Effect of Nambu-Goldstone modes on wave-number- and frequency-dependent longitudinal correlation functions

Gene F. Mazenko*
The James Franck Institute and Department of Physics, The University of Chicago, Chicago, Illinois 60637
(Received 10 March 1976)

We discuss the effect of Nambu-Goldstone modes on the longitudinal wave-number- and frequency-dependent order-parameter correlation functions in the ordered phase. We find that the static wave-number-dependent correlation function diverges as $q^{-\epsilon}$ ($\epsilon = 4-d$) for small q and the linear response function for a time-dependent Ginzburg-Landau model, with $q=0$, diverges as $\omega^{-\epsilon/2}$ for small frequencies. The consequences of these effects are discussed.

The effect of Nambu-Goldstone modes on transverse (to the direction of ordering) correlation functions for systems with a broken continuous symmetry is well known.[1] There has been far less study[2-5] of the unusual behavior of the longitudinal susceptibility caused by couplings between the longitudinal and Nambu-Goldstone modes. In this paper we discuss the consequences of this coupling of the longitudinal and transverse modes on the frequency- and wave-number-dependent correlation functions.

Our discussion will be within the context of the conventional time-dependent Ginzburg-Landau model. This model has the same static (equal-time correlation functions) as in the usual Landau-Ginzburg-Wilson theory of static critical phenomena. It is expected, however, that many of the qualitative results we find will persist in more sophisticated models. We will use renormalization-group techniques valid near the transition temperature T_c in our calculations, but again we expect that many of the qualitative features we find will persist for all temperatures below T_c where the continuous symmetry is broken. Thus the strong coupling between the longitudinal and transverse modes is not really a critical-phenomena effect.[6] We use critical-phenomena techniques because the scaling behavior near the transition allows one to perform calculations conveniently.

The work here will be, to some extent, a generalization of the work of Brézin, Wallace, and Wilson[2] and of Brézin and Wallace[3] on the static susceptibility $\chi_L = dM/dH = \chi_L(q=0)$ (M is the magnetization and H is an external magnetic field) to the case of wave-number- and frequency-dependent correlation functions. We will assume that their method for handling the new singularities that develop in χ_L is correct and that we can extend their ideas to the more general correlation functions. Brézin *et al.*[2,3] have conjectured that the longitudinal susceptibility has the form

$$\chi_L = \text{const} + (H/M)^{-\epsilon/2}. \tag{1}$$

This relation is supposed to hold for all $N>1$ (where N is the number of components of the order parameter) and to all orders in $\epsilon = 4-d$. Clearly as the field goes to zero the susceptibility will diverge. The assumed behavior in Eq. (1) is the weakest one compatible with the exact results of Lebowitz and Penrose.[5] The idea that the longitudinal susceptibility diverges for all temperatures below T_c for a system with broken continuous symmetry (which excludes $N=1$ Ising-like models) is not widely appreciated. Indeed, such behavior has not been observed experimentally.[4] This is discussed further below. We have support for Eq. (1) from ϵ calculations to $O(\epsilon^2)$, $1/N$ calculations to $O(1/N)$, and some simple but general heuristic arguments.[2,3,7,8]

Let us first discuss the extension of (1) to the case of finite wave numbers. We will calculate the longitudinal susceptibility for the usual Landau-Ginzburg effective free energy

$$F = \frac{1}{2}\int d^dx\{r_0\varphi^2(x) + [\nabla\varphi(x)]^2 + \tfrac{1}{4}u[\varphi^2(x)]^2\}, \tag{2}$$

where $\varphi^2(x) = \sum_{i=1}^N \varphi_i^2(x)$, $r_0 = a(T-T_c)$, and $u, a>0$. All wave numbers are restricted to magnitudes less than the cutoff Λ. We will work in the scaling region where it has been shown, using renormalization-group arguments,[9] that we can develop perturbation theory about four dimensions if we choose the quartic coupling u to be given by

$$u = 2\epsilon/(N+8)K_4 + O(\epsilon^2) \tag{3}$$

and K_d is the surface area of a unit sphere in d dimensions. When we work below T_c, where the longitudinal field has a nonzero average, $\langle\varphi_1(x)\rangle = M$, we must treat $M \sim O(\epsilon^{-1/2})$ in developing perturbation theory. A particularly elegant development of perturbation theory below T_c is given by

Amit and Zannetti.[10] The longitudinal self-energy to $O(\epsilon)$ is a sum of a Hartree term and a bubble with two magnetization lines. We obtain the result

$$\chi_L^{-1}(q) = r_0 + \tfrac{3}{2}uM^2 + q^2 - \Sigma_L^{(s)}(q)\,, \tag{4}$$

where

$$\Sigma_L^{(s)}(q) = -\frac{\epsilon}{2(N+8)}\left((N+2)\Lambda^2 - 3uM^2 \ln\frac{\Lambda^2}{uM^2}\right) + \frac{(N-1)uM^2\epsilon}{2(N+8)}\left(1+\ln\frac{\Lambda^2}{q^2}\right) + \frac{9uM^2\epsilon}{2(N+8)}\left(1+\ln\frac{\Lambda^2}{uM^2} + \frac{2(x^2+4)^{1/2}}{x}\ln\tfrac{1}{2}[(x^2+4)^{1/2}-x]\right), \tag{5}$$

where $x^2 = q^2/uM^2$ and $uM^2 = \xi^{-2}$ to lowest order in ϵ. We can combine (5) with the equation of state [correct to $O(\epsilon)$],

$$r_0 + \frac{uM^2}{2} + \frac{\epsilon}{2(N+8)}\left((N+2)\Lambda^2 - 3uM^2\ln\frac{\Lambda^2}{uM^2}\right) = 0\,, \tag{6}$$

to obtain

$$\chi_L^{-1}(q) = q^2 + uM^2 - \frac{(N-1)uM^2\epsilon}{2(N+8)}\left(1+\ln\frac{\Lambda^2}{q^2}\right) - \frac{9uM^2\epsilon}{2(N+8)}\left(1+\ln\frac{\Lambda^2}{uM^2} + \frac{2(x^2+4)^{1/2}}{x}\ln\tfrac{1}{2}[(x^2+4)^{1/2}-x]\right). \tag{7}$$

The interesting term in (7) is the one proportional to $N-1$. This term comes from a bubble with two transverse propagators $[\chi_T(q) \sim q^{-2}]$. We see that as $q\to 0$ we have a singularity for all temperatures below T_c. If we follow Brézin *et al.* and anticipate that $\chi_L^{-1}(q)\to 0$ as $q \to 0$ we can rewrite χ_L^{-1} in the form [consistent to $O(\epsilon)$]

$$\chi_L^{-1}(q) = q^2 - \frac{9\epsilon uM^2}{N+8}\left(1 + \frac{(x^2+4)^{1/2}}{x}\ln\tfrac{1}{2}[(x^2+4)^{1/2}-x]\right) + \Lambda^2\left(\frac{uM^2}{\Lambda^2}\right)^{1+\epsilon/2}\left(\frac{N+8+\frac{1}{2}(10-N)\epsilon}{9+(N-1)x^{-\epsilon}}\right). \tag{8}$$

Note that to $O(\epsilon)$ the inverse correlation length squared is related to the magnetization by

$$\xi^{-2} = \Lambda^2(uM^2/\Lambda^2)^{\nu/\beta} = \Lambda^2(uM^2/\Lambda^2)^{1+\epsilon/2}\,,$$

so Eq. (8) has the correct scaling form to $O(\epsilon)$. We have investigated the $O(\epsilon^2)$ contributions and have verified that the $\ln^2 q$ terms exponentiate to give a q^ϵ behavior for $\chi_L^{-1}(q)$ for small q. Ma[11] has shown that $\chi_L^{-1}(q)\sim q^\epsilon$ for small q in the large-N limit and Fisher *et al.*[4] suggested on phenomenological grounds that $\chi_L \sim r^{-4+2\epsilon}$ in coordinate space, which is consistent with our results. Note that for $N=1$ the susceptibility is perfectly regular as $q\to 0$:

$$\chi_L^{-1}(0) = uM^2(uM^2/\Lambda^2)^{\epsilon/2}(1+\tfrac{1}{2}\epsilon).$$

We now turn to a discussion of dynamics. There has been, to our knowledge, only one discussion of the critical dynamics of longitudinal order-parameter correlation functions for systems with $T<T_c$ using renormalization-group techniques. This was the discussion by Ma and Mazenko[12] for the isotropic ferromagnet. This case was somewhat special since the dynamics can be conveniently treated only by working near six dimensions. Thus while the dynamics below T_c were very interesting, it was not clear whether the results could be extrapolated to three dimensions. Here we want to discuss the simplest dynamical model, the time-dependent Ginzburg-Landau model, with a nonconserved order parameter. This model was studied in some detail by Halperin, Hohenberg, and Ma[13] for $T>T_c$ and is characterized by the equation of motion

$$\frac{\partial\varphi_i(x,t)}{\partial t} = -\Gamma\frac{\delta F}{\delta\varphi_i(x,t)} + \eta_i(x,t)\,, \tag{9}$$

where η_i is a Gaussian noise satisfying

$$\langle \eta_i(x,t)\eta_j(x',t')\rangle = 2\Gamma\delta_{i,j}\delta(x-x')\delta(t-t') \tag{10}$$

and Γ is a bare kinetic coefficient. The diagrammatics for this equation of motion can be carried out in several different but equivalent ways.[14] We have calculated the response function $G(q,\omega)$ as defined by Ma and Mazenko.[12] While we have calculated G as a function of q and ω, the interesting dynamical effects are highlighted in the $q\to 0$ limit of G. Again using the equation of state, we obtain

$$G_L^{-1}(0,\omega) = -\frac{i\omega}{\Gamma} + uM^2 - \frac{9uM^2\epsilon}{2(N+8)}\ln\frac{\Lambda^2}{uM^2} - \frac{(N-1)\epsilon uM^2}{2(N+8)}\ln\frac{\Lambda^2}{uM^2} - \frac{9uM^2\epsilon}{N+8}\left(-\frac{1}{2}+\frac{1}{i\nu^0}\right)\ln\left(1-\frac{i\nu^0}{2}\right) + \frac{(N-1)}{2(N+8)}uM^2\epsilon\ln\left(\frac{-i\nu^0}{2}\right), \tag{11}$$

where $\nu^0 = \omega/\Gamma uM^2$. We can rewrite (11) in the form [correct to $O(\epsilon)$]

$$G_L^{-1}(0,\omega)=uM^2\left(\frac{uM^2}{\Lambda^2}\right)^{\epsilon/2}\times\left(\frac{-i\nu}{\gamma(\nu)}+\frac{N+8+9\epsilon/2}{9+(N-1)(-i\nu/2)^{-\epsilon/2}}\right), \quad (12)$$

where

$$\gamma(\nu)=1-\frac{9\epsilon}{2(N+8)(i\nu)}\left[1+\frac{1}{i\nu}(2-i\nu)\ln\left(1-\frac{i\nu}{2}\right)\right] \quad (13)$$

and

$$\nu=\omega/[\Gamma uM^2(uM^2/\Lambda^2)^{\epsilon/2}] . \quad (14)$$

For small frequencies we find, for $N>1$,

$$G_L^{-1}(0,\omega)=uM^2\left(\frac{uM^2}{\Lambda^2}\right)^{\epsilon/2}\frac{N+8+9\epsilon/2}{N-1}\left(\frac{-i\nu}{2}\right)^{\epsilon/2} . \quad (15)$$

Since $\chi_L^{-1}(q=0)$ is given by[15] the $\omega\to 0$ limit of $G_L^{-1}(0,\omega)$ we see there is consistency with $\chi_L^{-1}(q=0)=0$. It is expected that the q^ϵ behavior in χ_L^{-1} would go over to a $(-i\omega/\Gamma)^{\epsilon/2}$ behavior in $G_L^{-1}(0,\omega)$ because the zero-order transverse propagator is $(-i\omega/\Gamma+q^2)^{-1}$. Consequently the infrared singularities can be cut off by nonzero external frequencies or wave numbers. We have also carried out a $1/N$ calculation and obtained to $O(1/N)$

$$G_L^{-1}(0,\omega)=\frac{4M^2\sin(\pi\epsilon/2)}{NK_d\,\pi}\left(\frac{-i\omega}{2\Gamma}\right)^{\epsilon/2}, \quad (16)$$

which agrees with our $O(\epsilon)$ calculation in the large-N limit.[16]

A major consequence of our analysis is that the correlation function (using the fluctuation-dissipation theorem)

$$C_L(t)=\lim_{q\to 0}\langle\varphi_1(q,t)\varphi_1(-q,0)\rangle =\lim_{q\to 0}\int\frac{d\omega}{2\pi}\,e^{+i\omega t}\,\frac{2}{\omega}\,\mathrm{Im}G_L(q,\omega)$$

does *not* decay exponentially for long times but goes as

$$C_L(t)\sim t^{-\epsilon/2}$$

for long times. We see then that $C_L(t)$ decays as $t^{-1/2}$ in three dimensions. Clearly $C_L(t)$ is not integrable and the usual ideas about exponential decay at long times and an associated kinetic coefficient are no longer applicable.

We have also treated a time-dependent Ginzburg-Landau model with a conserved order parameter. In this case the Nambu-Goldstone modes have no qualitative effect on the dynamics. This is expected since we know from the work of Halperin, Hohenberg, and Ma that this model is to some extent artificial (there is no mechanism for changing the transport coefficient). One must introduce mode coupling terms if one is to have a realistic model for a system with a conserved order parameter. The results of Ma and Mazenko for the ferromagnet below T_c indicate that the mode coupling terms do lead to a qualitative change in the behavior. In particular they found that the transport coefficient Γ_M does not exist in the sense that it develops an anamolous wave-number dependence for small wave numbers $\Gamma_M\sim q^{-(6-d)/6}$. At present we are investigating the case of the antiferromagnet where the order parameter is not conserved but is coupled to the noncritical magnetization which is conserved.

There is at present no experimental evidence for the divergences in the longitudinal susceptibility discussed here. There are several reasons why these effects have not been observed. In the case of systems with a physically realizable conjugate field (the external magnetic field for a ferromagnet) uncontrollable residual fields cut off the effect. This has been discussed by Brézin and Wallace.[3] One should, apparently, look at systems where there are no physically conjugate field, like antiferromagnets and helium, where we can probe the small-q and -ω behavior of the order-parameter correlation function. In helium, we can not couple directly to the order parameter, but in principle one could use neutron scattering from isotropic antiferromagnets to see the proposed behavior. The difficulty here is that one measures the total susceptibility $\overline{\chi}(q)=(N-1)\chi_T(q)+\chi_L(q)$. For small enough $x=q\xi$ we expect the total susceptibility will be proportional to $x^{-2}+ax^{-\epsilon}$ where the x^{-2} is from χ_T and $x^{-\epsilon}$ from χ_L. To first order in ϵ,

$$a=\frac{1}{N+8}\left(1+\frac{\epsilon}{4(N+8)}(2N-19)\right).$$

In the physically interesting case $N=3$, $\epsilon=1$, $a=0.06$. Because of the small value of a it may be very difficult to separate the x^{-2} and $x^{-\epsilon}$ behavior. We can estimate the range of x where χ_L will be controlled by the $x^{-\epsilon}$ behavior by comparing the small-q behavior of (8) with an Ornstein-Zernike form. We find their ratio is relatively constant down to $x\sim 0.35$ where a rapid variation in the ratio begins. Thus for $x<0.35$ we expect the $x^{-2}+ax^{-\epsilon}$ form to be reasonably accurate.

It seems to be important to understand whether these proposed infrared divergences have gone unobserved because of experimental limitations or because the usual isotropic models are unrealistic in the ordered phase. We know that by introducing a small asymmetry in the effective free energy we eliminate these divergences.[17] Physically, in this case, the spin-wave frequency will not go to zero as $q \to 0$ and the $q=0$ intercept will cut off the infrared divergences.

We would like to thank Professor P. Horn and Professor S. Ma for very useful discussions.

---

*Research supported in part by the National Science Foundation and the Louis Block Fund, The University of Chicago.

[1]D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, New York, 1975).

[2]E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973).

[3]E. Brézin and D. J. Wallace, Phys. Rev. B 7, 1967 (1973).

[4]See M. Barmatz, P. C. Hohenberg, and A. Kornblit [Phys. Rev. B 12, 1947 (1975)] for a discussion of reasons why the Nambu-Goldstone singularities have not been observed and a list of further references. M. E. Fisher, M. N. Barker, and D. Jasnow [Phys. Rev. A 8, 1111 (1973)] suggest that the work of B. E. Argyle, S. H. Charap, and E. W. Pugh [Phys. Rev. 132, 2051 (1963)] may provide experimental evidence for these effects.

[5]J. L. Lebowitz and O. Penrose, Phys. Rev. Lett. 35, 549 (1975). These authors show rigorously that for $N=2$, the divergence of χ_L is at least as strong as $H^{-1/2}$ for $d=3$ and $\ln H^{-1}$ for $d=4$ and $T<T_c$.

[6]See, for example, V. G. Vaks, A. I. Larkin, and S. A. Pikin, {Zh. Eksp. Teor. Fiz. 53, 1089 (1967) [Sov. Phys.-JETP 26, 647 (1968)]} for a discussion of spin-wave calculations.

[7]We thank Prof. S. Ma for discussions on this point.

[8]D. J. Wallace and R. K. P. Zia, Phys. Rev. B 12, 5340 (1975); E. Brézin and J. Zinn-Justin, Saclay report, 1975 (unpublished).

[9]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).

[10]D. J. Amit and M. Zannetti, J. Stat. Phys. 7, 31 (1973).

[11]S. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, to be published).

[12]S. Ma and G. F. Mazenko, Phys. Rev. B 11, 4077 (1975).

[13]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. B 10, 139 (1974).

[14]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973). C. de Dominicis, Nuovo Cimento Lett. 12, 576 (1975); or see Ref. 11.

[15]This follows from the fluctuation-dissipation theorem discussed in Ref. 11.

[16]It is very convenient in carrying out the $1/N$ expansion to use the Matsubara method discussed by de Dominicis (see Ref. 14).

[17]D. Nelson (unpublished).