Determination of the thermodynamics of He II from sound-velocity data*

J. Maynard
Department of Physics, University of California, Los Angeles, California 90024
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Recently measured sound-velocity data were used to completely determine the thermodynamics of superfluid helium at all temperatures above 1.2 K and at all pressures in the He II phase. The sound-velocity data consisted of first-, second-, and fourth-sound velocities measured simultaneously at more than 400 points in the P-T plane. Equations of two-fluid hydrodynamics and a small amount of published data were used to convert the sound velocities to thermodynamic quantities. Because of the high precision of the data (0.2%) and the fine mesh of data points, it was possible to obtain continuous functions (of the variables P and T) for the density, thermal-expansion coefficient, normal-fluid fraction, entropy, specific heats, and compressibility of He II. As an additional feature the correct asymptotic behavior near the λ line was incorporated into the thermodynamic functions. These functions were used to generate tables of the thermodynamic quantities and smoothed values of the sound velocities. The precision of the results, typically 0.3%, represents an order-of-magnitude improvement over a significant portion of the existing data and has eliminated some discrepancies as large as (10–15)%.

I. INTRODUCTION

In the preceding paper[1a] we reported simultaneous measurement of the velocities of first, second, and fourth sound[1b] (C_1, C_2, and C_4, respectively) in superfluid helium at more than 400 points in the He II phase diagram. Here we use these data to completely determine the thermodynamics of He II at temperatures from 1.2 K to the λ transition temperature T_λ and at all pressures from the saturated vapor pressure (SVP) to 25 bar.

The sound velocities determine directly the important thermodynamic derivatives that can be integrated to give the temperature and pressure dependence of the thermodynamic quantities. The constants of integration can be found from the expansion coefficient $\beta_\rho = -(1/\rho)(\partial\rho/\partial T)_\rho$ along the SVP line and the entropy and density at a single point at SVP. Only one thermodynamic quantity must be found by numerically taking a derivative, a procedure that is usually considered as yielding imprecise results. However, by using a Maxwell relation, this derivative can be calculated from the sound data in two independent ways. Hence the sound velocities contain an intrinsic cross check on the only weak calculation.

By using acoustic resonators, the sound velocities were measured to 0.2% precision. Data were taken in 50-mK steps from 1.2 K to within 10^{-2} K of T_λ and in 1-bar steps from SVP to 25 bar. Temperature was measured to within 1 mK, and pressure was accurate to within 0.1%. Because of the good precision and the fine mesh of data points it was possible to convert the data to continuous thermodynamic functions to a precision of typically 0.3%. Since the sound velocities were measured simultaneously in one experiment, and since data from outside experiments are needed only at SVP, the thermodynamic functions are completely self-consistent and suffer no problems due to conflicting systematic errors between different experiments.

In analyzing the sound data, we made use of a Landau elementary excitation theory[2] and a law of corresponding states[3] for the Landau parameters. As a result, most of the temperature dependence of the thermodynamic functions occurs explicitly in the statistical-mechanical expressions, and most of the pressure dependence is accounted for by expressing parameters as functions of the reduced temperature $T/T_\lambda(P)$ where $T_\lambda(P)$ is determined along the isobar P. By doing this, we have avoided the problem of using high-order polynomials to fit data, and the resulting thermodynamic functions and especially their derivatives have more physical significance.

Since the sound data approached T_λ to within 10^{-2} K, they just reached the critical transition region near the λ line. The asymptotic behavior of the thermodynamic quantities as $T \to T_\lambda$ has been theoretically[4-7] and experimentally[8-12] investigated. We have used available λ-line information and incorporated the correct asymptotic limits in our thermodynamic functions.

Thus, by using sound-velocity data, a small amount of data at SVP, and the asymptotic behavior near the λ line, the thermodynamics of superfluid helium above 1.2 K is completely determined. In Sec. II of this paper we outline the theory involved in calculating the thermodynamics from the sound velocities. In Sec. III we show how our sound data were used to find continuous functions (of P and T) for the density ρ, the expansion coefficient β_ρ, the normal-fluid fraction ρ_n/ρ, the

entropy S, the specific heat at constant pressure C_p, the ratio of C_p to the specific heat at constant volume ($\gamma = C_p/C_v$), and the isothermal compressibility κ_T. In Sec. IV we present tables of the thermodynamic quantities and the sound velocities calculated from the thermodynamic functions. The calculated velocities reproduce the measured velocities to well within the experimental precision and can be taken as a smoothing of the experimental data.

In Sec. V we compare the present results with the thermodynamic data from previous experiments. It is found that the tables represent nearly an order-of-magnitude increase in precision over a significant part of the existing data.

In Sec. VI we study the results in regard to the Landau elementary excitation theory, the law of corresponding states, and the Tisza approximation.[13]

II. THERMODYNAMICS FROM THE SOUND VELOCITIES

A. Two-fluid hydrodynamics

The relationship between the sound velocities and the thermodynamic functions is contained in the velocity expressions derived in the two-fluid hydrodynamic theory.[14] The fundamental assumptions of this theory and an outline of the derivation of the sound-velocity expressions are given in Ref. 1(b). The results, to first order in $\gamma - 1$, are:

$$C_1^2 = C_{10}^2\{1 + [(\gamma - 1)/\gamma]C_{20}^2/(C_{10}^2 - C_{20}^2)\}, \tag{1}$$

$$C_2^2 = C_{20}^2[1 + (\gamma - 1)C_{20}^2/(C_{10}^2 - C_{20}^2)], \tag{2}$$

$$C_4^2 = C_{10}^2(1 - \rho_n/\rho + \zeta'), \tag{3}$$

where

$$C_{10}^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \gamma\left(\frac{\partial P}{\partial \rho}\right)_T, \tag{4}$$

$$C_{20}^2 = [(\rho_n/\rho)^{-1} - 1]TS^2/C_p \tag{5}$$

and

$$\zeta' = \frac{\rho_n}{\rho}\left(\frac{C_{20}}{C_{10}}\right)^2\left(1 - \frac{2\beta_p C_{10}^2}{\gamma S}\right). \tag{6}$$

If we add to this list of equations, the thermodynamic identities

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p \tag{7}$$

and

$$\gamma = 1 + (T/C_p)(\beta_p C_{10})^2, \tag{8}$$

then it can be seen that Eqs. (1)–(8) contain only the sound velocities, ρ_n/ρ, S, ρ, and the first derivatives of S and ρ with respect to P and T. Thus the equations form a set of coupled partial differential equations which in principle can be solved to give S, ρ, and ρ_n/ρ in terms of C_1, C_2, and C_4. In order to obtain unique solutions, we must have some constants of integration. We shall see in Sec. IIB that it is sufficient to know β_p along the SVP line, and S and ρ at one point.

In order to completely specify the thermodynamics of He II, one must include the thermodynamic variable[15] $(\vec{v}_n - \vec{v}_s)^2$, where $\vec{v}_n$ and $\vec{v}_s$ are the normal and superfluid velocity fields, respectively. If one knows the sound velocities and integration constants in terms of these variables, then one can find ρ, S, and ρ_n/ρ in terms of P, T, and $(\vec{v}_n - \vec{v}_s)^2$. It is then possible to integrate the thermodynamic relations

$$\frac{1}{\rho} = \left(\frac{\partial \mu}{\partial P}\right)_{T,(\vec{v}_n - \vec{v}_s)^2}, \tag{9}$$

$$S = -\left(\frac{\partial \mu}{\partial T}\right)_{P,(\vec{v}_n - \vec{v}_s)^2}, \tag{10}$$

$$\frac{\rho_n}{\rho} = -2\left(\frac{\partial \mu}{\partial (\vec{v}_n - \vec{v}_s)^2}\right)_{T,P}, \tag{11}$$

to obtain the chemical potential μ. The constant of integration can be found by assuming that μ at SVP and low temperature is equal to the chemical potential of its ideal-gas vapor. Once μ as a function of P, T, and $(\vec{v}_n - \vec{v}_s)^2$ is found all the thermodynamics of superfluid helium are known since all the thermodynamic quantities can be expressed in terms of μ and its derivatives.

It is usually assumed that the dependence on the variable $(\vec{v}_n - \vec{v}_s)^2$ is weak, except as indicated in Eq. (11). There is some evidence[16] that the dependence of ρ_n on $(\vec{v}_n - \vec{v}_s)^2$ is below present-day experimental resolution. We shall ignore the thermodynamic variable $(\vec{v}_n - \vec{v}_s)^2$ and assume that knowledge of S, ρ, and ρ_n/ρ as functions of P and T is sufficient to completely determine the thermodynamics of He II.

There is another thermodynamic equation that has not been used with the sound-velocity equations: the Maxwell relation

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P = \rho\left(\frac{\partial S}{\partial P}\right)_T. \tag{12}$$

If this equation is used, it is not necessary to know all of the sound velocities at all pressures and temperatures. This will be made clear in Sec. IIB.

B. Simplifications

The sound-velocity equations can be simplified by using the fact that for all pressures and temperatures the quantity

$(\gamma - 1)C_{20}^2/(C_{10}^2 - C_{20}^2)$

is less than 2×10^{-4}. Hence for a precision of 0.02% this term in Eqs. (1) and (2) can be ignored. The sound-velocity equations can now be rewritten

$$C_1^2 = (1+\delta)\left(\frac{\partial P}{\partial \rho}\right)_T , \tag{13}$$

$$C_2^2 = [(\rho_n/\rho)^{-1} - 1]TS^2/C_p , \tag{14}$$

$$C_4^2 = C_1^2(1 - \rho_n/\rho + \zeta) , \tag{15}$$

where

$$\delta \equiv (\gamma - 1) = (T/C_p)(C_1\beta_p)^2 \tag{16}$$

and

$$\zeta = \frac{\rho_n}{\rho}\left(\frac{C_2}{C_1}\right)^2\left(1 - \frac{2\beta_p C_1^2}{\gamma S}\right). \tag{17}$$

The quantities δ and ζ are small, affecting the sound velocities by at most 10%. Although this is too large to ignore, we can take advantage of the small influence of these quantities and solve the equations by iteration. Thus, existing thermodynamic data can be used in Eqs. (16) and (17), and δ and ζ can be considered as known functions of P and T. Equations (13)–(15) can now be written

$$\left(\frac{\partial \rho}{\partial P}\right)_T = \frac{1+\delta}{C_1^2} , \tag{18}$$

$$\frac{1}{S^2}\left(\frac{\partial S}{\partial T}\right)_P = \frac{1}{C_2^2}\left[\left(1 + \zeta - \frac{C_4^2}{C_1^2}\right)^{-1} - 1\right], \tag{19}$$

$$\rho_n/\rho = 1 + \zeta - C_4^2/C_1^2 , \tag{20}$$

where $C_p = T(\partial S/\partial T)_P$ has been used. Since everything on the right-hand side is assumed known, then ρ_n/ρ is found directly and ρ and S can be found by integrating Eqs. (18) and (19). The required constants of integration can be taken in several ways, but a suitable choice is to use the expansion coefficient along the SVP line, $\beta_{\mathrm{SVP}}(T)$, and the entropy and density at a single point on the SVP line, $S(\mathrm{SVP}, T_0)$ and $\rho(\mathrm{SVP}, T_0)$. Thus

$$\rho(P,T) = \rho(\mathrm{SVP},T) + \int_{\mathrm{SVP}}^{P} \frac{1+\delta}{C_1^2} dP , \tag{21}$$

where

$$\rho(\mathrm{SVP},T) = \rho(\mathrm{SVP},T_0)\exp\left(-\int_{T_0}^{T} \beta_{\mathrm{SVP}}(T)\, dT\right).$$

Once $\rho(P,T)$ is known, the Maxwell relation [Eq. (12)] can be integrated to obtain the entropy

$$S(P,T) = S(\mathrm{SVP},T) + \int_{\mathrm{SVP}}^{P} \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P dP . \tag{22}$$

To find $S(\mathrm{SVP},T)$ it is necessary to use Eq. (19) and the C_2 data at SVP only

$$S(\mathrm{SVP},T) = \left\{\frac{1}{S(\mathrm{SVP},T_0)} - \int_{T_0}^{T} \frac{1}{C_2^2}\left[\left(1 + \zeta - \frac{C_4^2}{C_1^2}\right)^{-1} - 1\right] dT\right\}^{-1} . \tag{23}$$

[Equation (23) is exact for isobars; along the SVP line it is accurate to 0.05%.] Thus the minimum amount of data required to completely determine the thermodynamics is C_1 and C_4 at all P and T, C_2 and β_p at SVP, and ρ and S at one point on the SVP line.

When experimental sound data are being used, Eq. (22) is not used to determine S since it requires numerically taking a derivative to obtain $(\partial \rho/\partial T)_P$:

$$\left(\frac{\partial \rho}{\partial T}\right)_P \sim \int_{\mathrm{SVP}}^{P} (1+\delta)\frac{\partial}{\partial T}\left(\frac{1}{C_1^2}\right)_P dP . \tag{24}$$

It is far preferable to use Eq. (19) and the C_2 data at all P and T to obtain S. In this case Eq. (22) can be used along the isotherm T_0 to obtain the constant of integration required by Eq. (19).

When the new thermodynamic functions $\rho(P,T)$ and $S(P,T)$ have been determined, then δ and ζ can be recalculated and the entire procedure repeated. The iteration can be continued until the "input" and "output" thermodynamic functions agree to within an arbitrary precision. Because of the weak influence of δ and ζ, only two iterations are necessary to obtain 0.3% precision in ρ and S.

C. Summary and comments

The scheme for obtaining the thermodynamic functions $\rho(P,T)$ and $S(P,T)$ from the sound velocities is illustrated in Fig. 1. The quantities $\beta_{\mathrm{SVP}}(T)$ and $\rho(\mathrm{SVP},T_0)$ determine ρ along the SVP line and C_1 determines its variation with pressure along the isotherms. The Maxwell relation [Eq. (12)] and $S(\mathrm{SVP},T_0)$ determine S along the isotherm T_0, and C_2 and C_4/C_1 determine its variation with temperature along the isobars. It should be noted that the sound velocities determine the major variations of ρ and S with respect to P and T. That is, ρ varies by only 1 to 2% with temperature, but by 20% with pressure, and first sound determines its pressure variation. Similarly, S varies by a factor of only 2 with pressure, but by a factor of 30 with temperature (above 1.2 K), and the sound velocities determine its temperature variation.

When the thermodynamic quantities are determined from actual sound-velocity measurements, it is significant that the thermodynamic derivatives C_p and κ_T are found directly without the use of numerical derivatives. Thus the precision of

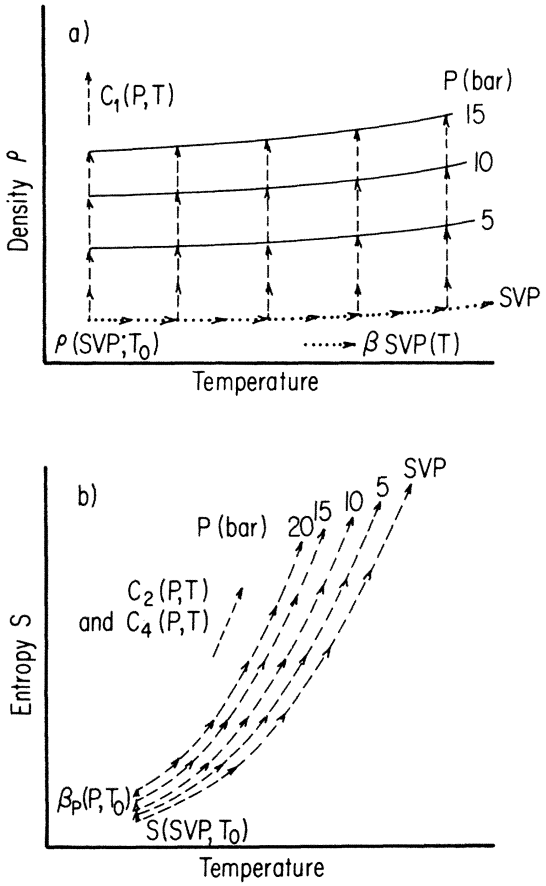

FIG. 1. Determination of the density and entropy from the sound velocities. First sound gives the pressure dependence of ρ; second and fourth sounds give the temperature dependence of S. Quantities $\beta_{SVP}(T)$, $\rho(\mathrm{SVP}, T_0)$, and $S(\mathrm{SVT}, T_0)$ are required for constants of integration.

these quantities is of the same order as that of the sound velocities. The only primary derivative that is not determined directly is the thermal expansion coefficient β_p. However, because of the Maxwell relation [Eq. (12)], β_p can be found from $(\partial\rho/\partial T)_P$ involving a numerical derivative of C_1, or from $(\partial S/\partial P)_T$ involving numerical derivatives of C_2 and C_4/C_1. Hence the sound velocities contain an intrinsic cross check on the only numerically imprecise calculation.

Experimentally there are some significant advantages in measuring sound velocities. By using acoustic resonators, the velocities can be measured to high precision. Only the length of the resonator and the scattering correction for fourth sound are needed to normalize the data. No special low-temperature techniques are required and the pressure and temperature variables can be measured in a straightforward manner.

III. ANALYSIS OF THE SOUND DATA

A. Objectives

In analyzing the sound-velocity data, we considered the following objectives:

(a) We wished to use the fine mesh of sound-velocity data points of the preceding paper to obtain ρ, S, and ρ_n/ρ as continuous analytic functions of P and T. Thus the derivatives C_p and β_p could be calculated exactly, and all the thermodynamic quantities could be found at arbitrary P-T points. These smooth thermodynamic functions, when inserted into the sound-velocity expressions [Eqs. (13)–(15)], would be required to reproduce the experimental sound-velocity data to within the precision of the measurements. The thermodynamic functions would be normalized to give $\beta_{SVP}(T)$, $S(\mathrm{SVP}, T_0)$, and $\rho(\mathrm{SVP}, T_0)$ as measured in other experiments.

(b) The thermodynamic functions should have the correct asymptotic limits as $T \to T_\lambda$ built in.

(c) As a test of the thermodynamic functions, the only derivative not determined directly by the sound-velocity data, β_p, would be compared with the direct measurements of other experiments.

B. Curve fitting

Since we wish to obtain continuous functions for the thermodynamic quantities, it is necessary to use some curve fitting with polynomials. In order to fit directly data that vary over a large range to a precision of 0.2%, it is necessary to use high-order polynomials. The problem of including the correct asymptotic limits increases the order. However, there are serious problems in using high-order polynomial fits. Such fits tend to "curl" near the end points; past the end points the fit might display totally unphysical behavior and could definitely not be used for extrapolations. The interior of the fit might also have small ripples that, although negligible in the original fit, may result in an unphysical structure in the derivative. The curling near the endpoints can produce large errors in the derivative. It is advantageous to avoid using high-order polynomials and use some fit based on more physical ideas. Hence we have used a parametrized Landau elementary excitation theory and the law of corresponding states to obtain continuous functions for ρ_n/ρ, S, C_p, and β_p. These theories have a further benefit in providing a natural means of including the correct asymptotic behavior.

C. Landau theory

In the Landau theory, the thermodynamic properties of He II are calculated by treating the liquid

as a system of elementary excitations with a known energy spectrum [excitation energy as a function of momentum, $E(p)$]. Basic statistical-mechanical expressions can be used to find S and ρ_n/ρ from $E(p)$. It is possible to measure $E(p)$ directly with neutron scattering and use this information to find the thermodynamic functions. This procedure has been thoroughly investigated.[17-21] Unfortunately, the thermodynamic functions are extremely sensitive to the excitation spectrum, and the neutron data are insufficient to accurately determine the thermodynamics over the entire P-T plane.

In our use of the Landau theory, we parametrized the excitation spectrum and adjusted the parameters to obtain thermodynamic functions, which subsequently fit the sound-velocity data. For $E(p)$, we used a smooth curve determined by the velocity of first sound, a maximum $E_{\max}$ at a momentum $p_{\max} = 1.1$ Å^{-1}, and a parabolic minimum Δ (the roton energy-gap parameter) at a momentum p_0 with an effective roton mass μ. The details of the parametrization of the excitation spectrum and the Landau theory calculations are given in the Appendix.

The parameters p_0 and μ were taken from the neutron scattering measurements of Dietrich *et al.*[20] and were not adjusted in the fit. Only two parameters, $E_{\max}$ and Δ, were adjusted; Δ was adjusted because it is the most critical parameter in the theory; it was necessary to adjust $E_{\max}$ because its temperature dependence was not available in the published neutron scattering data.

The Landau theory has been extended to include interactions among the elementary excitations.[21] The result is simply that the excitation spectrum parameters can be taken as functions of temperature. However, the temperature dependence is weak; most of the temperature dependence of the thermodynamic functions is contained explicitly in the statistical-mechanical expressions. This is the tremendous advantage in using the Landau theory: the parameters $E_{\max}$ and Δ can be fit with low-order polynomials in T. The Landau theory supplies the main temperature dependence of the thermodynamic functions and thus prevents unphysical structure from appearing in the derivatives.

D. Law of corresponding states

The roton energy-gap parameter, Δ, plays a role in the Landau theory similar to that of the energy gap Δ_c in the theory of superconductivity. It has been found that Δ_c obeys a law of corresponding states.[22] That is, for different superconductors or for superconductors under pressure, the reduced energy gap Δ_c/kT_c is a universal function of the reduced temperature T/T_c, where T_c is the superconductor's critical temperature and k is Boltzmann's constant. For He II, we might expect similar behavior. Thus as one varies temperature along any isobar, He II passes through a sequence of corresponding states such that the reduced Landau excitation spectrum parameters, Δ/kT_λ and $E_{\max}/kT_\lambda$, are universal functions of T/T_λ. Here T_λ would be determined by the isobar: $T_\lambda(P)$. If this were strictly true, then all of the pressure dependence of the thermodynamic functions would be given implicitly by $T_\lambda(P)$ in the various universal functions. If it is not strictly true, we can still expect that the pressure dependence of the reduced parameters can be fit with low-order polynomials.

E. Asymptotic limits

Neutron scattering experiments indicate that most of the temperature variation of the excitation spectrum occurs near T_λ, and this is the place where Δ and $E_{\max}$ have their greatest influence on S and ρ_n/ρ. This fact and the law of corresponding states make Δ and $E_{\max}$ ideal for including the asymptotic behavior.

Scaling laws indicate that as $T \to T_\lambda$

$$S = S_\lambda + A\epsilon \ln\epsilon \,, \tag{25}$$

$$\rho_s/\rho \equiv 1 - \rho_n/\rho = k'\epsilon^{2/3} \,, \tag{26}$$

where $\epsilon \equiv 1 - T/T_\lambda$. These limits have been experimentally investigated along the λ line,[11,12] and the quantities S_λ, A, and k' have been determined as functions of P. In the Landau theory, these limits can be realized by giving Δ/kT_λ and $E_{\max}/kT_\lambda$ the following leading terms:

$$\Delta/kT_\lambda = \Delta_\lambda/kT_\lambda + D_{23}\epsilon^{2/3} + D_{ln}\epsilon \ln\epsilon \,, \tag{27}$$

$$E_{\max}/kT_\lambda = E_\lambda/kT_\lambda + E_{23}\epsilon^{2/3} + E_{ln}\epsilon \ln\epsilon \,. \tag{28}$$

The six coefficients Δ_λ, E_λ, D_{23}, D_{ln}, E_{23}, and E_{ln} are uniquely determined as functions of P by the expressions (25) and (26) and the Landau theory. The details are given in the Appendix.

The quantities ρ and C_1, although not fit using the Landau theory, were given polynomial fits incorporating the correct asymptotic behavior. The leading terms in the expression for the density ρ were

$$\rho = \rho_\lambda + R\epsilon \ln\epsilon \,. \tag{29}$$

The quantity R as a function of P was taken so as to give the correct asymptotic behavior of β_ρ as measured by Elwell and Meyer.[23] The quantity ρ_λ was found using the expression given in the λ line work of Kierstead.[25]

The asymptotic behavior of C_1 is given by the

Pippard-Buckingham-Fairbank relation,[24]

$$C_1 = C_1^\lambda + C/C_p = C_1^\lambda + C/(B - A\ln\epsilon)\,. \tag{30}$$

We have determined C_1^λ, C, and B as functions of P from Greywall and Ahlers.[11] In all of the expressions involved in the asymptotic limits and the law of corresponding states, $T_\lambda(P)$ is found by inverting the expression for $P(T_\lambda)$ in the reference by Kierstead.[25] For reference, values of various quantities along the λ line are given in Table I.

F. Analysis procedure

To begin, the thermodynamic data of Refs. 23 and 26–28 were used to obtain the initial values of δ and ζ. Then Eq. (21) was used to find $\rho(P,T)$. For the normalization data we used $\rho(\mathrm{SVP}, T_0 = 1.2\ \mathrm{K}) = 0.1452\ \mathrm{g\,cm^{-3}}$ and $\beta_{\mathrm{SVP}}(T)$ from Ref. 28. For convenience in subsequent calculations involving ρ and C_1, and in order to include the asymptotic limits [Eqs. (29) and (30)], the data were fit with the following expressions:

$$\rho(P,T) = \rho_\lambda + R\epsilon\ln\epsilon + A_1\epsilon + A_2\epsilon^2\,, \tag{31}$$

$$C_1(P,T) = C_1^\lambda + \frac{C}{B - A\ln\epsilon} + \epsilon\left(\sum_{n=0}^{3} B_n\epsilon^n\right). \tag{32}$$

TABLE I. Values of the transition temperature T_λ, density ρ_λ, first-sound velocity C_1^λ, and entropy S_λ along the λ line. T_λ and ρ_λ are from Kierstead (Ref. 25); C_1^λ and S_λ are from Ahlers (Ref. 12).

P (bar)	T_λ (K)	ρ_λ ($\mathrm{g\,cm^{-3}}$)	C_1^λ ($\mathrm{m\,sec^{-1}}$)	S_λ ($\mathrm{J\,g^{-1}\,K^{-1}}$)
SVP	2.172	0.1462	217.1	1.559
1	2.163	0.1481	225.3	1.538
2	2.154	0.1499	233.4	1.517
3	2.143	0.1517	241.0	1.496
4	2.133	0.1533	248.1	1.477
5	2.122	0.1548	254.8	1.459
6	2.111	0.1563	261.0	1.442
7	2.099	0.1577	266.9	1.425
8	2.087	0.1590	272.5	1.409
9	2.075	0.1603	277.7	1.394
10	2.063	0.1615	282.6	1.379
11	2.051	0.1627	287.3	1.365
12	2.038	0.1638	291.8	1.351
13	2.025	0.1649	296.0	1.337
14	2.012	0.1660	300.1	1.324
15	1.998	0.1671	303.9	1.311
16	1.985	0.1681	307.6	1.298
17	1.971	0.1691	311.1	1.286
18	1.957	0.1701	314.5	1.273
19	1.942	0.1710	317.7	1.261
20	1.928	0.1719	320.7	1.249
21	1.913	0.1728	323.6	1.237
22	1.897	0.1737	326.3	1.225
23	1.882	0.1746	328.8	1.213
24	1.866	0.1755	331.1	1.201
25	1.850	0.1763	333.3	1.189

With the coefficients in the Appendix, these expressions fit the actual data to within 0.05%. However, the derivative $(\partial\rho/\partial T)_P$ was found using Eq. (21) rather than (31).

Next the parametrized Landau theory was used to fit the C_2 and C_4 data. To include weak pressure and temperature dependence in the adjustable Landau parameters, terms involving cubic polynomials in ϵ were added to expressions (27) and (28). An additional term was added to normalize the entropy along the isotherm T_0. The complete expressions for the adjustable Landau parameters were

$$\frac{\Delta}{kT_\lambda} = \frac{\Delta_\lambda}{kT_\lambda} + D_{23}\epsilon^{2/3} + D_{1n}\epsilon\ln\epsilon + D\epsilon + \epsilon^2\left(\sum_{n=0}^{3} D_n\epsilon^n\right), \tag{33}$$

$$\frac{E_{\max}}{kT_\lambda} = \frac{E_\lambda}{kT_\lambda} + E_{23}\epsilon^{2/3} + D_{1n}\epsilon\ln\epsilon + \epsilon\left(\sum_{n=0}^{3} E_n\epsilon^n\right). \tag{34}$$

At each pressure, the cubic polynomials in large parentheses were adjusted in a least-squares program to fit C_2 and C_4. Then the coefficients D_n and E_n were fit with cubic polynomials in pressure. The coefficient D, adjusted to normalize the entropy, was also fit with a cubic polynomial in P. The normalization point $S(\mathrm{SVP}, T_0 = 1.2\ \mathrm{K}) = 0.0515\ \mathrm{J\,g^{-1}\,K^{-1}}$ was taken as the average of Refs. 26 and 28.

When all the coefficients were determined, expressions (33) and (34) were used in the Landau theory to calculate S, ρ_n/ρ, C_p, and β_p. Then new values of δ and ζ were calculated and the entire procedure was repeated. After the second iteration, there was no significant change in any of the thermodynamic quantities. The final forms of expressions (33) and (34) were used to generate $S(P,T)$ and $\rho_n/\rho(P,T)$ and their derivatives.

It was found that by adjusting only cubic polynomials in the Landau theory it was possible to fit the C_2 and C_4 data at elevated pressures (~ 800 data points) to within 0.2% precision. This is a remarkable result and certainly could not be obtained using direct polynomial fits to the sound and thermodynamic data. Thus the parametrized Landau theory was a powerful tool in the analysis. For the SVP fit, the coefficients required very slight modifications of the cubic fit. The final form of all the coefficients is given in the Appendix.

IV. RESULTS

A. Thermodynamic tables

The results of the sound-velocity analysis are presented in Table II. The mass density $\rho(P,T)$

TABLE II. Density ρ, thermal expansion coefficient β_p, normal-fluid fraction ρ_n/ρ, entropy S, specific heat C_p, specific-heat ratio γ, isothermal compressibility κ_T, and first-, second-, and fourth-sound velocities (C_1, C_2, C_4) at various temperatures and pressures (T, P) in the He II phase. The entries are self-consistent in that thermodynamic identities [eg. $C_p = T(\partial S/\partial T)_P$, $\beta_p = -\rho(\partial S/\partial P)_T$] and the sound-velocity equations are obeyed exactly. The functions which generated the tables had the correct asymptotic behavior as $T \to T_\lambda$ built in.

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
				$P = 0.0$						
1.20	0.1452	0.02	0.0283	0.0515	0.318	0.000	1.222	237.4	18.55	234.0
1.25	0.1452	0.42	0.0368	0.0663	0.408	0.003	1.225	237.1	18.78	232.8
1.30	0.1452	0.78	0.0472	0.0843	0.515	0.009	1.228	236.8	19.03	231.2
1.35	0.1452	1.11	0.0598	0.1060	0.641	0.015	1.231	236.5	19.30	229.4
1.40	0.1453	1.45	0.0748	0.1319	0.786	0.021	1.236	236.1	19.58	227.2
1.45	0.1453	1.84	0.0924	0.1623	0.953	0.029	1.240	235.6	19.84	224.7
1.50	0.1453	2.32	0.1131	0.1978	1.142	0.039	1.245	235.2	20.07	221.7
1.55	0.1453	2.91	0.1369	0.2387	1.357	0.053	1.251	234.6	20.25	218.3
1.60	0.1453	3.62	0.1643	0.2855	1.598	0.072	1.257	234.0	20.37	214.3
1.65	0.1453	4.44	0.1955	0.3387	1.869	0.095	1.264	233.4	20.41	209.8
1.70	0.1454	5.30	0.2310	0.3990	2.174	0.119	1.272	232.7	20.36	204.6
1.75	0.1454	6.17	0.2712	0.4668	2.514	0.142	1.281	231.9	20.19	198.6
1.80	0.1454	7.00	0.3166	0.5429	2.896	0.162	1.291	231.0	19.89	191.7
1.85	0.1455	7.79	0.3677	0.6279	3.323	0.178	1.302	230.0	19.43	183.7
1.90	0.1455	8.63	0.4253	0.7228	3.804	0.195	1.315	228.8	18.78	174.4
1.95	0.1456	9.79	0.4901	0.8285	4.351	0.222	1.331	227.4	17.89	163.5
2.00	0.1457	11.72	0.5632	0.9465	4.990	0.281	1.350	225.8	16.69	150.4
2.05	0.1458	15.09	0.6469	1.0791	5.791	0.405	1.373	224.0	15.00	134.4
2.10	0.1459	19.61	0.7484	1.2316	6.972	0.570	1.401	221.8	12.39	112.6
2.15	0.1460	22.85	0.8856	1.4162	8.723	0.619	1.434	219.2	7.99	75.1
				$P = 1.0$						
1.20	0.1469	0.44	0.0291	0.0517	0.325	0.004	1.133	245.0	18.14	241.5
1.25	0.1469	0.88	0.0380	0.0667	0.415	0.014	1.135	244.8	18.41	240.2
1.30	0.1470	1.34	0.0489	0.0850	0.523	0.027	1.137	244.6	18.70	238.7
1.35	0.1470	1.83	0.0620	0.1071	0.649	0.042	1.140	244.4	19.00	236.8
1.40	0.1470	2.35	0.0775	0.1332	0.795	0.058	1.143	244.0	19.28	234.6
1.45	0.1470	2.93	0.0958	0.1640	0.964	0.077	1.147	243.7	19.55	232.0
1.50	0.1470	3.58	0.1170	0.1999	1.155	0.099	1.151	243.2	19.78	228.9
1.55	0.1470	4.33	0.1416	0.2412	1.372	0.125	1.156	242.7	19.96	225.3
1.60	0.1471	5.18	0.1698	0.2886	1.617	0.156	1.161	242.2	20.07	221.2
1.65	0.1471	6.12	0.2020	0.3424	1.890	0.191	1.167	241.6	20.11	216.4
1.70	0.1472	7.15	0.2386	0.4033	2.196	0.230	1.174	240.9	20.05	210.9
1.75	0.1472	8.25	0.2799	0.4718	2.537	0.270	1.182	240.1	19.87	204.5
1.80	0.1473	9.35	0.3266	0.5485	2.920	0.308	1.191	239.2	19.56	197.2
1.85	0.1473	10.53	0.3790	0.6343	3.349	0.348	1.201	238.2	19.08	188.7
1.90	0.1474	11.91	0.4379	0.7299	3.836	0.394	1.213	237.0	18.40	178.8
1.95	0.1475	13.70	0.5040	0.8366	4.399	0.462	1.227	235.6	17.47	167.2
2.00	0.1476	16.38	0.5787	0.9562	5.071	0.580	1.244	234.0	16.20	153.3
2.05	0.1477	20.70	0.6644	1.0914	5.927	0.799	1.266	232.2	14.43	136.1
2.10	0.1478	27.65	0.7679	1.2480	7.161	1.186	1.294	230.0	11.75	112.4
2.15	0.1480	41.51	0.9093	1.4387	9.313	2.056	1.334	227.4	6.91	69.7
				$P = 2.0$						
1.20	0.1486	0.78	0.0302	0.0521	0.332	0.014	1.058	252.3	17.76	248.5
1.25	0.1486	1.25	0.0395	0.0674	0.423	0.029	1.059	252.1	18.07	247.2
1.30	0.1486	1.76	0.0509	0.0861	0.532	0.048	1.061	252.0	18.39	245.6
1.35	0.1486	2.31	0.0645	0.1085	0.659	0.070	1.062	251.8	18.70	243.7
1.40	0.1486	2.91	0.0806	0.1350	0.807	0.093	1.065	251.5	19.00	241.4
1.45	0.1486	3.57	0.0995	0.1662	0.976	0.119	1.068	251.2	19.27	238.6
1.50	0.1487	4.30	0.1214	0.2025	1.170	0.149	1.071	250.8	19.51	235.4
1.55	0.1487	5.11	0.1468	0.2444	1.390	0.183	1.075	250.3	19.68	231.7
1.60	0.1487	6.02	0.1758	0.2924	1.637	0.221	1.080	249.8	19.79	227.3
1.65	0.1488	7.04	0.2090	0.3469	1.913	0.265	1.085	249.2	19.82	222.3

TABLE II *(Continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
1.70	0.1488	8.15	0.2467	0.4085	2.222	0.314	1.092	248.5	19.75	216.5
1.75	0.1489	9.38	0.2892	0.4778	2.566	0.368	1.099	247.7	19.56	209.7
1.80	0.1490	10.74	0.3372	0.5553	2.951	0.428	1.107	246.8	19.23	202.0
1.85	0.1490	12.27	0.3910	0.6420	3.385	0.497	1.116	245.8	18.73	193.0
1.90	0.1491	14.05	0.4513	0.7387	3.880	0.579	1.127	244.6	18.02	182.5
1.95	0.1492	16.27	0.5191	0.8468	4.461	0.685	1.140	243.2	17.04	170.1
2.00	0.1493	19.49	0.5959	0.9683	5.169	0.858	1.156	241.7	15.69	155.2
2.05	0.1495	24.57	0.6845	1.1067	6.091	1.168	1.177	239.8	13.78	136.4
2.10	0.1497	33.19	0.7923	1.2685	7.445	1.754	1.204	237.6	10.91	110.0
2.15	0.1499	59.43	0.9537	1.4719	11.014	3.800	1.256	234.8	4.53	51.5
					$P=3.0$					
1.20	0.1501	1.05	0.0313	0.0527	0.339	0.026	0.992	259.2	17.44	255.1
1.25	0.1501	1.54	0.0411	0.0684	0.432	0.046	0.993	259.0	17.77	253.8
1.30	0.1501	2.09	0.0530	0.0874	0.541	0.070	0.995	258.9	18.11	252.1
1.35	0.1501	2.68	0.0671	0.1101	0.670	0.097	0.996	258.7	18.44	250.1
1.40	0.1501	3.32	0.0838	0.1371	0.819	0.126	0.998	258.5	18.75	247.7
1.45	0.1502	4.03	0.1033	0.1688	0.990	0.158	1.000	258.2	19.02	244.8
1.50	0.1502	4.80	0.1260	0.2056	1.186	0.194	1.003	257.9	19.25	241.5
1.55	0.1502	5.67	0.1522	0.2480	1.408	0.234	1.007	257.4	19.42	237.5
1.60	0.1503	6.63	0.1821	0.2966	1.658	0.280	1.011	256.9	19.53	232.9
1.65	0.1503	7.71	0.2163	0.3518	1.937	0.333	1.016	256.3	19.54	227.6
1.70	0.1504	8.92	0.2550	0.4142	2.249	0.393	1.021	255.6	19.46	221.5
1.75	0.1505	10.25	0.2988	0.4843	2.598	0.460	1.028	254.9	19.26	214.4
1.80	0.1506	11.76	0.3480	0.5629	2.988	0.537	1.035	254.0	18.91	206.2
1.85	0.1507	13.49	0.4033	0.6506	3.429	0.628	1.044	252.9	18.38	196.7
1.90	0.1508	15.59	0.4653	0.7486	3.936	0.744	1.054	251.8	17.63	185.5
1.95	0.1509	18.34	0.5350	0.8583	4.535	0.907	1.067	250.4	16.59	172.3
2.00	0.1510	22.08	0.6141	0.9822	5.282	1.143	1.082	248.8	15.15	156.2
2.05	0.1512	27.91	0.7062	1.1242	6.280	1.550	1.102	246.9	13.10	135.6
2.10	0.1514	38.20	0.8203	1.2922	7.803	2.351	1.130	244.7	9.92	105.4
					$P=4.0$					
1.20	0.1515	1.28	0.0325	0.0535	0.347	0.040	0.935	265.7	17.16	261.4
1.25	0.1515	1.79	0.0428	0.0695	0.440	0.064	0.936	265.6	17.51	260.0
1.30	0.1515	2.36	0.0551	0.0889	0.551	0.093	0.937	265.5	17.86	258.2
1.35	0.1516	2.98	0.0698	0.1120	0.681	0.124	0.938	265.3	18.20	256.1
1.40	0.1516	3.66	0.0871	0.1394	0.831	0.159	0.940	265.1	18.52	253.6
1.45	0.1516	4.41	0.1074	0.1716	1.005	0.197	0.942	264.9	18.79	250.6
1.50	0.1517	5.23	0.1308	0.2089	1.203	0.238	0.945	264.5	19.02	247.0
1.55	0.1517	6.14	0.1577	0.2519	1.427	0.386	0.948	264.1	19.19	242.9
1.60	0.1518	7.17	0.1886	0.3012	1.680	0.340	0.952	263.6	19.28	238.1
1.65	0.1518	8.32	0.2237	0.3571	1.963	0.403	0.956	263.0	19.28	232.5
1.70	0.1519	9.61	0.2636	0.4203	2.279	0.474	0.961	262.3	19.19	226.0
1.75	0.1520	11.07	0.3086	0.4914	2.632	0.557	0.967	261.6	18.97	218.5
1.80	0.1521	12.71	0.3592	0.5709	3.028	0.653	0.974	260.7	18.60	209.9
1.85	0.1522	14.64	0.4159	0.6599	3.477	0.768	0.982	259.6	18.04	199.8
1.90	0.1523	16.98	0.4796	0.7593	3.998	0.916	0.992	258.5	17.24	187.9
1.95	0.1524	20.07	0.5515	0.8710	4.623	1.123	1.004	257.1	16.13	173.8
2.00	0.1526	24.48	0.6333	0.9975	5.411	1.446	1.019	255.4	14.59	156.5
2.05	0.1528	31.25	0.7294	1.1436	6.496	1.980	1.039	253.5	12.37	133.7
2.10	0.1530	43.53	0.8517	1.3190	8.245	3.046	1.067	251.2	8.79	98.5
					$P=5.0$					
1.20	0.1529	1.47	0.0338	0.0544	0.354	0.054	0.885	272.0	16.91	267.4
1.25	0.1529	2.00	0.0445	0.0707	0.450	0.082	0.886	271.9	17.28	265.9
1.30	0.1529	2.59	0.0574	0.0905	0.562	0.115	0.886	271.8	17.64	264.0
1.35	0.1529	3.25	0.0726	0.1141	0.693	0.152	0.888	271.6	17.99	261.8
1.40	0.1530	3.97	0.0906	0.1420	0.845	0.192	0.889	271.4	18.31	259.1

TABLE II *(Continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
1.45	0.1530	4.75	0.1115	0.1746	1.020	0.236	0.891	271.1	18.58	255.9
1.50	0.1531	5.62	0.1356	0.2125	1.220	0.285	0.894	270.8	18.80	252.2
1.55	0.1531	6.60	0.1634	0.2561	1.448	0.341	0.896	270.4	18.96	247.9
1.60	0.1532	7.70	0.1952	0.3060	1.704	0.405	0.900	269.9	19.04	242.8
1.65	0.1532	8.93	0.2314	0.3628	1.990	0.479	0.904	269.3	19.04	236.9
1.70	0.1533	10.32	0.2723	0.4268	2.310	0.566	0.909	268.7	18.93	230.1
1.75	0.1534	11.90	0.3185	0.4989	2.668	0.666	0.914	267.9	18.69	222.2
1.80	0.1535	13.70	0.3705	0.5796	3.071	0.784	0.921	267.0	18.29	213.1
1.85	0.1536	15.82	0.4288	0.6698	3.530	0.928	0.929	266.0	17.70	202.4
1.90	0.1538	18.43	0.4944	0.7709	4.067	1.112	0.938	264.7	16.85	189.9
1.95	0.1539	21.89	0.5684	0.8847	4.719	1.373	0.950	263.3	15.67	174.8
2.00	0.1541	26.88	0.6534	1.0142	5.560	1.779	0.965	261.7	14.01	155.9
2.05	0.1543	34.84	0.7542	1.1651	6.745	2.488	0.985	259.7	11.59	130.7
2.10	0.1546	50.08	0.8875	1.3493	8.819	3.954	1.016	257.3	7.41	87.9
					$P=6.0$					
1.20	0.1542	1.65	0.0352	0.0554	0.362	0.069	0.840	278.0	16.70	273.2
1.25	0.1542	2.20	0.0463	0.0721	0.459	0.101	0.841	277.9	17.08	271.5
1.30	0.1542	2.82	0.0597	0.0922	0.572	0.139	0.842	277.7	17.45	269.5
1.35	0.1543	3.51	0.0755	0.1163	0.705	0.182	0.843	277.6	17.80	267.1
1.40	0.1543	4.26	0.0941	0.1446	0.859	0.228	0.844	277.4	18.12	264.3
1.45	0.1543	5.10	0.1157	0.1778	1.036	0.279	0.846	277.1	18.39	261.0
1.50	0.1544	6.03	0.1406	0.2163	1.239	0.337	0.848	276.8	18.60	257.1
1.55	0.1545	7.07	0.1692	0.2606	1.469	0.403	0.851	276.3	18.75	252.5
1.60	0.1545	8.25	0.2020	0.3112	1.729	0.479	0.854	275.9	18.82	247.2
1.65	0.1546	9.59	0.2392	0.3688	2.019	0.570	0.858	275.3	18.80	241.0
1.70	0.1547	11.12	0.2813	0.4338	2.344	0.676	0.863	274.6	18.67	233.9
1.75	0.1548	12.86	0.3287	0.5069	2.708	0.801	0.868	273.9	18.41	225.6
1.80	0.1549	14.85	0.3821	0.5888	3.119	0.949	0.875	273.0	17.99	215.9
1.85	0.1550	17.21	0.4421	0.6805	3.589	1.129	0.882	271.9	17.36	204.6
1.90	0.1552	20.14	0.5096	0.7834	4.144	1.363	0.891	270.7	16.46	191.3
1.95	0.1554	24.03	0.5861	0.8995	4.828	1.691	0.903	269.2	15.19	175.1
2.00	0.1555	29.67	0.6744	1.0325	5.728	2.200	0.918	267.5	13.40	154.6
2.05	0.1558	38.83	0.7810	1.1888	7.039	3.095	0.939	265.5	10.74	126.2
2.10	0.1561	59.55	0.9310	1.3843	9.705	5.304	0.976	262.9	5.54	70.4
					$P=7.0$					
1.20	0.1555	1.82	0.0366	0.0565	0.370	0.086	0.799	283.8	16.52	278.6
1.25	0.1555	2.39	0.0482	0.0736	0.469	0.123	0.800	283.6	16.89	276.9
1.30	0.1555	3.05	0.0621	0.0941	0.584	0.166	0.802	283.5	17.27	274.7
1.35	0.1555	3.78	0.0785	0.1186	0.718	0.215	0.803	283.3	17.62	272.2
1.40	0.1556	4.58	0.0977	0.1475	0.874	0.270	0.804	283.1	17.94	269.2
1.45	0.1556	5.47	0.1200	0.1812	1.053	0.330	0.806	282.8	18.21	265.7
1.50	0.1557	6.47	0.1457	0.2203	1.259	0.398	0.808	282.4	18.41	261.6
1.55	0.1557	7.59	0.1752	0.2653	1.492	0.476	0.811	282.0	18.55	256.8
1.60	0.1558	8.86	0.2089	0.3167	1.755	0.567	0.814	281.5	18.61	251.2
1.65	0.1559	10.31	0.2472	0.3752	2.051	0.675	0.818	281.0	18.57	244.7
1.70	0.1560	11.97	0.2905	0.4412	2.382	0.804	0.822	280.3	18.42	237.2
1.75	0.1561	13.88	0.3393	0.5155	2.752	0.957	0.827	279.6	18.14	228.5
1.80	0.1562	16.11	0.3942	0.5988	3.171	1.143	0.834	278.6	17.69	218.3
1.85	0.1564	18.73	0.4558	0.6921	3.654	1.369	0.841	277.6	17.01	206.4
1.90	0.1565	22.01	0.5254	0.7969	4.229	1.661	0.851	276.3	16.05	192.2
1.95	0.1567	26.38	0.6046	0.9157	4.949	2.071	0.862	274.8	14.70	174.8
2.00	0.1569	32.78	0.6967	1.0525	5.917	2.706	0.878	273.0	12.76	152.4
2.05	0.1572	43.42	0.8100	1.2151	7.384	3.840	0.901	270.9	9.80	120.0

TABLE II *(Continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
				$P = 8.0$						
1.20	0.1567	1.99	0.0380	0.0577	0.379	0.105	0.763	289.3	16.35	283.9
1.25	0.1567	2.59	0.0501	0.0752	0.479	0.147	0.764	289.1	16.73	282.0
1.30	0.1567	3.28	0.0645	0.0962	0.596	0.196	0.765	289.0	17.11	279.7
1.35	0.1568	4.06	0.0816	0.1211	0.732	0.253	0.767	288.8	17.46	277.0
1.40	0.1568	4.91	0.1014	0.1505	0.889	0.317	0.768	288.5	17.77	273.9
1.45	0.1569	5.87	0.1245	0.1848	1.071	0.387	0.770	288.2	18.03	270.2
1.50	0.1569	6.94	0.1510	0.2246	1.279	0.467	0.773	287.9	18.23	265.8
1.55	0.1570	8.14	0.1814	0.2703	1.516	0.560	0.775	287.5	18.36	260.8
1.60	0.1571	9.51	0.2161	0.3226	1.784	0.669	0.778	287.0	18.40	254.9
1.65	0.1572	11.09	0.2555	0.3820	2.085	0.798	0.782	286.4	18.35	248.1
1.70	0.1573	12.90	0.3000	0.4492	2.422	0.953	0.786	285.8	18.18	240.2
1.75	0.1574	14.99	0.3502	0.5247	2.800	1.140	0.791	285.0	17.87	231.1
1.80	0.1575	17.44	0.4066	0.6095	3.229	1.368	0.798	284.1	17.38	220.4
1.85	0.1577	20.39	0.4702	0.7045	3.726	1.652	0.805	282.9	16.66	207.7
1.90	0.1578	24.07	0.5420	0.8115	4.324	2.019	0.815	281.6	15.64	192.5
1.95	0.1580	28.99	0.6241	0.9332	5.084	2.528	0.827	280.1	14.18	173.8
2.00	0.1583	36.26	0.7206	1.0743	6.132	3.319	0.844	278.2	12.08	149.2
2.05	0.1586	48.84	0.8422	1.2443	7.798	4.773	0.868	275.9	8.73	111.5
				$P = 9.0$						
1.20	0.1579	2.16	0.0396	0.0591	0.387	0.126	0.730	294.7	16.20	288.9
1.25	0.1579	2.80	0.0521	0.0769	0.489	0.173	0.731	294.5	16.57	286.9
1.30	0.1579	3.53	0.0671	0.0983	0.608	0.230	0.733	294.3	16.95	284.5
1.35	0.1580	4.35	0.0848	0.1238	0.746	0.296	0.734	294.1	17.31	281.6
1.40	0.1580	5.27	0.1053	0.1538	0.906	0.370	0.736	293.8	17.62	278.3
1.45	0.1581	6.29	0.1291	0.1887	1.090	0.453	0.738	293.5	17.87	274.4
1.50	0.1581	7.44	0.1565	0.2291	1.302	0.548	0.740	293.1	18.06	269.8
1.55	0.1582	8.74	0.1878	0.2757	1.543	0.657	0.743	292.7	18.17	264.5
1.60	0.1583	10.23	0.2235	0.3289	1.815	0.787	0.746	292.2	18.20	258.4
1.65	0.1584	11.93	0.2640	0.3893	2.122	0.942	0.749	291.6	18.13	251.2
1.70	0.1585	13.91	0.3099	0.4576	2.465	1.129	0.754	291.0	17.94	243.0
1.75	0.1586	16.20	0.3615	0.5346	2.852	1.356	0.759	290.2	17.60	233.3
1.80	0.1588	18.90	0.4196	0.6209	3.292	1.634	0.765	289.2	17.07	222.0
1.85	0.1589	22.18	0.4851	0.7179	3.806	1.984	0.773	288.1	16.31	208.6
1.90	0.1591	26.33	0.5594	0.8274	4.430	2.443	0.783	286.7	15.21	192.4
1.95	0.1593	31.89	0.6447	0.9524	5.236	3.077	0.796	285.0	13.64	172.1
2.00	0.1596	40.21	0.7462	1.0984	6.377	4.061	0.814	283.0	11.34	144.8
2.05	0.1599	55.58	0.8788	1.2770	8.327	5.985	0.842	280.5	7.44	99.5
				$P = 10.0$						
1.20	0.1590	2.34	0.0412	0.0605	0.396	0.149	0.701	299.8	16.06	293.7
1.25	0.1591	3.01	0.0542	0.0787	0.500	0.203	0.702	299.6	16.43	291.6
1.30	0.1591	3.78	0.0698	0.1006	0.621	0.269	0.703	299.4	16.81	289.0
1.35	0.1591	4.66	0.0881	0.1266	0.761	0.344	0.705	299.2	17.16	286.0
1.40	0.1592	5.64	0.1094	0.1572	0.924	0.431	0.706	298.9	17.46	282.5
1.45	0.1592	6.74	0.1340	0.1928	1.111	0.528	0.709	298.5	17.71	278.3
1.50	0.1593	7.98	0.1622	0.2340	1.326	0.640	0.711	298.1	17.89	273.5
1.55	0.1594	9.39	0.1945	0.2814	1.571	0.770	0.713	297.7	17.99	268.0
1.60	0.1595	11.00	0.2313	0.3355	1.848	0.925	0.717	297.2	18.00	261.5
1.65	0.1596	12.86	0.2730	0.3971	2.161	1.111	0.720	296.6	17.91	254.1
1.70	0.1597	15.01	0.3201	0.4667	2.512	1.336	0.725	296.0	17.69	245.4
1.75	0.1598	17.53	0.3733	0.5452	2.909	1.611	0.730	295.1	17.32	235.2
1.80	0.1600	20.52	0.4332	0.6333	3.362	1.951	0.737	294.2	16.76	223.2
1.85	0.1601	24.16	0.5009	0.7325	3.895	2.380	0.745	293.0	15.94	209.0
1.90	0.1603	28.80	0.5778	0.8447	4.548	2.944	0.756	291.5	14.76	191.6
1.95	0.1606	35.13	0.6668	0.9733	5.407	3.736	0.770	289.7	13.07	169.6
2.00	0.1608	44.71	0.7741	1.1248	6.661	4.964	0.789	287.6	10.53	138.9
2.05	0.1612	65.08	0.9227	1.3143	9.110	7.730	0.824	284.8	5.71	80.8

TABLE II *(Continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
					$P = 11.0$					
1.20	0.1602	2.52	0.0429	0.0620	0.406	0.174	0.673	304.8	15.92	298.3
1.25	0.1602	3.23	0.0565	0.0807	0.512	0.236	0.674	304.6	16.29	296.1
1.30	0.1602	4.05	0.0726	0.1031	0.635	0.311	0.676	304.4	16.67	293.4
1.35	0.1602	4.99	0.0916	0.1297	0.778	0.399	0.678	304.1	17.02	290.2
1.40	0.1603	6.04	0.1136	0.1608	0.942	0.500	0.680	303.8	17.32	286.5
1.45	0.1603	7.22	0.1390	0.1972	1.133	0.615	0.682	303.4	17.56	282.1
1.50	0.1604	8.56	0.1682	0.2392	1.351	0.747	0.684	303.0	17.72	277.1
1.55	0.1605	10.08	0.2015	0.2874	1.601	0.901	0.687	302.6	17.81	271.2
1.60	0.1606	11.83	0.2393	0.3427	1.884	1.085	0.690	302.1	17.80	264.5
1.65	0.1607	13.86	0.2823	0.4054	2.204	1.307	0.694	301.5	17.69	256.6
1.70	0.1608	16.22	0.3309	0.4765	2.564	1.579	0.698	300.8	17.45	247.5
1.75	0.1610	19.00	0.3857	0.5565	2.971	1.912	0.704	299.9	17.05	236.8
1.80	0.1611	22.30	0.4475	0.6466	3.439	2.325	0.711	298.9	16.44	224.1
1.85	0.1613	26.34	0.5175	0.7482	3.993	2.849	0.720	297.7	15.55	208.9
1.90	0.1615	31.54	0.5973	0.8634	4.681	3.540	0.731	296.1	14.28	190.2
1.95	0.1618	38.71	0.6904	0.9962	5.601	4.515	0.746	294.2	12.45	166.1
2.00	0.1621	49.92	0.8048	1.1541	6.995	6.066	0.769	291.8	9.61	131.2
2.05	0.1625	101.17	0.9895	1.3607	12.522	13.940	0.843	288.4	1.79	30.1
					$P = 12.0$					
1.20	0.1612	2.71	0.0447	0.0636	0.416	0.202	0.648	309.6	15.80	302.8
1.25	0.1612	3.46	0.0588	0.0828	0.524	0.273	0.649	309.5	16.16	300.4
1.30	0.1613	4.33	0.0756	0.1057	0.650	0.359	0.651	309.2	16.54	297.6
1.35	0.1613	5.32	0.0952	0.1329	0.795	0.460	0.653	308.9	16.88	294.2
1.40	0.1614	6.45	0.1180	0.1647	0.962	0.576	0.655	308.5	17.18	290.3
1.45	0.1614	7.72	0.1443	0.2018	1.156	0.710	0.657	308.2	17.41	285.7
1.50	0.1615	9.17	0.1744	0.2447	1.379	0.866	0.659	307.7	17.56	280.4
1.55	0.1616	10.83	0.2087	0.2939	1.634	1.050	0.662	307.3	17.63	274.3
1.60	0.1617	12.74	0.2478	0.3503	1.923	1.270	0.666	306.8	17.60	267.2
1.65	0.1618	14.95	0.2921	0.4144	2.251	1.536	0.669	306.2	17.47	258.9
1.70	0.1619	17.54	0.3422	0.4869	2.620	1.862	0.674	305.4	17.20	249.3
1.75	0.1621	20.59	0.3987	0.5688	3.039	2.264	0.680	304.6	16.76	238.0
1.80	0.1623	24.24	0.4626	0.6610	3.523	2.765	0.688	303.5	16.10	224.5
1.85	0.1625	28.75	0.5350	0.7652	4.101	3.403	0.697	302.1	15.15	208.3
1.90	0.1627	34.57	0.6181	0.8838	4.828	4.245	0.710	300.5	13.78	188.1
1.95	0.1630	42.72	0.7160	1.0213	5.822	5.441	0.727	298.4	11.77	161.5
2.00	0.1633	56.08	0.8391	1.1866	7.401	7.431	0.752	295.7	8.54	120.8
					$P = 13.0$					
1.20	0.1623	2.90	0.0466	0.0654	0.426	0.234	0.625	314.3	15.68	307.0
1.25	0.1623	3.70	0.0613	0.0850	0.537	0.314	0.626	314.1	16.04	304.6
1.30	0.1623	4.62	0.0787	0.1085	0.665	0.412	0.628	313.9	16.41	301.6
1.35	0.1624	5.68	0.0990	0.1363	0.813	0.527	0.630	313.6	16.75	298.1
1.40	0.1624	6.88	0.1226	0.1688	0.984	0.661	0.632	313.2	17.04	293.9
1.45	0.1625	8.25	0.1498	0.2067	1.181	0.817	0.634	312.8	17.26	289.1
1.50	0.1626	9.80	0.1809	0.2505	1.409	0.999	0.637	312.3	17.40	283.5
1.55	0.1627	11.60	0.2163	0.3008	1.669	1.215	0.640	311.9	17.45	277.1
1.60	0.1628	13.68	0.2566	0.3584	1.965	1.476	0.643	311.3	17.40	269.6
1.65	0.1629	16.11	0.3023	0.4239	2.301	1.797	0.647	310.7	17.24	260.9
1.70	0.1630	18.96	0.3540	0.4981	2.681	2.189	0.652	310.0	16.94	250.8
1.75	0.1632	22.32	0.4124	0.5820	3.114	2.673	0.659	309.0	16.47	238.8
1.80	0.1634	26.36	0.4785	0.6765	3.617	3.279	0.667	307.9	15.76	224.5
1.85	0.1636	31.38	0.5537	0.7836	4.222	4.052	0.677	306.5	14.73	207.1
1.90	0.1638	37.91	0.6405	0.9059	4.994	5.075	0.691	304.6	13.24	185.2
1.95	0.1641	47.23	0.7438	1.0487	6.076	6.542	0.710	302.3	11.03	155.6
2.00	0.1645	63.76	0.8787	1.2231	7.929	9.185	0.741	299.3	7.22	106.3

TABLE II *(Continued)*

T (K)	($g\,cm^{-3}$)	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S ($J\,g^{-1}\,K^{-1}$)	C_P ($J\,g^{-1}\,K^{-1}$)	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 ($m\,sec^{-1}$)	C_2 ($m\,sec^{-1}$)	C_4 ($m\,sec^{-1}$)
					$P = 14.0$					
1.20	0.1633	3.10	0.0487	0.0672	0.438	0.268	0.604	318.8	15.56	311.1
1.25	0.1633	3.95	0.0639	0.0873	0.551	0.360	0.605	318.7	15.92	308.6
1.30	0.1633	4.93	0.0819	0.1114	0.681	0.471	0.606	318.5	16.29	305.5
1.35	0.1634	6.06	0.1030	0.1399	0.832	0.603	0.608	318.1	16.62	301.8
1.40	0.1634	7.34	0.1274	0.1732	1.006	0.757	0.611	317.7	16.90	297.4
1.45	0.1635	8.80	0.1555	0.2120	1.208	0.937	0.613	317.3	17.11	292.3
1.50	0.1636	10.48	0.1877	0.2567	1.440	1.148	0.616	316.8	17.24	286.4
1.55	0.1637	12.42	0.2243	0.3082	1.707	1.401	0.619	316.3	17.27	279.7
1.60	0.1638	14.67	0.2659	0.3671	2.011	1.708	0.623	315.8	17.20	271.8
1.65	0.1639	17.33	0.3132	0.4342	2.356	2.087	0.627	315.1	17.01	262.7
1.70	0.1641	20.46	0.3666	0.5102	2.748	2.560	0.633	314.3	16.68	252.0
1.75	0.1643	24.18	0.4270	0.5962	3.196	3.144	0.639	313.4	16.16	239.3
1.80	0.1645	28.66	0.4955	0.6933	3.719	3.873	0.648	312.2	15.39	224.0
1.85	0.1647	34.25	0.5737	0.8036	4.355	4.805	0.660	310.6	14.28	205.3
1.90	0.1650	41.60	0.6645	0.9301	5.180	6.045	0.675	308.6	12.66	181.4
1.95	0.1653	52.34	0.7743	1.0789	6.370	7.849	0.697	305.9	10.19	147.9
2.00	0.1657	74.99	0.9271	1.2648	8.759	11.744	0.737	302.4	5.36	83.4
					$P = 15.0$					
1.20	0.1643	3.32	0.0508	0.0692	0.449	0.307	0.585	323.2	15.45	315.1
1.25	0.1643	4.22	0.0666	0.0898	0.565	0.411	0.585	323.2	15.81	312.5
1.30	0.1643	5.26	0.0853	0.1145	0.699	0.537	0.587	322.9	16.17	309.2
1.35	0.1644	6.46	0.1072	0.1437	0.853	0.687	0.589	322.6	16.50	305.3
1.40	0.1644	7.82	0.1325	0.1778	1.031	0.863	0.591	322.2	16.77	300.7
1.45	0.1645	9.39	0.1616	0.2175	1.236	1.070	0.594	321.7	16.97	295.3
1.50	0.1646	11.19	0.1948	0.2633	1.474	1.314	0.597	321.2	17.08	289.2
1.55	0.1647	13.28	0.2327	0.3160	1.748	1.609	0.600	320.7	17.09	282.0
1.60	0.1648	15.73	0.2757	0.3763	2.060	1.968	0.604	320.1	17.00	273.8
1.65	0.1650	18.61	0.3245	0.4451	2.416	2.414	0.608	319.4	16.78	264.1
1.70	0.1651	22.05	0.3798	0.5231	2.821	2.973	0.614	318.6	16.41	252.8
1.75	0.1653	26.16	0.4424	0.6114	3.286	3.677	0.622	317.6	15.84	239.3
1.80	0.1655	31.13	0.5135	0.7114	3.832	4.553	0.632	316.3	15.01	223.0
1.85	0.1658	37.36	0.5950	0.8252	4.504	5.673	0.644	314.5	13.80	202.8
1.90	0.1660	45.68	0.6905	0.9565	5.390	7.173	0.662	312.2	12.02	176.5
1.95	0.1664	58.27	0.8083	1.1122	6.720	9.424	0.688	309.3	9.23	138.0
					$P = 16.0$					
1.20	0.1652	3.54	0.0531	0.0712	0.462	0.349	0.566	327.4	15.34	318.9
1.25	0.1652	4.49	0.0695	0.0924	0.580	0.466	0.567	327.5	15.70	316.2
1.30	0.1653	5.60	0.0889	0.1178	0.717	0.608	0.568	327.3	16.06	312.8
1.35	0.1653	6.87	0.1116	0.1477	0.874	0.778	0.570	327.0	16.38	308.7
1.40	0.1654	8.32	0.1378	0.1827	1.056	0.979	0.573	326.5	16.64	303.8
1.45	0.1655	10.00	0.1679	0.2234	1.267	1.216	0.576	326.0	16.82	298.2
1.50	0.1656	11.93	0.2023	0.2703	1.511	1.498	0.579	325.5	16.91	291.7
1.55	0.1657	14.19	0.2415	0.3243	1.792	1.838	0.582	324.9	16.91	284.2
1.60	0.1658	16.83	0.2860	0.3862	2.114	2.256	0.586	324.3	16.79	275.5
1.65	0.1660	19.97	0.3365	0.4567	2.481	2.778	0.591	323.6	16.54	265.3
1.70	0.1661	23.73	0.3938	0.5369	2.902	3.436	0.598	322.7	16.12	253.3
1.75	0.1663	28.26	0.4587	0.6278	3.386	4.272	0.606	321.7	15.50	238.9
1.80	0.1665	33.82	0.5327	0.7310	3.958	5.334	0.617	320.2	14.60	221.5
1.85	0.1668	40.80	0.6180	0.8487	4.670	6.680	0.631	318.3	13.28	199.5
1.90	0.1671	50.26	0.7189	0.9852	5.629	8.496	0.651	315.7	11.32	170.2
1.95	0.1675	65.39	0.8469	1.1492	7.154	11.360	0.682	312.2	8.07	124.6
					$P = 17.0$					
1.20	0.1661	3.77	0.0555	0.0734	0.475	0.395	0.550	331.6	15.23	322.5
1.25	0.1662	4.78	0.0726	0.0952	0.596	0.526	0.550	331.7	15.59	319.8
1.30	0.1662	5.94	0.0927	0.1213	0.736	0.686	0.551	331.6	15.95	316.3

TABLE II *(Continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
1.35	0.1663	7.29	0.1162	0.1520	0.897	0.878	0.553	331.2	16.26	311.9
1.40	0.1663	8.84	0.1434	0.1879	1.083	1.106	0.556	330.8	16.51	306.8
1.45	0.1664	10.63	0.1746	0.2296	1.299	1.376	0.559	330.2	16.68	300.9
1.50	0.1665	12.71	0.2102	0.2778	1.550	1.699	0.562	329.7	16.75	294.0
1.55	0.1666	15.14	0.2508	0.3332	1.839	2.091	0.566	329.1	16.72	286.1
1.60	0.1668	18.00	0.2969	0.3967	2.171	2.576	0.570	328.4	16.57	276.9
1.65	0.1669	21.41	0.3492	0.4692	2.552	3.184	0.576	327.7	16.29	266.2
1.70	0.1671	25.52	0.4086	0.5517	2.989	3.955	0.583	326.8	15.83	253.4
1.75	0.1673	30.51	0.4761	0.6454	3.495	4.941	0.592	325.6	15.15	238.1
1.80	0.1676	36.69	0.5533	0.7521	4.097	6.206	0.604	324.0	14.16	219.3
1.85	0.1678	44.53	0.6428	0.8742	4.855	7.822	0.620	321.8	12.72	195.2
1.90	0.1682	55.34	0.7501	1.0167	5.904	10.018	0.644	318.8	10.53	162.2
1.95	0.1686	74.60	0.8923	1.1907	7.749	13.873	0.682	314.7	6.56	105.5
					$P = 18.0$					
1.20	0.1670	4.01	0.0580	0.0758	0.489	0.444	0.534	335.7	15.13	326.0
1.25	0.1671	5.07	0.0758	0.0982	0.613	0.592	0.534	335.8	15.49	323.2
1.30	0.1671	6.30	0.0967	0.1250	0.756	0.771	0.535	335.7	15.84	319.6
1.35	0.1672	7.73	0.1211	0.1565	0.921	0.986	0.537	335.4	16.14	315.0
1.40	0.1672	9.38	0.1493	0.1934	1.112	1.243	0.540	334.9	16.38	309.7
1.45	0.1673	11.29	0.1816	0.2362	1.334	1.549	0.543	334.3	16.53	303.4
1.50	0.1674	13.51	0.2185	0.2856	1.591	1.916	0.547	333.7	16.58	296.2
1.55	0.1676	16.11	0.2605	0.3425	1.889	2.364	0.551	333.1	16.53	287.8
1.60	0.1677	19.20	0.3083	0.4078	2.233	2.919	0.555	332.4	16.35	278.1
1.65	0.1679	22.90	0.3626	0.4824	2.628	3.621	0.561	331.6	16.02	266.7
1.70	0.1681	27.38	0.4243	0.5675	3.084	4.519	0.569	330.7	15.52	253.2
1.75	0.1683	32.87	0.4946	0.6643	3.615	5.674	0.579	329.4	14.77	236.7
1.80	0.1686	39.72	0.5753	0.7748	4.252	7.167	0.592	327.6	13.70	216.3
1.85	0.1689	48.55	0.6696	0.9018	5.064	9.102	0.611	325.1	12.11	189.8
1.90	0.1692	61.06	0.7846	1.0511	6.226	11.770	0.638	321.6	9.62	152.1
1.95	0.1698	90.93	0.9510	1.2388	8.899	18.170	0.694	316.7	4.16	71.7
					$P = 19.0$					
1.20	0.1679	4.25	0.0607	0.0782	0.503	0.496	0.519	339.6	15.03	329.5
1.25	0.1680	5.37	0.0791	0.1013	0.630	0.660	0.519	339.9	15.39	326.5
1.30	0.1680	6.67	0.1009	0.1288	0.777	0.858	0.520	339.8	15.73	322.7
1.35	0.1681	8.17	0.1262	0.1612	0.946	1.099	0.522	339.5	16.02	318.0
1.40	0.1681	9.92	0.1554	0.1991	1.142	1.386	0.525	339.0	16.25	312.3
1.45	0.1682	11.95	0.1889	0.2431	1.370	1.730	0.528	338.3	16.38	305.7
1.50	0.1683	14.31	0.2272	0.2939	1.635	2.144	0.532	337.7	16.42	298.1
1.55	0.1685	17.10	0.2708	0.3524	1.943	2.649	0.536	337.0	16.33	289.3
1.60	0.1686	20.41	0.3204	0.4196	2.299	3.280	0.541	336.3	16.12	279.0
1.65	0.1688	24.41	0.3767	0.4964	2.711	4.082	0.548	335.5	15.75	266.9
1.70	0.1690	29.29	0.4409	0.5843	3.188	5.118	0.556	334.5	15.19	252.4
1.75	0.1693	35.33	0.5142	0.6845	3.747	6.465	0.567	333.0	14.38	234.8
1.80	0.1695	42.94	0.5989	0.7992	4.424	8.217	0.583	331.0	13.19	212.6
1.85	0.1699	52.91	0.6988	0.9317	5.299	10.523	0.604	328.1	11.43	183.1
1.90	0.1703	67.69	0.8236	1.0890	6.616	13.816	0.637	324.0	8.54	138.8
					$P = 20.0$					
1.20	0.1688	4.48	0.0635	0.0808	0.518	0.549	0.505	343.5	14.93	332.7
1.25	0.1688	5.66	0.0827	0.1046	0.649	0.728	0.505	343.8	15.29	329.7
1.30	0.1689	7.02	0.1053	0.1329	0.799	0.947	0.506	343.8	15.63	325.7
1.35	0.1689	8.61	0.1316	0.1662	0.973	1.213	0.508	343.5	15.91	320.8
1.40	0.1690	10.46	0.1619	0.2052	1.175	1.532	0.511	342.9	16.12	314.8
1.45	0.1691	12.60	0.1967	0.2504	1.409	1.915	0.515	342.2	16.23	307.8
1.50	0.1692	15.11	0.2364	0.3026	1.682	2.375	0.519	341.5	16.24	299.7
1.55	0.1694	18.07	0.2816	0.3628	2.000	2.940	0.523	340.8	16.13	290.4

TABLE II. *(continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
					$P=20.0$					
1.60	0.1695	21.61	0.3331	0.4320	2.370	3.647	0.529	340.1	15.88	279.6
1.65	0.1697	25.92	0.3916	0.5113	2.799	4.557	0.535	339.2	15.47	266.7
1.70	0.1700	31.23	0.4585	0.6021	3.300	5.744	0.544	338.1	14.85	251.3
1.75	0.1702	37.87	0.5351	0.7061	3.892	7.303	0.557	336.5	13.95	232.2
1.80	0.1705	46.34	0.6242	0.8255	4.615	9.351	0.574	334.2	12.65	207.9
1.85	0.1709	57.66	0.7307	0.9642	5.567	12.086	0.599	330.8	10.67	174.7
1.90	0.1714	75.92	0.8686	1.1309	7.123	16.328	0.639	325.9	7.18	120.6
					$P=21.0$					
1.20	0.1696	4.71	0.0665	0.0836	0.534	0.602	0.492	347.3	14.84	335.9
1.25	0.1697	5.93	0.0864	0.1080	0.668	0.797	0.491	347.7	15.20	332.7
1.30	0.1697	7.36	0.1099	0.1371	0.822	1.036	0.492	347.7	15.52	328.6
1.35	0.1698	9.03	0.1372	0.1714	1.001	1.326	0.495	347.3	15.79	323.4
1.40	0.1699	10.97	0.1687	0.2115	1.208	1.676	0.498	346.7	15.98	317.0
1.45	0.1700	13.23	0.2048	0.2580	1.449	2.096	0.502	346.0	16.08	309.7
1.50	0.1701	15.87	0.2460	0.3117	1.731	2.603	0.506	345.2	16.07	301.1
1.55	0.1703	19.00	0.2929	0.3737	2.060	3.225	0.511	344.5	15.93	291.3
1.60	0.1704	22.77	0.3464	0.4450	2.445	4.010	0.516	343.7	15.64	279.8
1.65	0.1706	27.40	0.4073	0.5270	2.895	5.029	0.524	342.8	15.18	266.2
1.70	0.1709	33.18	0.4770	0.6210	3.423	6.378	0.534	341.6	14.49	249.6
1.75	0.1712	40.49	0.5574	0.7290	4.051	8.174	0.547	339.8	13.50	228.9
1.80	0.1715	49.93	0.6515	0.8536	4.828	10.558	0.567	337.1	12.05	202.0
1.85	0.1719	62.90	0.7660	0.9993	5.880	13.808	0.597	333.0	9.80	164.1
1.90	0.1724	88.34	0.9240	1.1784	7.931	19.997	0.651	327.1	5.23	92.2
					$P=22.0$					
1.20	0.1705	4.93	0.0696	0.0864	0.550	0.653	0.479	351.0	14.75	338.9
1.25	0.1705	6.20	0.0903	0.1116	0.687	0.862	0.479	351.4	15.10	335.6
1.30	0.1706	7.68	0.1146	0.1416	0.846	1.120	0.480	351.5	15.42	331.3
1.35	0.1706	9.42	0.1430	0.1769	1.030	1.434	0.482	351.1	15.67	325.8
1.40	0.1707	11.45	0.1757	0.2181	1.243	1.812	0.486	350.4	15.85	319.1
1.45	0.1709	13.81	0.2132	0.2659	1.492	2.266	0.490	349.6	15.93	311.3
1.50	0.1710	16.58	0.2560	0.3213	1.782	2.814	0.494	348.8	15.89	302.3
1.55	0.1711	19.87	0.3048	0.3851	2.124	3.490	0.499	348.0	15.71	291.9
1.60	0.1713	23.86	0.3603	0.4587	2.525	4.349	0.505	347.2	15.38	279.7
1.65	0.1716	28.82	0.4238	0.5434	2.997	5.481	0.513	346.3	14.87	265.2
1.70	0.1718	35.09	0.4966	0.6409	3.556	7.003	0.524	344.9	14.11	247.4
1.75	0.1721	43.15	0.5810	0.7534	4.227	9.060	0.539	342.8	13.02	224.8
1.80	0.1725	53.73	0.6810	0.8837	5.067	11.827	0.562	339.6	11.40	194.9
1.85	0.1729	68.91	0.8054	1.0375	6.252	15.746	0.597	334.8	8.77	150.5
					$P=23.0$					
1.20	0.1713	5.12	0.0728	0.0893	0.567	0.699	0.467	354.6	14.66	341.8
1.25	0.1714	6.43	0.0943	0.1153	0.708	0.921	0.467	355.1	15.01	338.4
1.30	0.1714	7.97	0.1196	0.1461	0.871	1.196	0.468	355.1	15.32	333.8
1.35	0.1715	9.78	0.1491	0.1825	1.060	1.531	0.471	354.7	15.56	328.0
1.40	0.1716	11.88	0.1830	0.2249	1.280	1.934	0.474	353.9	15.72	320.9
1.45	0.1717	14.33	0.2220	0.2741	1.535	2.416	0.479	353.1	15.77	312.6
1.50	0.1719	17.19	0.2664	0.3311	1.836	2.996	0.483	352.2	15.70	303.1
1.55	0.1720	20.62	0.3171	0.3969	2.190	3.716	0.488	351.4	15.50	292.1
1.60	0.1722	24.83	0.3749	0.4729	2.609	4.645	0.494	350.6	15.12	279.2
1.65	0.1725	30.13	0.4410	0.5606	3.107	5.889	0.503	349.5	14.54	263.7
1.70	0.1727	36.95	0.5172	0.6619	3.701	7.595	0.514	348.0	13.71	244.5
1.75	0.1730	45.86	0.6062	0.7792	4.421	9.942	0.532	345.6	12.49	219.8
1.80	0.1734	57.80	0.7131	0.9159	5.338	13.154	0.559	341.7	10.67	186.1
1.85	0.1739	76.33	0.8508	1.0793	6.729	18.066	0.602	335.8	7.49	132.4

TABLE II. *(continued)*

T (K)	ρ (g cm^{-3})	$-10^3\beta_P$ (K^{-1})	ρ_n/ρ	S (J g^{-1} K^{-1})	C_P (J g^{-1} K^{-1})	$10^2(\gamma-1)$	$10^2\kappa_T$ (bar^{-1})	C_1 (m sec^{-1})	C_2 (m sec^{-1})	C_4 (m sec^{-1})
					$P=24.0$					
1.20	0.1722	5.29	0.0762	0.0924	0.584	0.737	0.456	358.1	14.58	344.6
1.25	0.1722	6.63	0.0985	0.1191	0.729	0.970	0.456	358.6	14.92	341.0
1.30	0.1723	8.22	0.1247	0.1508	0.897	1.259	0.457	358.6	15.21	336.1
1.35	0.1723	10.08	0.1553	0.1882	1.091	1.611	0.460	358.1	15.44	329.9
1.40	0.1724	12.24	0.1906	0.2319	1.317	2.032	0.464	357.2	15.58	322.4
1.45	0.1726	14.74	0.2311	0.2826	1.580	2.533	0.468	356.3	15.61	313.7
1.50	0.1727	17.68	0.2773	0.3412	1.890	3.135	0.473	355.4	15.52	303.7
1.55	0.1729	21.22	0.3299	0.4091	2.259	3.886	0.478	354.6	15.27	292.1
1.60	0.1731	25.62	0.3900	0.4875	2.698	4.873	0.484	353.8	14.85	278.4
1.65	0.1734	31.29	0.4589	0.5783	3.225	6.229	0.493	352.6	14.21	261.8
1.70	0.1736	38.72	0.5388	0.6837	3.859	8.131	0.506	350.9	13.28	241.0
1.75	0.1740	48.61	0.6330	0.8064	4.637	10.800	0.526	348.0	11.93	213.8
1.80	0.1744	62.25	0.7483	0.9504	5.650	14.552	0.557	343.3	9.84	175.3
1.85	0.1750	87.42	0.9057	1.1259	7.450	21.428	0.615	336.0	5.73	105.4
					$P=25.0$					
1.20	0.1730	5.42	0.0796	0.0955	0.602	0.766	0.446	361.5	14.50	347.2
1.25	0.1730	6.79	0.1027	0.1230	0.750	1.006	0.446	361.9	14.84	343.3
1.30	0.1731	8.41	0.1300	0.1557	0.923	1.304	0.447	361.9	15.11	338.2
1.35	0.1732	10.30	0.1618	0.1941	1.123	1.666	0.450	361.3	15.32	331.6
1.40	0.1733	12.50	0.1984	0.2391	1.355	2.096	0.454	360.4	15.45	323.7
1.45	0.1734	15.03	0.2405	0.2912	1.626	2.604	0.458	359.4	15.46	314.5
1.50	0.1736	18.01	0.2884	0.3516	1.946	3.213	0.463	358.5	15.33	303.9
1.55	0.1738	21.62	0.3430	0.4214	2.329	3.980	0.468	357.6	15.04	291.7
1.60	0.1740	26.21	0.4055	0.5024	2.791	5.011	0.474	356.8	14.57	277.2
1.65	0.1743	32.26	0.4774	0.5966	3.351	6.475	0.483	355.5	13.85	259.4
1.70	0.1746	40.39	0.5614	0.7064	4.033	8.589	0.498	353.4	12.82	236.9
1.75	0.1749	51.43	0.6617	0.8351	4.878	11.623	0.521	350.0	11.31	206.5
1.80	0.1754	67.41	0.7876	0.9874	6.023	16.090	0.559	344.2	8.86	161.5
1.85	0.1761	147.28	0.9902	1.1836	11.322	39.703	0.708	334.7	1.50	33.8

and the velocity of first sound $C_1(P,T)$ are from Eqs. (31) and (32), respectively. The entropy per gram $S(P,T)$ and the normal-fluid fraction $\rho_n/\rho(P,T)$ are calculated using Eqs. (33) and (34) in the Landau theory. The specific heat at constant pressure $C_p(P,T)$ and the thermal-expansion coefficient $\beta_p(P,T)$ are calculated from the derivatives of $S(P,T)$. The quantity $\gamma-1$ is calculated with Eq. (16). The isothermal compressibility $\kappa_T \equiv (1/\rho)(\partial\rho/\partial P)_T$ is calculated with

$$\kappa_T = \gamma/\rho C_1^2 . \tag{35}$$

The second-sound velocity $C_2(P,T)$ and the fourth-sound velocity $C_4(P,T)$ are calculated from the thermodynamic quantities using Eqs. (14) and (15).

The data in Table II can be used to take higher derivatives with respect to P or T to a precision of a few percent. More precise derivatives can be found by using the equations of Sec. II and the Appendix. In Table III we present some derivatives at SVP which are of use in calculations such as Doppler shifts and second-order effects.

We have calculated the chemical potential $\mu(P,T)$ for $(\vec{v}_n-\vec{v}_s)^2=0$ by integrating Eqs. (9) and (10); the results are given in Table IV. The constant of integration was found by evaluating the expression for the chemical potential (Gibbs potential) of the ideal He gas at 1.2 K and SVP. All other thermodynamic potentials can be found from $\mu(P,T)$ and the quantities in Table II.

B. Precision

The expressions that generated Table II reproduce the measured sound-velocity data (a total of almost 1300 data points) to within the measured precision (typically 0.2%). This represents directly a precision in ρ of better than 0.1% throughout the table. The precisions of ρ_n/ρ, S, and C_p are not so easy to determine due to the smoothing effects of the Landau theory and the use of only cubic polynomials in the fit. How-

TABLE III. Second-order derivatives evaluated at SVP.

T (K)	$\frac{\rho}{C_1}\frac{\partial C_1}{\partial \rho}$	$10^2\frac{\partial(\rho_n/\rho)}{\partial P}$ (bar^{-1})	$\frac{\partial(\rho_n/\rho)}{\partial T}$ (K^{-1})	$-10^3\frac{\partial \beta_\rho}{\partial P}$ (bar^{-1}K^{-1})	$-10^3\frac{\partial \beta_\rho}{\partial T}$ (K^{-2})	$\frac{\partial C_\rho}{\partial T}$ (J g^{-1}K^{-2})
1.20	2.724	0.064	0.152	0.47	8.45	1.64
1.25	2.749	0.111	0.188	0.51	7.57	1.97
1.30	2.779	0.158	0.229	0.66	6.79	2.33
1.35	2.808	0.204	0.275	0.90	6.56	2.71
1.40	2.831	0.250	0.326	1.18	7.16	3.12
1.45	2.849	0.300	0.382	1.57	8.50	3.55
1.50	2.862	0.356	0.444	1.84	10.64	4.03
1.55	2.869	0.422	0.511	2.06	13.10	4.55
1.60	2.872	0.502	0.585	2.23	15.37	5.12
1.65	2.871	0.597	0.666	2.37	16.95	5.74
1.70	2.867	0.706	0.755	2.52	17.50	6.44
1.75	2.860	0.826	0.854	2.74	17.02	7.21
1.80	2.851	0.951	0.963	3.09	16.04	8.07
1.85	2.839	1.075	1.085	3.61	15.87	9.05
1.90	2.825	1.192	1.221	4.30	18.87	10.22
1.95	2.809	1.304	1.374	5.14	28.88	11.74
2.00	2.789	1.423	1.557	6.09	50.91	14.04
2.05	2.763	1.563	1.813	7.27	84.61	18.64
2.10	2.731	1.592	2.315	10.69	81.25	30.43
2.15	2.687	1.273	3.126	25.16	190.35	24.08

TABLE IV. Chemical potential. Entries are $-\mu(P,T)$ (J g^{-1}).

P (bar)	T (K) 1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00	2.10
SVP	14.891	14.897	14.906	14.919	14.939	14.969	15.010	15.068	15.146	15.252
1	14.206	14.212	14.221	14.235	14.255	14.285	14.327	14.385	14.464	14.571
2	13.529	13.536	13.545	13.559	13.579	13.609	13.652	13.710	13.790	13.898
3	12.860	12.866	12.875	12.890	12.910	12.941	12.984	13.043	13.125	13.234
4	12.196	12.203	12.212	12.227	12.248	12.279	12.323	12.383	12.466	12.577
5	11.540	11.546	11.556	11.571	11.593	11.624	11.669	11.730	11.814	11.927
6	10.888	10.895	10.905	10.920	10.943	10.976	11.020	11.083	11.168	11.283
7	10.243	10.250	10.260	10.275	10.298	10.331	10.377	10.441	10.528	
8	9.602	9.609	9.619	9.635	9.659	9.693	9.740	9.805	9.893	
9	8.966	8.974	8.984	9.001	9.025	9.059	9.108	9.174	9.264	
10	8.335	8.343	8.354	8.371	8.395	8.431	8.480	8.548	8.640	
11	7.709	7.716	7.728	7.745	7.771	7.807	7.857	7.927	8.021	
12	7.086	7.094	7.106	7.124	7.150	7.187	7.239	7.310	7.406	
13	6.468	6.476	6.488	6.507	6.534	6.572	6.625	6.697	6.796	
14	5.854	5.862	5.874	5.893	5.921	5.960	6.015	6.089	6.190	
15	5.243	5.252	5.264	5.284	5.313	5.353	5.409	5.485		
16	4.636	4.645	4.658	4.678	4.708	4.749	4.806	4.884		
17	4.033	4.042	4.055	4.076	4.106	4.149	4.208	4.288		
18	3.432	3.442	3.456	3.477	3.508	3.552	3.613	3.695		
19	2.835	2.845	2.859	2.882	2.914	2.959	3.021	3.106		
20	2.241	2.251	2.266	2.289	2.322	2.369	2.433	2.521		
21	1.651	1.660	1.676	1.700	1.734	1.782	1.848	1.939		
22	1.063	1.073	1.089	1.113	1.149	1.199	1.267			
23	0.477	0.488	0.505	0.530	0.567	0.618	0.689			
24	−0.105	−0.094	−0.077	−0.050	−0.012	0.041	0.114			
25	−0.684	−0.673	−0.655	−0.628	−0.589	−0.534	−0.458			

ever, by slightly manipulating the fit within the confines of the C_2 and C_4 experimental precision and observing the excursions of the thermodynamic quantities, representative precisions can be determined. Limits on the precision resulting from errors in the measurement of P and T occur only for those few points very near T_λ. However, it should be kept in mind that the expressions that generated the thermodynamic quantities had the correct asymptotic behavior built in.

The precision of β_ρ can be estimated by comparing the results obtained using C_2 and C_4 [i.e., using $\beta_\rho = -\rho(\partial S/\partial P)_T$] with the results using C_1 in Eq. (24). However, the crucial test of β_ρ and the pressure dependence of S, C_ρ, and[29] ρ_n/ρ is in the comparison of β_ρ with values measured directly in other experiments. Figure 2 is a plot of β_ρ together with data of Ref. 28 at SVP and Elwell and Meyer[23] at elevated pressures. In view of the difficulty in obtaining small derivatives precisely from numerical data, we feel that the agreement is excellent. In general, the values are in agreement within the experimental precision of the directly measured β_ρ over a range of more than two orders of magnitude. (The precision of the Elwell and Meyer data is given by the maximum of 0.3×10^{-3} K^{-1} or 3%.) The agreement of β_ρ indicates a precision of 0.2% in the pressure dependence of S, ρ_n/ρ, and C_ρ.

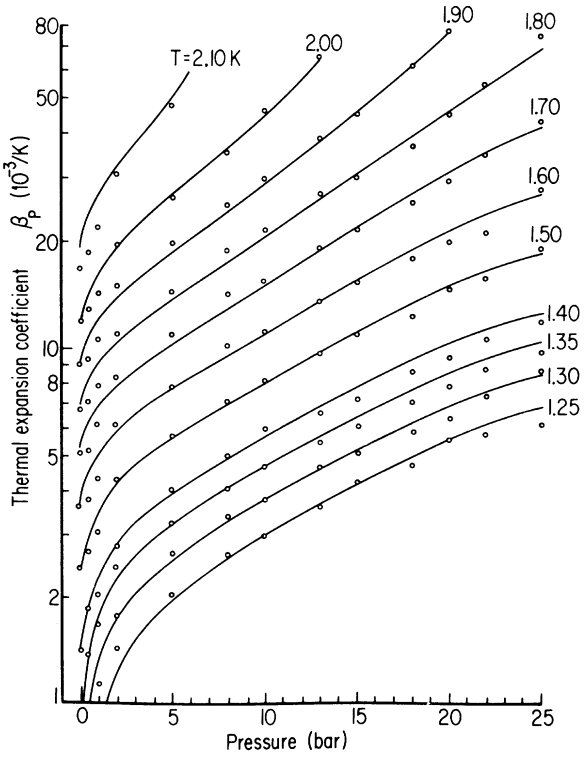

FIG. 2. Thermal-expansion coefficient vs pressure along isotherms. Lines are from this work; points at SVP are from Ref. 28; other points are from Elwell and Meyer (Ref. 23).

By considering all of the above factors, the precisions of the quantities in Table II have been estimated for various regions of the P-T plane; the results are given in Table V. The absolute accuracy of ρ, β_ρ, S, and C_ρ depends on the accuracy of the normalization data $\beta_{S\,VP}(T)$, $\rho(\mathrm{SVP}, T_0)$, and $S(\mathrm{SVP}, T_0)$. The relative accuracy of ρ_n/ρ and the sound velocities is determined by the accuracy of the experimental sound-velocity data and is within the precisions indicated in Table V. The accuracy of the derivatives in Table III is estimated at 3%. The values of the chemical potential in Table IV are accurate to ~0.01 Jg^{-1} relative to the normalization value.

V. COMPARISON WITH PREVIOUS MEASUREMENTS

Figures 2–6 show graphs of our thermodynamic functions (solid lines) together with data measured in other experiments. The data points at SVP in all figures are from Ref. 28. Figure 2 showing β_ρ has already been discussed. Elwell and Meyer[23] determined β_ρ graphically from their precision (0.1%) measurements of the molar volume. These measurements were made by observing the change in the dielectric constant of the fluid as the temperature was varied at constant pressure. We have converted their molar volume data to density data and plotted the results in Fig. 3. Their data agree with our $\rho(P, T)$ to 0.1%.

Figure 4 shows the smoothed entropy data of Van den Meijdenberg *et al.*[26] measured with the fountain effect. Their data have a precision of (2–3)% and in general fit our entropy to within this precision.

The C_v data of Lounasmaa[27] have been converted to C_ρ and presented in Fig. 5. The precision above 1.5 K was reported as 2% and is in reasonable agreement with the present results. However, below 1.6 K significant deviations (reaching 10%) appear. The data of Wiebes and Kramers[30] show better agreement, but their measurements do not extend above 1.6 K.

The ρ_n/ρ data points in Fig. 6 are based on several different experiments. At SVP, the data points are based on the second-sound measurements of Peshkov[31] and S and C_ρ from Ref. 28. The data points at elevated pressures for $\rho_n/\rho > \sim 0.3$ (triangles) are from the oscillating disk measurements of Romer and Duffy.[32] The precision of their data is reported as ±0.01, which corresponds to a precision of (1–3)% in ρ_n/ρ and (2–10)% in ρ_s/ρ. The data points for $\rho_n/\rho < \sim 0.3$ (circles) were calculated using the second-sound

TABLE V. Precision of the results in Table II.

	Saturated vapor pressure			Elevated pressures		
	1.2 K	Intermediate T	Highest T	1.2 K	Intermediate T	Highest T
ρ	0.1%	0.1%	0.1%	0.1%	0.1%	0.1%
β_p	$0.2\times10^{-3}\,\mathrm{K}^{-1}$	3%	8%	$0.3\times10^{-3}\,\mathrm{K}^{-1}$	3%	3%
ρ_n/ρ	0.0005	0.3%	0.05%	0.0005	0.3%	0.05%
S	0.0002 $\mathrm{J\,g^{-1}\,K^{-1}}$	0.3%	0.6%	0.0002 $\mathrm{J\,g^{-1}\,K^{-1}}$	0.3%	0.6%
C_p	0.002 $\mathrm{J\,g^{-1}\,K^{-1}}$	0.3%	0.6%	0.002 $\mathrm{J\,g^{-1}\,K^{-1}}$	0.3%	0.6%
$\gamma-1$	50%	6%	10%	50%	6%	6%
κ_T	0.2%	0.2%	0.8%	0.2%	0.1%	0.2%
C_1	0.1%	0.1%	0.4%	0.1%	0.05%	0.1%
C_2	0.2%	0.1%	0.4%	0.2%	0.05%	0.4%
C_4	0.2%	0.1%	0.4%	0.2%	0.1%	0.4%

data of Maurer and Herlin,[33] the entropy data of Van den Meijdenberg *et al.*,[26] and the specific-heat data to Lounasmaa.[27] The errors in the data combine produce discrepancies of (10–15)%. Prior to the present results, this represented the best available data for ρ_n/ρ at elevated pressures. It should be noted that in Fig. 6 the ρ_n/ρ data points from the different sources do not match smoothly.

There are other discrepancies in the previously available thermodynamic data. For example, derivatives calculated with the published entropy tables deviate significantly from the measured values of C_p and β_p. An important feature of the tables presented here is that they are completely self-consistent. The thermodynamic derivatives and the sound-velocity expressions are evaluated

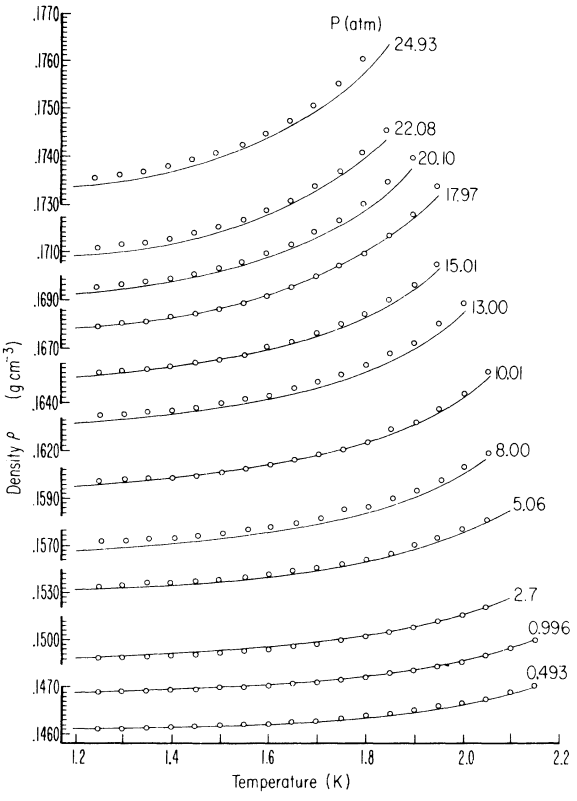

FIG. 3. Density vs temperature along isobars. Lines are from this work; points at SVP are from Ref. 28; other points are from Elwell and Meyer (Ref. 23). Vertical scale has been broken so that an expanded scale could be used.

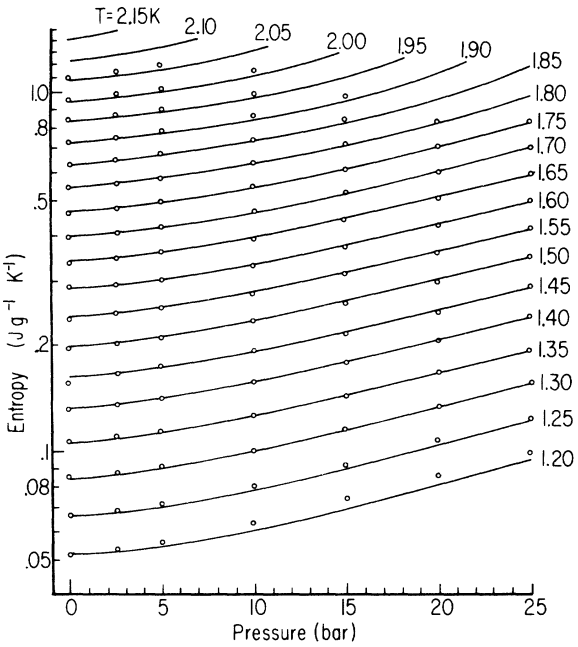

FIG. 4. Entropy vs pressure along isotherms. Lines are from this work; points at SVP are from Ref. (28); other points are from van den Meijdenberg *et al.* (Ref. 26).

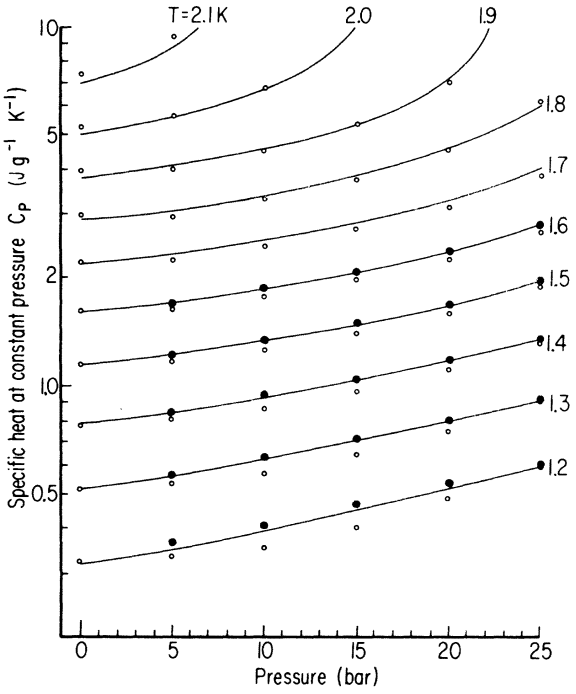

FIG. 5. Specific heat at constant pressure vs pressure along isotherms. Lines are from this work; points at SVP are from Ref. 28; open circles are from Lounasmaa (Ref. 27); solid points are from Wiebes and Kramers (Ref. 30).

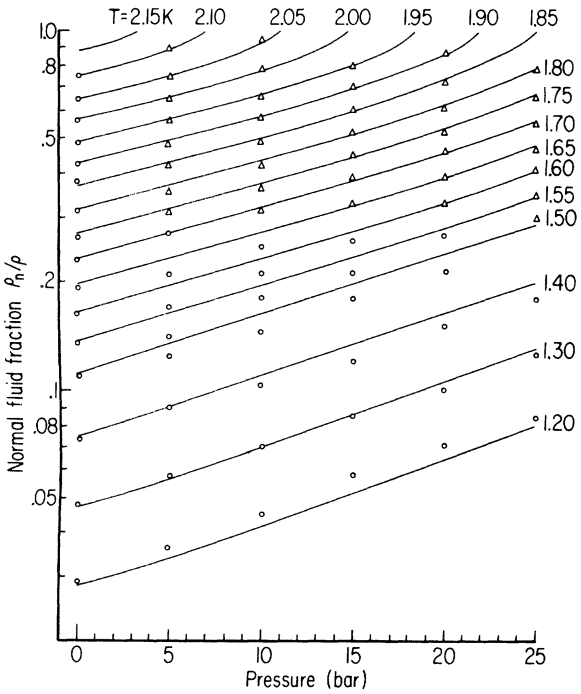

FIG. 6. Normal-fluid fraction vs pressure along isotherms. Lines are from this work; points at SVP are from Ref. 28; triangles are from Romer and Duffy (Ref. 32); other points are calculated using C_2 from Maurer and Herlin (Ref. 33), S from van den Meijdenberg *et al.* (Ref. 26), and C_p from Lounasmaa (Ref. 27).

exactly.

In addition to being self-consistent, the data in Table II represent an excellent experimental precision. In Figs. 2–6, the precision for most of the graphs is better than the width of the line. The precision of ρ and β_p is comparable with that of the Elwell and Meyer measurements. The precisions of S, C_p, and ρ_n/ρ have been increased by nearly an order of magnitude over the existing measurements. A comparison of the sound velocities with previous measurements can be found in the preceding paper.

VI. COMMENTS

A. Landau theory

The parametrized Landau theory proved to be a useful tool in analyzing the sound-velocity data. However, there is the possibility that the theory provides more than just a curve-fitting technique; there may be some physical significance to the resulting elementary excitation spectrum. Since the Landau parameters E_{max} and Δ were adjusted in the fit without any constraints, quite radical looking excitation spectra could have resulted. We obtain, however, very typical spectra, and the critical parameter Δ is in reasonable agreement with the neutron scattering data. In Fig. 7(a) the solid lines give the temperature dependence of our Δ along several isobars; the data points are from the neutron scattering measurements of Dietrich *et al.*[20] At low temperatures ($T/T_\lambda < 0.8$) the data agree to within the precision of the experimental points. As T_λ is approached, there appears to be a systematic discrepancy. This has been previously observed[18,20] in studies of the Landau theory and is usually attributed to the failure of the Landau theory for large densities of elementary excitations. However, the work by Roberts and Donnelly[21] indicates that the interactions of the elementary excitations should be accounted for by the temperature dependence of Δ.

There has been some theoretical work[34] that suggests an alternate method of analyzing the raw neutron scattering data, which gives closer agreement between the neutron Δ and the parameter Δ. We have observed that any analysis that averages the energy eigenstates with a Boltzmann weighting factor $e^{-E/kT}$ will shift the center of the broadened excitation energy by an amount of order $\Gamma^2/2kT$, where Γ is the width of the broadening. By using the Γ data of Dietrich, *et al.* we have calculated the shift and applied it to our Δ. The results,

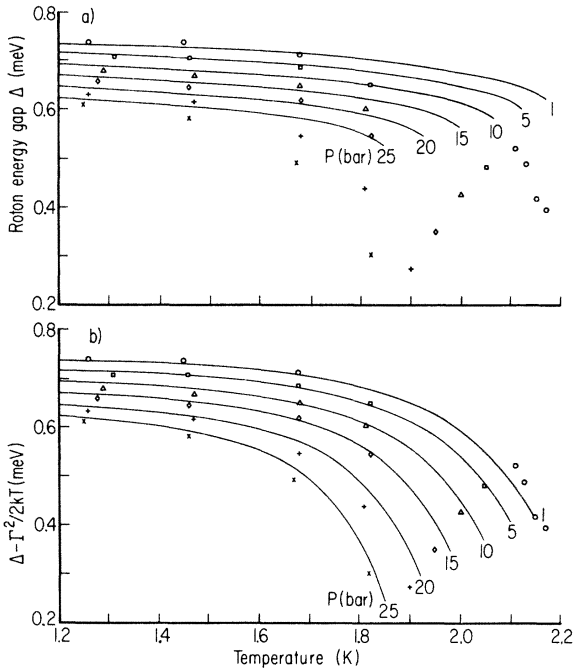

FIG. 7. Unshifted and shifted roton energy-gap parameter (solid lines) vs temperature along several isobars. Points are from the neutron scattering measurements of Dietrich *et al.* (Ref. 20). (a) Unshifted parameter Δ. (b) Parameter Δ shifted by $\Gamma^2/2kT$.

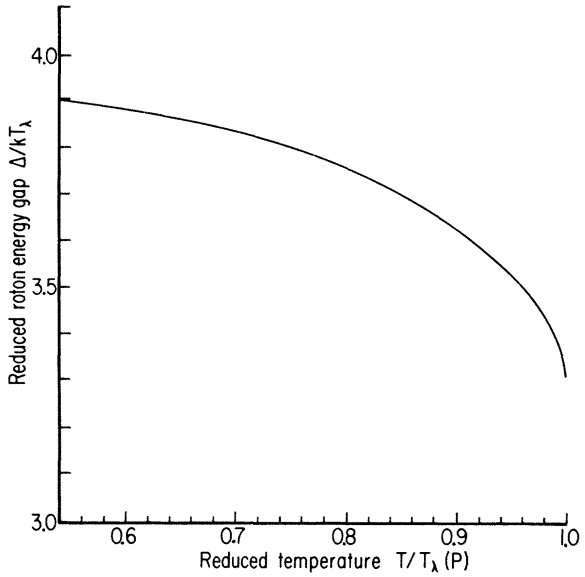

FIG. 8. Reduced roton energy gap Δ/kT_λ vs reduced temperature $T/T_\lambda(P)$. All pressures fall on this curve to within 0.2%.

shown in Fig. 7(b), fit the neutron data to within a few percent. We present this only as a numerical result. The problem of reconciling the neutron data with the Landau theory is still open to theoretical analysis.

B. Law of corresponding states

One of the most striking results of the analysis of the sound-velocity data has been the observation of the law of corresponding states. In Sec. III C we noted that an approximation to this law by Δ would be beneficial by permitting the use of low-order polynomials in P. However, Δ was in no way constrained to obey such a law. After Δ had been adjusted, we examined the quantity Δ/kT_λ as a function of T/T_λ and found universal behavior for all 26 pressures to within a standard deviation of 0.2%. The universal function Δ/kT_λ is shown in Fig. 8; the 0.2% standard deviation is roughly represented by the width of the line.

We examined other quantities for universal behavior and found no striking results except for the superfluid fraction ρ_s/ρ (and equivalently ρ_n/ρ). This quantity scaled to within ±0.002 near T_λ and to better than ±0.001 at low temperatures. At elevated pressures ρ_s/ρ scaled to within the experimental precision, and the possibility exists that ρ_s/ρ obeys the law of corresponding states exactly. This would not be unreasonable since ρ_s/ρ, being closely related to the order parameter,[35] may be considered the single most significant thermohydrodynamic quantity characterizing He II. Figure 9 shows a plot of ρ_s/ρ data points determined from Table II versus $T/T_\lambda(P)$ for all pressures above SVP. The ~400 data points give the appearance of a single line. This universal behavior at points far from the λ line is interesting as well as convenient.

C. Tisza approximation

Now that precise, self-consistent thermodynamic data are available for He II, it is of interest to examine some of the numerical properties. One well-known numerical approximation is based on the so-called Tisza approximation. If one assumes that the superfluid component of He II carries no entropy, then one may write

$$\rho S = \rho_n S_n + \rho_s S_s = \rho_n S_n . \tag{36}$$

In the Tisza approximation, S_n and consequently the ratio of S to ρ_n/ρ are taken to be independent of temperature. At SVP, the Tisza approximation appears valid to 3% above 1.7 K, but to only 10% at lower temperatures. However, for pressures above 3 bar, the Tisza approximation is valid to 2% down to 1.2 K.

By studying Table II we have found other useful numerical approximations. For example, C_p/S is constant in temperature to a few percent. This

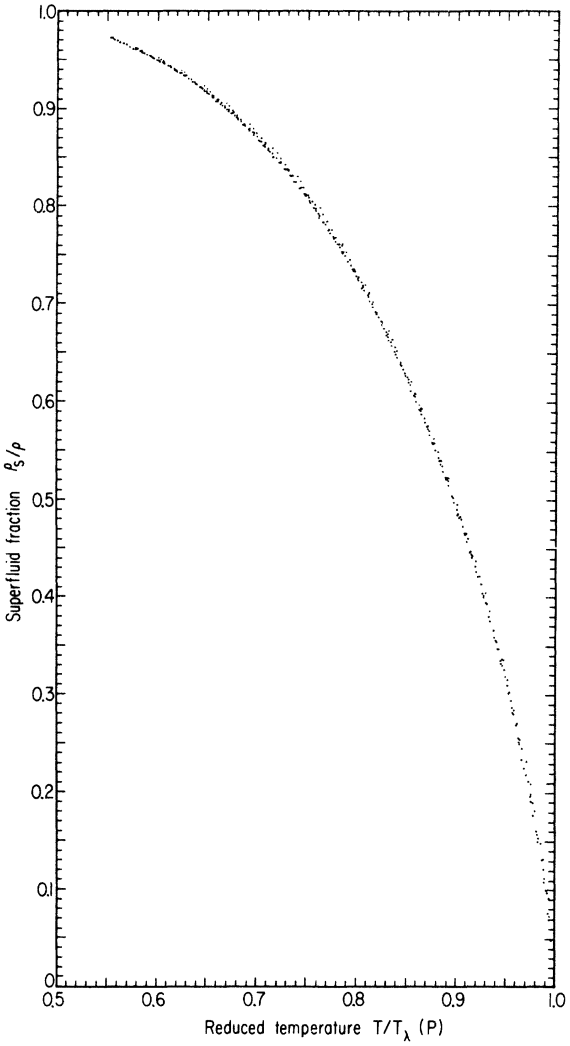

FIG. 9. Plot of ρ_s/ρ data points vs reduced temperature $T/T_\lambda(P)$ for all pressures above SVP. The ~400 data points giving the appearance of a single line are a striking manifestation of the law of corresponding states.

can be integrated to give a power law for S (and C_p):

$$S = \text{const}\, T^{C_p/S}. \tag{37}$$

Except for the lowest temperatures, we have found that the ratio of any two of ρ_n/ρ, S, C_p, and β_p is approximately constant in temperature and only slowly varying in pressure. This is evident in the similarity of the graphs in Figs. 2–6.

ACKNOWLEDGMENT

The very capable assistance of Emily Lin in taking and analyzing the sound-velocity data is gratefully acknowledged.

APPENDIX

In this appendix we present the details of the numerical analysis of the sound-velocity data. The analysis was performed with a computer, and for various programming reasons some of the expressions of Sec. III were given slightly different forms. Here we give the expressions and the numerical values of all coefficients exactly as they appear in the computer program.

The numerical fits for ρ and C_1 are

$$\rho/\rho_\lambda = 1 + (R/\rho_\lambda)\epsilon \ln\epsilon + A_1'\epsilon + A_2'\epsilon^2, \tag{A1}$$

$$\frac{C_1}{C_1^\lambda} = 1 + \frac{C/C_1^\lambda}{B - A\ln\epsilon} + \epsilon\left(\sum_{n=0}^{3} B_n'\epsilon^n\right). \tag{A2}$$

A slight curling (~0.1%) of the polynomial fit for C_1 was corrected using

$$C_1^{\text{final}}/C_1 = 1 + \{(15 + 9.868P - 2.254\,47P^2 + 0.138\,062\,8P^3 - 0.002\,414\,345P^4)\exp[-(12.4 + 0.7P)(T - 1.187)]\} \times 10^{-4}.$$

The quantity ρ_λ was evaluated using the expression from Kierstead:

$$\rho_\lambda = d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4 + d_5e^{d_6x}, \tag{A3}$$

where $x = T_\lambda - 2.172$ K. The λ transition temperature was found by inverting the following expression from Kierstead:

$$P_\lambda = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5e^{b_6x}. \tag{A4}$$

The values of the coefficients d_n and b_n are given in Table VI. All other coefficients in Eqs. (A1) and (A2) are fit with polynomials in P. The coefficients of these polynomials are given in Table VII.

After the first iteration of the program, the value of δ in Eq. (21) changed enough to warrant modifying the fit for ρ by ~0.1%. The final ρ is given by

$$\rho_{\text{final}}/\rho = 1 - (1/\rho_\lambda)[3(1 - e^{-0.2P}) - 1] \times 10^{-4}. \tag{A5}$$

In the Landau-theory calculations we used for the excitation spectrum $E(p)$ a smooth curve determined by the five parameters C_1, $E_{\max}$, Δ, p_0, and μ. The curve was pieced together in four smoothly joining sections.

TABLE VI. Coefficients for Eqs. (A3), (A4), (A20, and (A21). Coefficients b_n and d_n are from Kierstead (Ref. 25).

Coefficient subscript n	Eq. (A3) d_n ($g\,cm^{-3}\,K^{-n}$)	Eq. (A4) b_n (at K^{-n})	Eq. (A20) e_n	Eq. (A21) f_n
0	0.148 413 88	0.428 007 49	1.459 05	−0.309 06
1	−0.150 735	−95.0719	−29.703 46	4.975 19
2	−0.329 822 5	−86.417	201.399 54	−29.9201
3	−0.530 313 33	−103.341	−556.3564	74.517 67
4	−0.383 035	−77.521 75	549.236 93	−66.078 84
5	−0.002 263 88	−0.378 270 65	41.128	−7.235 43
6	36.7348	42.2507	−93.4187	−93.226 07

In the region $0 \leq p \leq 1.1$ Å^{-1},

$$E(p) = C_1 p + ap^2 + bp^3 , \tag{A6}$$

where a and b are determined with the conditions $E(1.1\ \text{Å}^{-1}) = E_{max}$ and $dE/dP(1.1\ \text{Å}^{-1}) = 0$. In the region $1.1\ \text{Å}^{-1} \leq p \leq (p_0 - 0.1\ \text{Å}^{-1})$,

$$E(p) = E_{max} - c(p - 1.1\ \text{Å}^{-1})^2 + d(p - 1.1\ \text{Å}^{-1})^3 , \tag{A7}$$

where c and d are chosen so that E and dE/dp are continuous with the roton curve at $(p_0 - 0.1\ \text{Å}^{-1})$. The roton region is given by $(p_0 - 0.1\ \text{Å}^{-1}) \leq p \leq p_c$, where p_c is the point above p_0 where $dE/dp = C_1$. In the roton region, $E(p)$ is parabolic:

$$E(p) = \Delta + (1/2\mu)(p - p_0)^2 . \tag{A8}$$

For $p \geq p_c$, the curve was continued with a straight line of slope C_1:

$$E(p) = [\Delta + (1/2\mu)(p_c - p_0)^2] + C_1(p - p_c) . \tag{A9}$$

The parameter C_1 is given by Eq. (A1). The parameters p_0 and μ were taken from the neutron scattering measurements of Dietrich *et al.* We used the following analytic approximations:

TABLE VII. Coefficients of the polynomials in P.

	P^0	P^1	P^2	P^3	P^4
R ($g\,cm^{-3}$)	0.016 616	0.4986×10^{-3}	0.1051×10^{-4}		
A_1	$0.264\,56\times10^{-2}$	-0.6758×10^{-2}	$0.689\,06\times10^{-3}$	$-0.318\,95\times10^{-4}$	$0.544\,99\times10^{-6}$
A_2	$-0.715\,78\times10^{-2}$	0.744×10^{-2}	$-0.799\,71\times10^{-3}$	0.3914×10^{-4}	$-0.685\,92\times10^{-6}$
C_1^{λ} ($m\,sec^{-1}$)	216.6	8.998	−0.3118	0.8375×10^{-2}	0.1133×10^{-4}
A ($J\,g^{-1}\,K^{-1}$)	1.275	−0.014 12	0.2409×10^{-3}		
B ($J\,g^{-1}\,K^{-1}$)	3.8896	−0.089 96	0.1125×10^{-2}		
C ($W\,m\,g^{-1}\,K^{-1}$)	14.3	−0.2554	0.0118		
B_0	0.537 51	−0.038 614	$0.249\,94\times10^{-2}$	0.556×10^{-6}	
B_1	−1.7138	0.117 43	$-0.560\,86\times10^{-2}$	$-0.388\,91\times10^{-3}$	
B_2	3.1858	−0.151 19	−0.010 746	$0.228\,74\times10^{-2}$	
B_3	−2.5374	−0.051 365	0.046 553	$-0.386\,92\times10^{-2}$	
Δ_λ (meV)	0.627 99	$-0.619\,18\times10^{-2}$	$0.262\,99\times10^{-3}$	$-0.113\,92\times10^{-4}$	0.1792×10^{-6}
D_{23}	3.354	−0.040 36			
D	1.373 47	0.144 307	$-0.796\,015\,5\times10^{-2}$	$0.237\,418\,78\times10^{-3}$	
D_0	8.204 29	−1.205 109	0.099 416 33	$-0.285\,386\,7\times10^{-2}$	
D_1	−7.725 865	5.690 282	−0.528 854 6	0.014 576 4	
D_2	11.143 69	−13.833 91	1.244 774	0.031 156 16	
D_3	7.355 31	13.395 59	−1.029 535	0.022 269 65	
E_λ (meV)	1.176	−0.012 66	0.1036×10^{-2}	-0.6156×10^{-4}	0.1098×10^{-5}
E_{23}	31.73	−0.1061	−0.011 18		
E_{in}	19.04	-0.272×10^{-2}	-0.9035×10^{-2}		
E_0	28.318 93	1.056 605	−0.085 928 8	$0.123\,011\,94\times10^{-2}$	
E_1	−22.746 175	−6.081 797	0.354 215 83	$0.108\,991\,59\times10^{-2}$	
E_2	38.713 83	14.359 708	0.273 766 4	−0.049 153 267 8	
E_3	−40.056 795	−9.8299	−1.952 864 7	0.103 196 458 9	
μ_λ (He^4 mass)	0.1197	-0.1356×10^{-2}	0.2318×10^{-4}		

$$p_0 = 3.64\rho^{1/3}(\text{Å}^{-1}), \tag{A10}$$

$$\mu/\mu_\lambda = 1 + 0.35[1 - (T/T_\lambda)^7]. \tag{A11}$$

The coefficients of the polynomial fit for μ_λ are given in Table VII.

The parameters $E_{\max}$ and Δ are given by the expressions

$$\frac{\Delta}{kT} = \frac{\Delta_\lambda}{kT_\lambda} + D_{23}\epsilon^{2/3} + D_{1n}\epsilon\ln\epsilon + D\epsilon + \epsilon^2\left(\sum_{n=0}^{3} D_n\epsilon^n\right), \tag{A12}$$

$$\frac{E_{\max}}{kT} = \frac{E_\lambda}{kT_\lambda} + E_{23}\epsilon^{2/3} + E_{1n}\epsilon\ln\epsilon + \epsilon\frac{T_\lambda}{T}\left(\sum_{n=0}^{3} E_n\epsilon^n\right). \tag{A13}$$

The quantities Δ_λ and E_λ are determined by the conditions $S(T_\lambda) = S_\lambda$ and $\rho_n/\rho(T_\lambda) = 1$. The quantity S_λ is found from Ahlers,[12]

$$S_\lambda = 0.011\,513 + 0.492\,13T_\lambda + 0.139\,36(1 - T_\lambda/2.26)^{-0.38}. \tag{A14}$$

The polynomial fits for Δ_λ and E_λ are indicated in Table VII.

The coefficients D_{23}, D_{1n}, E_{23}, and E_{1n} are determined by considering the Landau calculation of S and $\rho_s/\rho = 1 - \rho_n/\rho$ as a function of $y = \Delta/kT$ and $z = E_{\max}/kT$. Then the Landau calculation can be expanded for small ϵ:

$$S - S_\lambda = \left(\frac{\partial S}{\partial y}D_{23} + \frac{\partial S}{\partial z}E_{23}\right)\epsilon^{2/3} + \left(\frac{\partial S}{\partial y}D_{1n} + \frac{\partial S}{\partial z}E_{1n}\right)\epsilon\ln\epsilon, \tag{A15}$$

$$\frac{\rho_s}{\rho} = -\left(\frac{\partial(\rho_n/\rho)}{\partial y}D_{23} + \frac{\partial(\rho_n/\rho)}{\partial z}E_{23}\right)\epsilon^{2/3} - \left(\frac{\partial(\rho_n/\rho)}{\partial y}D_{1n} + \frac{\partial(\rho_n/\rho)}{\partial z}E_{1n}\right)\epsilon\ln\epsilon. \tag{A16}$$

The partial derivatives of S and ρ_n/ρ are evaluated numerically at T_λ. These expressions are equated to the expressions of Ahlers:

$$S - S_\lambda = A\epsilon\ln\epsilon, \tag{A17}$$

$$\rho_s/\rho = k'\epsilon^{2/3} \tag{A18}$$

and the coefficients D_{23}, D_{1n}, E_{23}, and E_{1n} are calculated. The polynomial fits to these coefficients are given in Table VII. The fit for D_{1n} is given by

$$D_{1n} = E_{1n}D_{23}/E_{23} - 1.178 + 0.63\times10^{-3}P + 0.205\times10^{-3}P^2. \tag{A19}$$

The other coefficients in Eqs. (A12) and (A13) are determined from the sound-velocity data. The polynomial fits for these coefficients are given in Table VII. By using only cubic polynomials in P these coefficients could be fit at all pressures except SVP. Consequently small corrections to Δ/kT and $E_{\max}/kT$ were used:

$$\delta(E_{\max}/kT) = (e_0 + e_1\epsilon + e_2\epsilon^2 + e_3\epsilon^3 + e_4\epsilon^4 + e_5e^{e_6\epsilon})e^{-P}, \tag{A20}$$

$$\delta(\Delta/kT) = (f_0 + f_1\epsilon + f_2\epsilon^2 + f_3\epsilon^3 + f_4\epsilon^4 + f_5e^{f_6\epsilon})e^{-P}. \tag{A21}$$

The e_n and f_n coefficients are given in Table VI.

When C_1, $E_{\max}$, Δ, p_0, and μ are specified, $E(p)$ is determined, and S and ρ_n/ρ are calculated with[2,17]

$$S = \frac{4\pi k}{\rho h^3}\int\left(\frac{E/kT}{e^{E/kT}-1} - \ln(1 - e^{-E/kT})\right)p^2\,dp, \tag{A22}$$

$$\frac{\rho_n}{\rho} = \frac{4\pi}{3kT\rho h^3}\int\frac{e^{E/kT}}{(e^{E/kT}-1)^2}p^4\,dp, \tag{A23}$$

where h is Planck's constant. The integrals are evaluated using Simpson's rule with an increment of 0.05 Å^{-1}. The integration was terminated at $p = 3$ Å^{-1} since above this point the integrands were too small to significantly contribute.

In the computer program, derivatives were evaluated by calculating differences due to small changes in T or P (10^{-5} K or 10^{-4} bar, respectively).

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