the surface, σ the surface electron density, v the electron velocity (with the subscript x denoting the x component), φ the electrostatic potential, ϵ_0 the vacuum dielectric constant, and $-e$ and m are the charge and mass of an electron.

The linear response of the electrons to the pump electric field is obtained from Eq. (2),

$$v_{x0} = -(eE_0/\omega_0 m)\sin\omega_0 t . \tag{4}$$

The equilibrium solution of Eq. (3) gives the relation between the externally applied vertical electric field $F_0\vec{e}_z$ and the unperturbed electron charge density $-e\sigma_0$,

$$E_{0z} = F_0 - 4\pi e \sigma_0 U(z - z_0) , \tag{5}$$

where $U(z)$ is the unit step function. From Eq. (5) we see that the electrons may be confined on the liquid surface by F_0 if

$$F_0 > 2\pi e \sigma_0 . \tag{6}$$

We now derive the coupled equation describing the interaction of the plasma wave and surface waves in the presence of a pump electric field. For this it is convenient to use a complex amplitude for oscillatory quantities, i.e.,

$$\varphi(x, z, t) = \mathrm{Re}\,\varphi(z) e^{i(kx - \omega t)} . \tag{7}$$

Here $\varphi(z)$ is obtained from Eq. (3),

$$\varphi(z) = -(2\pi e \sigma_{k,\omega}/|k|) e^{-|k|\,|z - z_0|} \tag{8}$$

$$\equiv \varphi_{k,\omega} e^{-|k|\,|z - z_0|} , \tag{9}$$

where

$$\varphi_{k,\omega} = -2\pi e \sigma_{k,\omega}/|k| , \tag{10}$$

and

$$\sigma(x, t) = \mathrm{Re}\,\sigma_{k,\omega} e^{i(kx - \omega t)} . \tag{11}$$

Between the two terms that contribute to the coupling, σv_x in Eq. (1) and $v_x \partial v_x/\partial x$, the former can be shown to contribute dominantly. The coupled equation then becomes, from Eqs. (1), (2), and (10),

$$\epsilon_H(\omega, k)\varphi_{k,\omega} = \Lambda_H \varphi^*_{-k,\Omega} v_{x0} , \tag{12}$$

where

$$\epsilon_H(\omega, k) = 1 - \omega_p^2 |k| / \omega^2 , \tag{13}$$

$$\Lambda_H = k/\omega , \tag{14}$$

$$\omega_p^2 = 2\pi e^2 \sigma_0 / m , \quad v_{x0} = eE_0/im\omega_0 , \tag{15}$$

and Ω is the frequency associated with the surface-wave perturbation $\Omega = \omega_0 - \omega$. Without the pump field, Eq. (12) gives $\epsilon_H = 0$, which represents the eigenmode associated with the two-dimensional plasma wave, whose dispersion relation is given by

$$\omega^2 = \omega_p^2 |k| . \tag{16}$$

Note here ω_p^2 has a unit of cm sec^{-2}.

We now consider the electron dynamics associated with the low-frequency surface-wave perturbation. Compared with the plasma wave, one major difference is that the electrons face additional force due to the ripples on the liquid surface.

If we designate the deviation of the liquid surface from the equilibrium position z_0 by $a(x, t)$ this additional force F_x on the electrons in the hori-

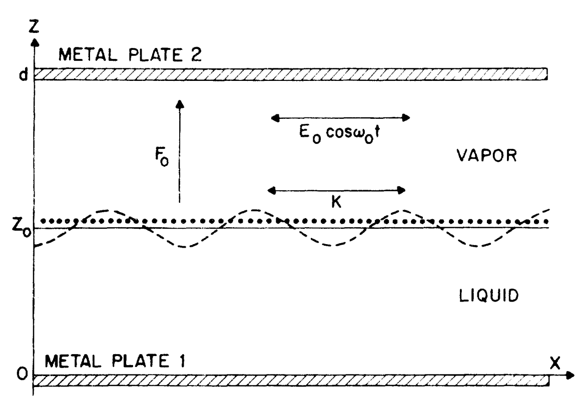

FIG. 1. Geometrical configuration of situation that is under consideration. The dotted line indicates the electron monolayer on an equilibrium surface. The solid line and the broken line denote the liquid surface in the equilibrium state and in a perturbed state. z_0 is the depth of the liquid and d is the distance between two metal frames. k is a wave number of a surface wave or a plasma wave.

zontal direction is given by, referring Fig. 1,

$$F_x = -eF_0\theta \,, \tag{17}$$

with

$$\theta = \frac{\partial a}{\partial x} \ (\ll 1)\,. \tag{18}$$

The expression (17) is approximate and holds when $F_0 \gg 2\pi\sigma e$. Otherwise F_x is reduced by the shielding effect of the electron layer.

The nonlinear term that provides the coupling to the plasma wave is also different. The dominant coupling is provided here by the $v_x \partial v_x/\partial x$ term (the ponderomotive force). If we take these into account, we can obtain the coupled equation for the electrostatic potential associated with the surface-wave perturbation $\varphi_{k,\Omega}$ with the quantity $a_{k,\Omega}$ and the plasma-wave potential $\varphi_{k,\omega}$,

$$-\frac{\omega_p^2|k|}{\Omega^2}\varphi_{-k,\Omega} + \frac{\omega_p^2|k|}{\Omega^2}F_0 a_{-k,\Omega} = \frac{\omega}{\Omega}\frac{|k|v_{x0}}{\Omega}\varphi^*_{k,\omega} - \frac{\omega}{\Omega}\frac{|k|v^*_{x0}}{\Omega}\varphi_{-k,\omega+2\Omega}\,. \tag{19}$$

Finally we obtain the relation between $a_{-k,\Omega}$ and $\varphi_{-k,\Omega}$ by applying the hydrodynamic equation to the motion of the liquid,

$$\rho_0\frac{\partial\vec{v}}{\partial t} = -\vec{\nabla}p + \rho g[U(z - z_0 - a) - 1]\vec{e}_z + \left(\tau_0\frac{\partial^2 a}{\partial x^2} - e\sigma F_0\right)\delta(z - z_0 - a)\vec{e}_z\,. \tag{20}$$

Here ρ_0, $\vec{v}, p$, g, and τ_0 are the mass density, velocity, pressure of the fluid, the gravitation constant, and the surface tension, respectively. The forces acting upon the fluid are a pressure gradient, gravitational, surface tension, and the electric force applied through the surface electrons as represented by the respective terms on the right-hand side. The externally applied vertical electric field pushes electrons down on the liquid surface, which acts as a vertical pressure on the liquid. Because this force depends on the electron-density perturbation, it produces the coupling to the electric field perturbation.

To solve Eq. (20), we assume the liquid is incompressible,[10] i.e.,

$$\vec{\nabla}\cdot\vec{v} = 0\,. \tag{21}$$

We first look at the equilibrium state, in which $a = 0$ and $\partial\vec{v}/\partial t = 0$. Because of the electrostatic force acting upon the liquid surface, the pressure p can be discontinuous at the surface. This indicates that the macroscopic equilibrium can exist even if the area $z > z_0$ is a vacuum. The electrons provide sizable pressure upon the liquid surface due to the applied vertical electric field F_0.

The equilibrium pressure p_0 at $z < z_0$ is obtained by integrating Eq. (20) by putting $\partial\vec{v}/\partial t = a = 0$,

$$p_0 = e\sigma_0 F_0 + \rho g(z_0 - z)\,. \tag{22}$$

We now look at the dynamics of the surface wave. The wave equation for the perturbed pressure p_1 at $z < z_0$ can be obtained by taking a divergence of Eq. (20) and using $\vec{\nabla}\cdot\vec{v}_1 = 0$,

$$\nabla^2 p_1 = 0\,. \tag{23}$$

The appropriate solution of this equation is obtained in the form,

$$p_1 = \overline{p}e^{|k|(z - z_0)}e^{i(kx - \omega t)}\,. \tag{24}$$

On the other hand at $z > z_0$ we can assume $p_1 = 0$.

Now the value $\overline{p}$ can be obtained from the boundary condition on the surface. To obtain the suitable boundary condition, we linearize the z component of the equation of motion (20) to give

$$\rho_0\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \left(\tau_0\frac{\partial^2 a}{\partial x^2} - e\sigma_1 F_0 - \rho_0 g a\right)\delta(z - z_0)\,, \tag{25}$$

where σ_1 is the perturbed surface density of electrons at the low frequency.

We integrate Eq. (25) from $z_0 - 0$ to $z_0 + 0$, and have

$$0 = p(z_0 - 0) + \tau_0\frac{\partial^2 a}{\partial x^2} - e\sigma_1 F_0 - \rho_0 g a\,. \tag{26}$$

Consequently $\overline{p}$ becomes

$$\overline{p} = (k^2\tau_0 + \rho_0 g)a_{k,\Omega} + eF_0\sigma_{k,\Omega}\,. \tag{27}$$

If we now substitute this expression back to the equation of motion (25) and use the fact $v_z = \partial a/\partial t$, we can finally obtain the relation between $a_{-k,\Omega}$ and $\sigma_{-k,\Omega}$:

$$\left(\Omega^2 - \frac{(\tau_0 k^2 + \rho_0 g)|k|}{\rho_0}\right)a_{-k,\Omega} = \frac{eF_0|k|}{\rho_0}\sigma_{-k,\Omega}$$

or, in terms of $\varphi_{-k,\Omega}$,

$$\left(1 - \frac{(\tau_0 k^2 + \rho_0 g)|k|}{\rho_0\Omega^2}\right)a_{-k,\Omega} = -\frac{k^2 F_0}{2\pi\rho_0\Omega^2}\varphi_{-k,\Omega}\,. \tag{28}$$

The above expression gives the relation between the electrostatic potential φ and the surface-wave amplitude a associated with the surface wave of the liquid. We substitute $a_{-k,\Omega}$ obtained above into the equation of the low-frequency electron dynamics, Eq. (19). We can then obtain the coupled equation between the surface wave and the plasma wave, which is reduced to

$$\epsilon_L(\Omega, k)\varphi_{-k,\Omega} = \Lambda_L(\varphi^*_{k,\omega}v_{x0} - \varphi_{-k,\omega+2\Omega}v^*_{x0})\,, \tag{29}$$

where

$$\epsilon_L(\Omega, k) = 1 - \frac{|k|}{\rho_0\Omega^2}\left(\tau_0 k^2 + \rho_0 g - \frac{|k|F_0^2}{2\pi}\right) \tag{30}$$

and

$$\Lambda_L = -\left(\frac{|k|g + |k|^3\tau_0/\rho_0}{\Omega^2} - 1\right)\frac{k}{2\omega}. \tag{31}$$

As usual $\epsilon_L = 0$ gives the linear response of the system. In the absence of the vertical electrostatic field F_0, one obtains the ordinary surface-wave dispersion relation,

$$\Omega^2 = |k|\rho_0 g + \tau_0|k|^3/\rho_0. \tag{32}$$

However, in the presence of F_0, Ω becomes an imaginary number when

$$F_0^2 > (8\pi\tau_0\rho_0 g)^{1/2}. \tag{33}$$

This is a form of the Rayleigh-Taylor instability, since this inequality is achieved when the effective electron mass density (which is increased owing to the electric force) becomes larger than the fluid mass density. This type of instability has been discussed in a recent publication.[7] For the present study of the parametric excitation of the surface wave we assume that F_0 is below the threshold value for the Rayleigh-Taylor instability given by Eq. (33).

Equations (12) and (29) give the desired coupled equations between the plasma and surface wave. These equations are valid in a range of the wavelength which is much shorter than the depth of the liquid.

The surface wave modifies the plasma wave. The oscillating electron velocity in the pump electric field times the low-frequency density perturbation associated with the surface wave produces a high-frequency current component. On the other hand the plasma wave modifies the surface wave because the ponderomotive force of the high-frequency field produces the low-frequency bunching of electrons which act to produce a low-frequency pressure on the liquid surface.

III. COUPLED EQUATIONS FOR WAVELENGTH OF AN ARBITRARY SIZE

We have seen the basic coupling processes between the plasma and the surface waves. For the result to be applied to some real experiment, we extend the result so that it can be applicable to waves with the wavelength comparable to the depth of the liquids. The extension does not alter the basic coupling processes; however, the linear properties of the waves are modified.

First let us consider the plasma wave. The field and the kinetic equations shown in Eqs. (1)–(3) remain the same. The difference arises in the structure of the potential field. We take a boundary condition as shown in Fig. 1 such that the liquid is placed within two parallel conductors at $z = 0$ and d $(> z_0)$. We ignore boundaries in the x direction. Then the appropriate solution, for $\varphi(z)$ at $z > z_0$ and $z < z_0$ may be written

$$\varphi(z) = A\sinh|k|(z-d), \quad \varphi(z) = B\sinh|k|z, \tag{34}$$

where

$$A\sinh|k|(z_0 - d) = B\sinh|k|z_0 \equiv \varphi_{k,\omega}. \tag{35}$$

The boundary condition that gives the jump in $\partial\varphi/\partial z$ by the amount of the surface electron charge density remains the same. Hence the major difference which originates from the finite size of the boundary is the effect of the image charge appearing at both conductors. The coupled equation takes the form

$$\epsilon_H'(\omega, k)\varphi_{k,\omega} = \Lambda_H'\varphi^*_{-k,\Omega}v_{x0}, \tag{36}$$

where

$$\epsilon_H'(\omega, k) = 1 - \frac{\omega_p^2|k|}{\omega^2}\,\frac{\sinh|k|z_0\sinh|k|(d-z_0)}{\sinh|k|d} \tag{37}$$

and

$$\Lambda_H' = k/\omega. \tag{38}$$

Again, $\epsilon_H' = 0$ gives the dispersion relation of the plasma wave which is now modified by the boundaries. The metal boundaries terminate the electric field associated with the wave and reduce the intensity of the x component. This reduces the restoring force in the plasma oscillation of the electrons particularly in the long-wavelength regime. At $k \to 0$ the dispersion relation is modified to $\omega \propto k$ instead of $\omega \propto (|k|)^{1/2}$, as was found previously.[3]

Let us now look at the low-frequency electron dynamics associated with the liquid surface wave. If we ignore the inertia term as before, the low-frequency potential field $\varphi_{-k,\Omega}$ can be expressed

$$\varphi_{-k,\Omega} = \begin{cases} \dfrac{\sinh|k|z}{\sinh|k|z_0}\left(F_0 a_{-k,\Omega} - \dfrac{|k|}{\omega}v_{x0}\varphi^*_{k,\omega} + \dfrac{|k|}{\omega}v^*_{x0}\varphi_{k,\omega+2\Omega}\right), & z \le z_0 \\[2ex] \dfrac{\sinh|k|(d-z)}{\sinh|k|(d-z_0)}\left(F_0 a_{-k,\Omega} - \dfrac{|k|}{\omega}v_{x0}\varphi^*_{k,\omega} - \dfrac{|k|}{\omega}v^*_{x0}\varphi_{k,\omega+2\Omega}\right), & z \ge z_0. \end{cases} \tag{39}$$

Finally we consider the dynamics of the liquid. Here, the force equation, Eq. (20), should be modified to include the additional force acting on the liquid surface through the electrons being affected by the force due to the image charge:

$$\rho_0\frac{\partial\vec{v}}{\partial t}=-\vec{\nabla}p+\rho g[U(z-z_0-a)-1]\vec{e}_z + \left[\tau_0\frac{\partial^2 a}{\partial x^2}-e\sigma(x,t)F_0+e^2\sigma(x,t)\iint dx'\,dy'\frac{\partial}{\partial z_0}\left(\frac{\sigma_A(x',t)}{[(z_0+a)^2+(x-x')^2+y'^2]^{1/2}} -\frac{1}{2}\sum_{\epsilon=\pm 0}\frac{\sigma(x',t)}{\{(x-x')^2+[A(x)-A(x')+\epsilon]^2+y'^2\}^{1/2}} +\frac{\sigma_B(x',t)}{[(d-z_0-a)^2+(x-x')^2+y'^2]^{1/2}}\right)\right]\delta(z-z_0-a)\,. \quad (40)$$

Here, σ_A and σ_B are the number densities of the induced charges on the lower and the upper conductors, respectively. It is convenient here to use the velocity potential ψ defined by

$$\vec{v}=-\vec{\nabla}\psi\,. \quad (41)$$

Then the boundary condition at the bottom of the liquid gives

$$\rho_0\frac{\partial\psi}{\partial t}\bigg|_{z=0}=p(z=0)\,. \quad (42)$$

If we now integrate the z component of the equation of motion (40), and take a derivative with respect to time and use Eq. (42), we have

$$\rho_0\frac{\partial^2\psi}{\partial t^2}\bigg|_{z=z_0}=-\rho_0 g\frac{\partial\psi}{\partial z}\bigg|_{z=z_0}+\tau_0\frac{\partial^2}{\partial x^2}\frac{\partial\psi}{\partial z}\bigg|_{z=z_0}+eF_0\frac{\partial\sigma}{\partial t} +2e^2\frac{\partial}{\partial t}\left[\sigma(x,t)\int_{-\infty}^{\infty}dx'\left(\frac{\sigma_A(x',t)(z_0+a)}{(x-x')^2+(z_0+a)^2}-\frac{1}{2}\sum_{\epsilon=\pm 0}\frac{\sigma(x',t)[A(x)-A(x')]}{(x-x')^2+[A(x)-A(x')+\epsilon]^2} -\frac{\sigma_B(x',t)(d-z_0-a)}{(x-x')^2+(d-z_0-a)^2}\right)\right]\,. \quad (43)$$

The induced densities σ_A and σ_B are, of course, functions of the electron density σ itself. After some manipulation, they can be expressed as

$$\left.\begin{matrix}\sigma_A\\ \sigma_B\end{matrix}\right\}=\frac{1}{2d}\int_{-\infty}^{\infty}\frac{\sigma(x',t)\sin[(\pi/d)(z_0+a)]}{\cosh[(\pi/d)(x-x')]\mp\cos[(\pi/d)(z_0+a)]}\,dx'\,. \quad (44)$$

In a special case in which the equilibrium surface of the liquid is exactly in between the two conductors, whereby $d=2z_0$, σ_A and σ_B are further reduced to

$$\left.\begin{matrix}\sigma_A\\ \sigma_B\end{matrix}\right\}=\frac{\sigma_0}{2}\mp\frac{\sigma_0}{4z_0}\int_{-\infty}^{\infty}\frac{\pi a(x',t)/2z_0}{\cosh^2[(\pi/2z_0)(x-x')]}\,dx'+\frac{1}{4z_0}\int_{-\infty}^{\infty}\frac{\sigma_1(x',t)}{\cosh[(\pi/2z_0)(x-x')]}\,dx'\,, \quad (45)$$

where we neglect higher-order terms with respect to a and σ_1.

Now, the wave equation for the potential field ψ is given by

$$\nabla^2\psi=0,\quad z_0>z>0\,, \quad (46)$$

which should compare with Eq. (23). The appropriate solution which is consistent with the boundary condition that $\partial\psi/\partial z=0$ at $z=0$ is

$$\psi(x,z,t)=\mathrm{Re}\,\psi_{k,\Omega}\cosh kz\, e^{i(kx-\Omega t)}\,. \quad (47)$$

If we now substitute Eqs. (47) and (45) into (43), linearize the result and use the relation $v_z=\partial a/\partial t=-\partial\psi/\partial z$, we obtain the relation between the potential ψ and the surface charge density $-e\sigma_{k,\Omega}$:

$$\left[\Omega^2-\left(g+\frac{k^2\tau_0}{\rho_0}-\frac{2\pi e^2\sigma_0^2}{\rho_0}|k|\coth|k|z_0\right)\times|k|\tanh|k|z_0\right]\psi_{k,\Omega}=i\omega\frac{e\sigma_{k,\Omega}F_0}{\rho_0}\,. \quad (48)$$

The set of Eqs. (39) and (48), and the relation

$$\sigma=\frac{1}{4\pi e}\left(\frac{\partial\varphi}{\partial z}\bigg|_{z_0+0}-\frac{\partial\varphi}{\partial z}\bigg|_{z_0-0}\right) \tag{49}$$

produce finally the coupled equation between the low-frequency potential $\varphi_{-k,\Omega}$ and the high-frequency potential $\varphi_{k,\omega}$ through the pump field v_{x0},

$$\epsilon_L'(\Omega,k)\varphi_{-k,\Omega}=\Lambda_L'(\varphi^*_{k,\omega}v_{x0}-\varphi_{-k,\omega+2\Omega}v^*_{x0}), \tag{50}$$

where ϵ_L' and Λ_L' are given by

$$\epsilon_L'(\Omega,k)=1-\Omega_k^2/\Omega^2, \tag{51}$$

$$\Omega_k^2=k\tanh kz_0 \times\left[g+\frac{k^2\tau_0}{\rho_0}-\left(\frac{2\pi\sigma_0^2e^2}{\rho_0}-\frac{F_0^2}{2\pi\rho_0}\right)|k|\coth|k|z_0\right],$$

$$\Lambda_L'=-\left(1-\frac{\tilde\Omega_k^2}{\Omega^2}\right)\frac{k}{2\omega}, \qquad -\omega+\omega_0=\Omega, \tag{52}$$

and

$$\tilde\Omega_k^2=|k|\tanh|k|z_0 \times\left(g+\frac{k^2\tau_0}{\rho_0}-\frac{2\pi\sigma_0^2e}{\rho_0}|k|\coth|k|z_0\right).$$

IV. PROPERTIES OF COUPLED-WAVE SYSTEM–PARAMETRIC INSTABILITIES

Equations (12)–(15) [or Eqs. (36)–(38)] for the plasma wave and Eqs. (29)–(31) [or Eqs. (50)–(52)] for the surface wave constitute a coupled system. From these sets of equations, one obtains the following dispersion relation:

$$1=\frac{\Lambda(\tilde\Omega_k^2/\Omega^2-1)}{\epsilon_L(\Omega,-k)}\left(\frac{1}{\epsilon_H^*(\omega_0-\Omega,k)}+\frac{1}{\epsilon_H(\omega_0+\Omega,-k)}\right), \tag{53}$$

where

$$\Lambda=(1/2\omega_0^2)k^2|v_{x0}|^2 .$$

We investigate the parametric instability derived from this dispersion relation. We find two types of instabilities. One of them occurs when $\epsilon_L(\Omega,-k)\approx 0$ and $\epsilon_H^*(\omega_0-\Omega,k)\approx 0$. This type of instability is called the "decay instability." The other type of instability, which is called the "oscillating two-stream instability" (OTSI), takes place when $\epsilon_H^*(\omega_0-\Omega,k)\approx 0$ and $\epsilon_H(\omega_0+\Omega,-k)\approx 0$. One easily finds that these matching conditions are satisfied when $|\omega_0-\omega_k|\ll\omega_0$, because $|\Omega_k|\ll\omega_0$. In this case, we can simplify Eq. (53) to

$$(\Omega^2-\Omega_k^2)(\Omega^2-\Delta_k^2)=-D_k(\tilde\Omega_k^2-\Omega^2) \tag{54}$$

by using the approximation $\epsilon_H(\omega_0\pm\Omega,\pm k)\simeq 2(\omega_0-\omega_k\mp\Omega)/\omega_0$. Here, $D_k=(\Delta_k/2\omega_0)k^2|v_{x0}|^2$ and $\Delta_k=\omega_0-\omega_k$. We solve Eq. (54) for Ω to obtain

$$\Omega=\pm(1/\sqrt2)\{\Delta_k^2+\Omega_k^2+D_k \pm[(\Delta_k^2+D_k-\Omega_k^2)^2-4D_k(\tilde\Omega_k^2-\Omega_k^2)]^{1/2}\}^{1/2}. \tag{55}$$

This expression indicates that the coupled waves grow, i.e., $\mathrm{Im}\Omega>0$, under one of the following two conditions:

(i) $\Delta_k>0$ and

$$|\Delta_k^2+D_k-\Omega_k^2|<2[D_k(\tilde\Omega_k^2-\Omega_k^2)]^{1/2}; \tag{56}$$

(ii) $\Delta_k<0$ and

$$(\Delta_k^2+D_k)\Omega_k^2<|D_k(\tilde\Omega_k^2-\Omega_k^2)| . \tag{57}$$

The decay instability appears when Eq. (56) is satisfied, while the oscillating two-stream instability takes place when Eq. (57) is met. Note that the surface wave generated by OTSI is a purely growing mode, i.e., $\mathrm{Re}\Omega_k=0$.

With realistic parameters, we find that condition (57) is satisfied over a much wider range of ω_k compared to the range of ω_k satisfying (56). In fact, if we choose $z_0=0.1$ cm, $\sigma_0=10^8$ cm^{-2}, $F_0=3$ kV/cm, $E_0=3$ V/cm, and $\omega_0=4\times10^9$ rad/sec, one estimates the coupling coefficient appearing in (54) as $D_k/\omega_0^2\simeq10^{-3}\,\Delta_k/\omega_0$ and $\Omega_{k_0}/\omega_{k_0}=3\times10^{-7}$, with k_0 defined by $\omega_0=\omega_{k_0}$. We obtain from Eqs. (56) and (57) that the unstable frequency range of the decay instability is given by $|(\omega_0-\omega_k)/\omega_0|\lesssim4\times10^{-10}$, and the OTSI occurs in a range given by $|(\omega_0-\omega_k)/\omega_0|\lesssim10^{-3}$. We also estimate from (55) that the growth rate of OTSI $\mathrm{Im}\Omega\simeq[D_k(\tilde\Omega_k^2-\Omega^2)]^{1/4}\simeq10^{-5}\omega_0$, and the growth rate for the decay instability $\mathrm{Im}\Omega\simeq10^{-7}\omega_0$. Therefore, OTSI dominates over decay instability once it starts growing.

We have ignored damping of the waves in the above discussions. The wave damping gives rise to a finite threshold value for the pump amplitude to drive the instability. We phenomenologically introduce damping by adding dissipation terms $-2\gamma v_x$ and $-2\Gamma v_z$ to the right-hand side of Eqs. (2) and (25) or (40), respectively. We will discuss the physical processes included in γ and Γ later. The dispersion relation (53) is now rewritten as

$$1=\frac{\Lambda\omega_0^2[\tilde\Omega_k^2-(\Omega+2i\Gamma)^2]}{\Omega(\Omega+2i\Gamma)-\Omega_k^2}\left(\frac{1}{(\omega_0-\Omega)(\omega_0-\Omega-2i\gamma)-\omega_k^2}+\frac{1}{(\omega_0+\Omega)(\omega_0+\Omega+2i\gamma)-\omega_k^2}\right). \tag{58}$$

Following the prescription given by Ref. 8, we calculate the threshold pump intensities. We obtain the minimum threshold

$$\Lambda_{\mathrm{OTSI}} = 2\frac{\gamma}{\omega_0}\left(\frac{\Omega_k}{\tilde{\Omega}_k}\right)^2 \tag{59}$$

for OTSI when $\Delta_k = -\gamma$. The threshold for the decay instability is given by

$$\Lambda_{\mathrm{decay}} = \frac{4\Gamma\gamma}{\Omega_k\omega_0}\,\frac{\Omega_{k_0}^2}{\tilde{\Omega}_{k_0}^2 - \Omega_{k_0}^2}\,, \tag{60}$$

provided $\tilde{\Omega}_k^2 - \Omega_{k_0}^2 \gg \gamma\Gamma$. If $\gamma\Gamma$ becomes larger than $\tilde{\Omega}_k^2 - \Omega_{k_0}^2$, the decay instability does not overcome the damping. Hereafter, we discuss only the OTSI because it is the dominant instability. The growth rate of OTSI is calculated from Eq. (58) as

$$\mathrm{Im}\Omega = -\gamma + (-D_k - \Delta_k^2)^{1/2}\,. \tag{61}$$

We plot ImΩ in Fig. 2 as a function of $|\Delta_k|$ for the parameters given in the caption. Figure 3 shows the threshold pump amplitude E_0 as a function of γ/ω_0. If the threshold value of E_0 is obtained experimentally then we can find γ as a function of k. Here, k can be specified by choosing pump frequency ω_0.

V. DISCUSSIONS

We now discuss the physical processes which determine the damping rate γ and the expected threshold pump intensity. According to the recent experiment by Grimes and Adams, the mobility of the electrons is determined by electron–helium-gas-atom collisions in the high-temperature regime ($T \gtrsim 1\,^\circ$K). In the low-temperature regime ($T \lesssim 1\,^\circ$K), the effect of scattering of the electrons by the thermally excited surface wave, ripplon, dominates over the electron-atom collisions.

If we employ the mobility determined by the electron-ripplon interaction, the threshold pump amplitude is a few tens of volts per cm. However, the electron-ripplon interaction strongly depends on the external field amplitude to excite the plasma wave. In fact Ref. 11 reports that a significant increase in the mobility is found when the applied electric field exceeds a few tens of millivolts per cm. We consider that this enhancement of mobility is due to the heating of electrons. Since the electron-helium temperature relaxation time τ_r is much longer than the wave-damping time, only electrons are heated. Once electrons are heated above a few tens of $^\circ$K, they decouple from the ripplon because the thermal energy of the electrons exceeds the energy gaps between the bound states of electrons. The energy-balance equation gives that the electrons are heated up to this temperature when

$$v_{x0} \gtrsim 10^6 \tau_r \gamma\,.$$

If we choose $\tau_r\gamma \simeq 10^2$ and $\omega_0 = 4\times10^9$ rad/sec, this condition is satisfied when $E_0 \gtrsim 0.1$ V/cm. We, therefore, estimate the threshold pump amplitude by employing the electron–gas-atom collisions. If we do so, from Eq. (59) we find that the threshold amplitude becomes a few hundreds of mV/cm.

The dips on the surface produced by each electron in a "clamping" electric field F_0 may also de-

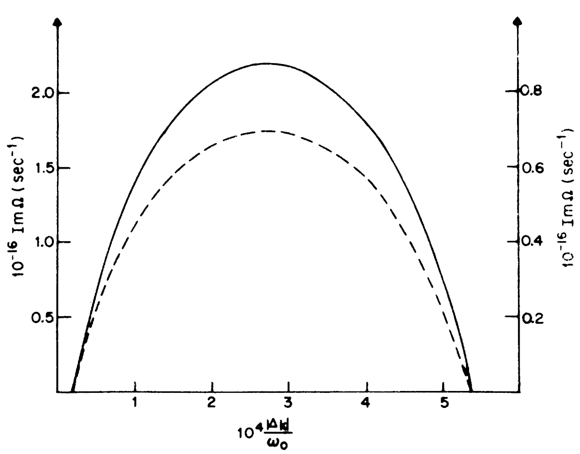

FIG. 2. Growth rate ImΩ is plotted versus $|\Delta_k|/\omega_k$. Parameters of the electron layer and the liquid surface are as follows: $z_0 = 0.1$ cm, $\sigma_0 = 10^8$ cm^{-2}, $F_0 = 3$ kV/cm, $E_0 = 3$ V/cm, $\omega_0 = 4\times10^9$ rad/sec (broken line), 1.26×10^{10} rad/sec (solid line), $\gamma/\omega_0 = 10^{-5}$.

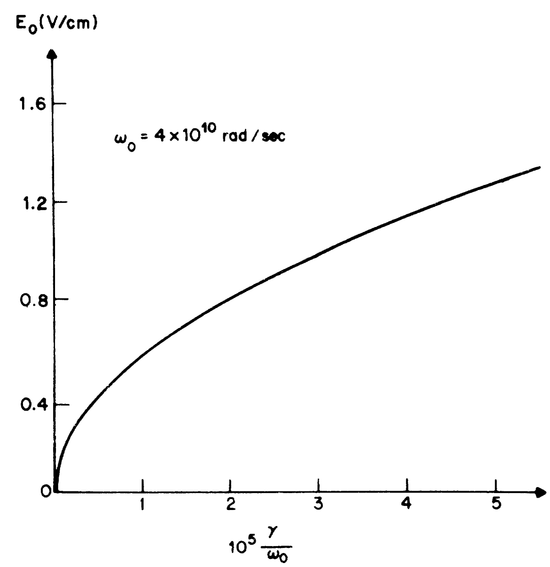

FIG. 3. For the same parameters as Fig. 2, the threshold amplitude of rf field is shown against the damping rate of the electron plasma wave.

crease the mobility of electrons. The binding energy of the electron to a dip is given by

$$\epsilon \simeq 7F_0^2(4+2\ln F_0)\times 10^{-20}\ \text{erg}.$$

(F_0 is in cgs-esu units.)

If the electrons are heated to above 10 °K as discussed previously, then one easily finds that ϵ is much smaller than the kinetic energy of electrons for realistic values of F_0 ($\lesssim 50$ cgs esu = 15 kV/cm). Therefore, the dips decouple to the electrons. Once electrons start moving, the dips should disappear because the surface deformation does not respond to fast electron movement.

Finally, we note that one can detect the instability by observing plasma-wave sidebands to the pump signal.

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