

Concentration dependence of the ^3He quasiparticle interaction in dilute solutions of ^3He in superfluid ^4He †

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On the basis of an analysis of transport-coefficient data for dilute solutions of ^3He in superfluid ^4He , Kuenhold and Ebner found evidence that the interaction between ^3He quasiparticles depends on the concentration of ^3He . We have estimated the size of concentration-dependent contributions to the effective interaction, assuming the zero-concentration form of the interaction to be known. These contributions vary as $x^{1/3}$, where x is the concentration of ^3He atoms, and are appreciable even for concentrations of a few percent. We find that the theoretical estimates of the spin-diffusion coefficient and thermal conductivity in the low-temperature limit are generally reduced by the concentration-dependent contribution to the interaction. For a simple form of the zero-concentration interaction the experimental data on these transport coefficients for 1.3- and 5-at. % solutions can be fitted significantly better if the concentration-dependent terms are included than if they are neglected.

I. INTRODUCTION

A wide range of properties of dilute solutions of ^3He in superfluid ^4He can be accounted for rather well on the assumption that the interaction between ^3He quasiparticles is independent of concentration.^{1,2} Recently, however, Kuenhold and Ebner³ have shown that the experimental low-temperature thermal conductivity and spin-diffusion coefficient data can be fitted much better if the effective interaction between ^3He quasiparticles is assumed to depend on concentration. From their fits they find that the long-wavelength effective interaction for 5-at. % solutions is 1.3 times the value for the potential that fits best the experimental data for both 1.3- and 5-at. % solutions.

In this paper we investigate theoretically finite-concentration contributions to the effective ^3He quasiparticle interaction. The dimensionless parameter which gives a measure of the magnitude of the concentration-dependent terms is $N(0)V(0)$, where $N(0) = m^* p_F / \pi^2 \hbar^3$ is the density of ^3He quasiparticle states of both spins at the Fermi surface (m^* is the quasiparticle effective mass and p_F is the Fermi momentum) and $V(0)$ is the long-wavelength ^3He quasiparticle interaction. With $V(0)$ given by the Bardeen, Baym, and Pines⁴ expression

$$V_{\text{BBP}}(0) = -\alpha^2 m_4 s^2 / n_4, \quad (1)$$

where α is the relative excess molar volume of an ^3He atom in liquid ^4He , m_4 is the mass of an ^4He atom, s is the velocity of sound in ^4He , and

n_4 is the number density of ^4He atoms, this dimensionless parameter at the vapor pressure is $1.3x^{1/3}$, where x is the fractional concentration of ^3He . Thus, because of the $x^{1/3}$ dependence, this parameter is large even for rather small concentrations: it is ≈ 0.5 for 5-at. % solutions, and for a concentration as small as 1.3 at. % the parameter is 0.3. In retrospect then it is hardly surprising that the effective interaction is found to depend on concentration.

To make quantitative estimates we shall assume the "bare" interaction, that is the zero-concentration form of the effective interaction between quasiparticles to be known and will use perturbation theory to evaluate concentration-dependent contributions. This is described in Sec. II, where we evaluate the effective interaction to second order in the bare quasiparticle interaction, one which we take to be independent of velocity. The calculation of the transport coefficients in the low-temperature limit is described in Sec. III, where we also fit our results to the experimental data. The magnitudes of the various concentration-dependent contributions to the effective interaction are compared. The long-wavelength interaction is found to be enhanced by about 10% for 1.3-at. % solutions and by about 40% for 5-at. % solutions. The results are discussed in Sec. IV.

II. EFFECTIVE INTERACTION

Landau Fermi-liquid theory has proved to be a powerful way of parametrizing the properties of strongly interacting Fermi systems at low temperatures. In this one expands quantities in

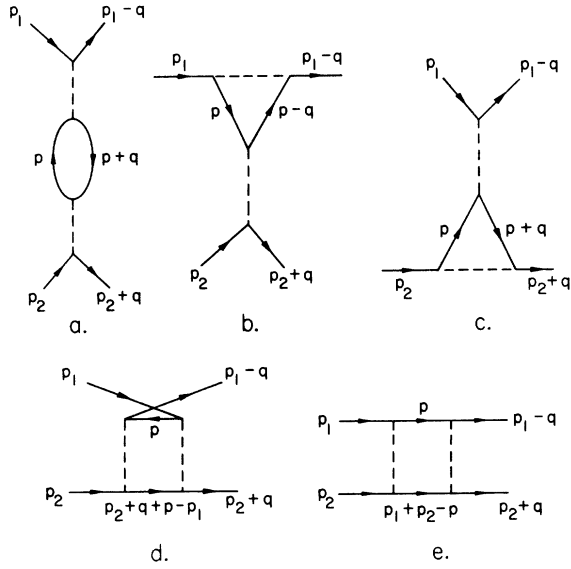


FIG. 1. Contributions to the concentration-dependent interaction of second order in the zero-concentration interaction.

terms of the *deviation* of the quasiparticle distribution function from its value in the ground state. For dilute solutions of liquid ^3He in superfluid ^4He at low temperatures one can use a variant of Landau theory in which one treats as the expansion parameter the ^3He quasiparticle distribution itself, and not just its deviation from the ground-state value. Bardeen, Baym, and Pines¹ have discussed in detail the terms in this expansion up to the second-order term, which takes into account the interaction of a pair of ^3He quasiparticles in pure ^4He . At finite concentrations of ^3He the effective interaction between quasiparticles differs from the zero-concentration interaction as a consequence of the modification of the scattering processes by the presence of the filled Fermi sea of quasiparticles. One class of corrections is represented diagrammatically in Fig. 1. These processes are ones in which the zero-concentration interaction, denoted by the dashed line, acts twice. Figure 1(a) represents a screening of the bare interaction, and Figs. 1(b) and 1(c) represent vertex corrections. Figure 1(d), an exchange correction to the simple screening, is the same as Fig. 1(a), but with exchange interactions in place of the direct interaction. Finally, Fig. 1(e) is the modification, due to the Fermi sea, of the repeated scattering of two quasiparticles. In evaluating these terms one must be careful to remove all contributions which are independent of concentration, since these are (implicitly) included in the bare interaction. What this amounts to in practice is that all contributions which are not

proportional to at least one power of the ^3He quasiparticle distribution $n_{\vec{p}}$ must be discarded. The contributions from all the processes 1(a)–1(e) are of order $N(0)V(0) \sim x^{1/3}$ relative to the bare zero-concentration interaction.

A further class of corrections comes from higher-order terms in the Landau expansion. The simplest of these is the intrinsic three-quasiparticle interaction, which will give rise to a contribution to the effective interaction between pairs of quasiparticles. This term is of order x relative to the bare interaction, and can therefore be neglected relative to those shown in Fig. 1. A further contribution which is of relative order x and hence can be neglected comes from the fact that for the solutions the Landau expansion is usually made keeping μ_4 , the ^4He chemical potential, fixed, whereas experimentally one usually compares results for different concentrations at fixed pressure. Thus the leading finite concentration contributions to the effective interaction between pairs of quasiparticles come from the processes represented in Fig. 1. We further note that for calculating transport coefficients at temperatures well below the Fermi temperature, only scattering of pairs of quasiparticles is important.

For calculating low-temperature transport properties one needs the scattering amplitude only for initial and final quasiparticles very close to the Fermi surface, and therefore in evaluating scattering amplitudes we shall put the momenta of all quasiparticles in initial and final states equal to p_F . Consider first the scattering of two particles with opposite spins. Using the standard rules for evaluating contributions from Feynman graphs, one finds the following expressions for the various contributions to the effective interaction:

$$V_a = 2V_B^2(q) \int \frac{d^3p}{(2\pi\hbar)^3} \frac{n_{\vec{p}+\vec{q}} - n_{\vec{p}}}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}}, \quad (2)$$

$$V_b = -V_B(q) \int \frac{d^3p}{(2\pi\hbar)^3} \frac{n_{\vec{p}} - n_{\vec{p}-\vec{q}}}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}} V_B(|\vec{p}_1 - \vec{p}|), \quad (3)$$

$$V_c = -V_B(q) \int \frac{d^3p}{(2\pi\hbar)^3} \frac{n_{\vec{p}+\vec{q}} - n_{\vec{p}}}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}} V_B(|\vec{p}_2 - \vec{p}|), \quad (4)$$

$$V_d = - \int \frac{d^3p}{(2\pi\hbar)^3} \frac{n_{\vec{p}} - n_{\vec{p}-\vec{k}}}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{k}}} \times V_B(|\vec{p}_1 - \vec{p}|) V_B(|\vec{p} - \vec{p}_1 + \vec{q}|), \quad (5)$$

$$V_e = - \int \frac{d^3p}{(2\pi\hbar)^3} \frac{n_{\vec{p}} + n_{\vec{p}_1 + \vec{p}_2 - \vec{p}}}{\epsilon_{\vec{p}_1} + \epsilon_{\vec{p}_2} - \epsilon_{\vec{p}} - \epsilon_{\vec{p}_1 + \vec{p}_2 - \vec{p}}} \times V_B(|\vec{p}_1 - \vec{p}|) V_B(|-\vec{p}_1 + \vec{p} + \vec{q}|), \quad (6)$$

where $\epsilon_{\vec{p}}$ is the ^3He quasiparticle energy, \vec{q} is the momentum transfer, $\vec{k} \equiv \vec{p}_1 - \vec{p}_2 - \vec{q}$ is the momentum transfer for exchange scattering, and V_B is the bare (zero-concentration) quasiparticle interaction. For simplicity we assume the volume of the system to be unity.

To evaluate the integrals in (3)–(6) one needs an explicit form for the bare quasiparticle interaction. We shall follow most previous workers in assuming that this depends only on the momentum transfer, and is independent of the center-of-mass velocity of the pair of quasiparticles. The term depending on the total momentum must be of order $v_F^2/s^2 \sim x^{2/3}$, where v_F is the Fermi velocity, times the other terms, and therefore it is not important in the low-concentration limit. We further note in passing that one velocity-dependent contribution, that coming from the usual dipolar interaction between quasiparticles due to exchange of longitudinal current fluctuations, must vanish identically for processes in which the initial and final quasiparticles are on the Fermi surface. This follows immediately from the fact that in such a process the energy of a quasiparticle is unchanged, and therefore, from the equation of continuity, creates no longitudinal current.

In the limit $x \rightarrow 0$ one may put the bare interaction equal to $V(0)$, since all momentum transfers then tend to zero. However for the 1.3- and 5-at.-% solutions the q dependence cannot be neglected since the phenomenological interactions for $q \sim 2p_F$ differ appreciably from their values at $q = 0$, even though the q -dependent terms are formally of order $x^{2/3}$ relative to the q -independent ones. To estimate the integrals in (3)–(6) analytically we shall take the bare potential to be of the quadratic form

$$V_B(q) = V_0[1 - 2f(q/2p_F)^2], \quad (7)$$

which, for suitable choices of f , is a rather good approximation to the phenomenological potentials commonly used.^{1,3,4} The choice of f will be discussed in Sec. III. Since approximation (7) is made only in the integrals in the concentration-dependent terms it is unlikely to lead to serious errors. A comparison of this approximate potential with the Bardeen, Baym, and Pines¹ (BBP) potential will be given in Sec. III.

At zero temperature, $n_{\vec{p}}$ is just a step function at the Fermi surface, and the integrations in (3)–(6) may be carried out straightforwardly. The details of the integrations are somewhat lengthy, and are given in the Appendix. We find

$$V_a(\eta) = -N(0)V_B^2(\eta)\chi(\eta), \quad (8)$$

$$V_b(\eta) + V_c(\eta) = N(0)V_B(\eta)V_0\{\chi(\eta) - \frac{1}{4}f[1 + 3\chi(\eta)(1 - \eta^2)]\}, \quad (9)$$

$$V_d(\eta, \theta) = \frac{1}{4}N(0)V_0^2\{-f + f^2(1 - \frac{5}{9}\xi^2 - \frac{1}{3}\eta^2) + \chi(\xi)[2 - 3f(1 - \xi^2) + f^2(1 - \xi^2)(\frac{5}{3}(1 - \xi^2) - \eta^2)]\}, \quad (10)$$

$$V_e(\eta, \theta) = N(0)V_0^2\left\{\frac{1}{2} + f\left(-\frac{4}{3} + \cos^2\frac{\theta}{2}\right) + f^2\left(\frac{46}{45} - \frac{14}{9}\cos^2\frac{\theta}{2} + \frac{2}{3}\cos^4\frac{\theta}{2}\right) + \left(-\frac{1}{4} + \frac{f}{2}\sin^2\frac{\theta}{2} - \frac{f^2}{3}\sin^4\frac{\theta}{2}\right)\sin\frac{\theta}{2}\ln\left(\frac{1 + \sin\frac{1}{2}\theta}{1 - \sin\frac{1}{2}\theta}\right) + f^2\eta^2\left[-\frac{4}{9} + \frac{\cos^2\frac{1}{2}\theta}{3} + \frac{\sin^3\frac{1}{2}\theta}{6}\ln\left(\frac{1 + \sin\frac{1}{2}\theta}{1 - \sin\frac{1}{2}\theta}\right)\right]\right\}. \quad (11)$$

Here $\eta = q/2p_F$, $q = |\vec{p}_1 - \vec{p}_3|$ being the momentum transfer, $\xi = k/2p_F$, $k = |\vec{p}_1 - \vec{p}_4|$ being the momentum transfer in the exchange channel, and θ is the angle between \vec{p}_1 and \vec{p}_2 . \vec{p}_3 and \vec{p}_4 are the momenta of the quasiparticles in the final state. Note that for quasiparticles on the Fermi surface

$$\xi^2 + \eta^2 = \sin^2\frac{1}{2}\theta, \quad (12)$$

and we may therefore express quantities in terms of η and θ alone. $\chi(\eta)$ is the Lindhard function, normalized to unity at long wavelengths, which has the form

$$\chi(\eta) = \frac{1}{2}\left(1 + \frac{1 - \eta^2}{2\eta}\ln\left|\frac{\eta + 1}{\eta - 1}\right|\right). \quad (13)$$

Adding (8)–(11) and the bare interaction one finds for the total effective interaction between quasiparticles of opposite spin to second order in the bare interaction

$$V_{\text{eff}}(\eta, \theta, x) = V_B(\eta)(1 - N(0)V_B(\eta)\chi(\eta) + N(0)V_0\{\chi(\eta) - \frac{1}{4}f[1 + 3\chi(\eta)(1 - \eta^2)]\}) + V_d(\eta, \theta, x) + V_e(\eta, \theta, x). \quad (14)$$

There are a number of points to note about this result. First, V_{eff} is velocity dependent; it depends on the total momentum as well as the momentum transfer, even though the bare potential depends only on the momentum transfer. Second, in the low-concentration limit $V_B(q)$ tends to $V_B(0)$ since all momentum transfers tend to zero, and f tends to zero. V_{eff} is then identical with Abrikosov and Khalatnikov's⁵ result for a dilute Fermi gas of particles of mass m , if one makes the identification

$$V_0 = (4\pi\hbar^2/m)a, \quad (15)$$

where a is the scattering length. Note that in this limit the screening and vertex correction contributions V_a , V_b , and V_c cancel each other identically. Third, V_{eff} diverges logarithmically as θ tends to π , or equivalently the total momentum $|\vec{p}_1 + \vec{p}_2|$ tends to zero. This is of course the same singularity that gives rise to the instability of a normal metal in the presence of a weak attraction between electrons, as in the theory of superconductivity.^{6,7} As far as we are concerned the singularity is not important, since all the averages of V_{eff} we need for calculating transport coefficients converge. It should also be pointed out that for a repulsive interaction the singularity is removed when repeated scattering of two quasiparticles is taken into account, and for an attractive interaction (as in the dilute solutions) repeated scattering will presumably give rise to superfluidity at sufficiently low temperatures. At temperatures close to the superfluid transition temperature the repeated scattering could affect the transport coefficients appreciably, as Emery⁸ has pointed out, but we are concerned here with temperatures large compared with T_c , where the effects of repeated scattering are small.

III. TRANSPORT COEFFICIENTS

The spin-diffusion coefficient D and the thermal conductivity K of dilute solutions of ^3He in ^4He have been measured for concentrations of 1.3 and 5 at.%.^{9,10} In this section we calculate the transport coefficients in terms of V_{eff} and compare experimental and theoretical values. The quasiparticle Boltzmann equation has been solved in the low-temperature limit by Brooker and Sykes¹¹ and by Højgaard Jensen, Smith, and Wilkins,¹² who find

$$K = \frac{1}{3}c_V v_F^2 \tau_K, \quad (16)$$

$$D = \frac{1}{3}(1 + F_0^a) v_F^2 \tau_D, \quad (17)$$

and

$$\eta = \frac{1}{5}n_3 m^* v_F^2 \tau_\eta, \quad (18)$$

where c_V is the specific heat per unit volume,

$1 + F_0^a$ is the susceptibility enhancement factor, and n_3 is the number density of ^3He atoms. The relaxation times τ in (16)–(18) are given by

$$\begin{aligned} \frac{\tau_K}{\tau_0} &= \frac{6}{\pi^2} \sum_{\nu=2,4,6,\dots} \frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\lambda_K]} \\ &= \frac{12-\pi^2}{2\pi^2} + \frac{12}{\pi^2} \lambda_K \sum_{\nu=2,4,6,\dots} \frac{2\nu+1}{\nu^2(\nu+1)^2} \\ &\quad \times \frac{1}{\nu(\nu+1)-2\lambda_K}, \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{\tau_D}{\tau_0} &= \frac{2}{\pi^2} \sum_{\nu=1,3,5,\dots} \frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\lambda_D]} \\ &= \frac{1}{6} + \frac{4}{\pi^2} \lambda_D \sum_{\nu=1,3,5,\dots} \frac{2\nu+1}{\nu^2(\nu+1)^2} \\ &\quad \times \frac{1}{\nu(\nu+1)-2\lambda_D}, \end{aligned} \quad (20)$$

and

$$\begin{aligned} \frac{\tau_\eta}{\tau_0} &= \frac{2}{\pi^2} \sum_{\nu=1,3,5,\dots} \frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\lambda_\eta]} \\ &= \frac{1}{6} + \frac{4}{\pi^2} \lambda_\eta \sum_{\nu=1,3,5,\dots} \frac{2\nu+1}{\nu^2(\nu+1)^2} \\ &\quad \times \frac{1}{\nu(\nu+1)-2\lambda_\eta}, \end{aligned} \quad (21)$$

with the characteristic time given by

$$\tau_0 \equiv \frac{8\pi^4\hbar^6}{m^* \langle W(\theta, \phi, x) / \cos \frac{1}{2}\theta \rangle k_B^2 T^2}. \quad (22)$$

The symbol $\langle \dots \rangle$ denotes an angular average; θ is the angle between \vec{p}_1 and \vec{p}_2 , ϕ is the angle by which the plane containing the final quasiparticle momenta $\vec{p}_3 (= \vec{p}_1 - \vec{q})$ and $\vec{p}_4 (= \vec{p}_2 + \vec{q})$ is rotated relative to the plane containing the initial momenta \vec{p}_1 and \vec{p}_2 , and k_B is Boltzmann's constant.

The effective interaction for quasiparticles of opposite spin is just $V_{\text{eff}}(\eta, \theta, x)$, and for quasiparticles of the same spin both direct and exchange terms must be included, so the effective interaction is $V_{\text{eff}}(\eta, \theta, x) - V_{\text{eff}}(\xi, \theta, x)$. Thus the corresponding transition probabilities are

$$W_{\uparrow\downarrow}(\theta, \phi, x) = (2\pi/\hbar) |V_{\text{eff}}(\eta, \theta, x)|^2, \quad (23)$$

$$W_{\uparrow\uparrow}(\theta, \phi, x) = (2\pi/\hbar) |V_{\text{eff}}(\eta, \theta, x) - V_{\text{eff}}(\xi, \theta, x)|^2. \quad (24)$$

The usual spin-averaged transition probability is

$$W(\theta, \phi, x) = \frac{1}{2} W_{\uparrow\downarrow}(\theta, \phi, x) + \frac{1}{4} W_{\uparrow\uparrow}(\theta, \phi, x). \quad (25)$$

The parameters λ_K , λ_D , and λ_η , are defined as follows:

$$\lambda_K = 1 + 2 \frac{\langle W(\theta, \phi, x) \cos \theta / \cos \frac{1}{2} \theta \rangle}{\langle W(\theta, \phi, x) / \cos \frac{1}{2} \theta \rangle}, \quad (26)$$

$$\lambda_D = 1 - \frac{\langle W_{\uparrow\uparrow}(\theta, \phi, x) \sin^2 \frac{1}{2} \theta (1 - \cos \phi) / \cos \frac{1}{2} \theta \rangle}{\langle W(\theta, \phi, x) / \cos \frac{1}{2} \theta \rangle}, \quad (27)$$

and

$$\lambda_\eta = 1 - 3 \frac{\langle W(\theta, \phi, x) \sin^4 \frac{1}{2} \theta \sin^2 \phi / \cos \frac{1}{2} \theta \rangle}{\langle W(\theta, \phi, x) / \cos \frac{1}{2} \theta \rangle}. \quad (28)$$

The purpose of this paper is to explore the size of the concentration-dependent contributions to the effective interaction, so we use the simple two-parameter form of the bare interaction suggested by BBP,

$$V_B(q) = V_0 \cos(\beta q / \hbar). \quad (29)$$

More-complicated expressions for the potential have been used, such as polynomials,^{3,13} but here we choose a simple form since our aim is to investigate concentration-dependent effects rather than to obtain the best possible fit to the data. Nevertheless, as we shall see, even with the two-parameter bare potential one can, allowing for concentration dependence, fit the transport-coefficient data much better than with a concentration-independent potential with a larger number of adjustable parameters.¹³

In evaluating the concentration-dependent terms in Sec. II we assumed that in integrals the bare interaction may be replaced by the quadratic form (7). For low concentrations, clearly the appropriate choice for f is $\beta^2 p_F^2 / \hbar^2$, which gives correctly the q^2 term in the BBP potential. However for concentrations of 1.3 or 5 at.% this is a poor approximation, as one can see by inspecting Fig. 2. There we show the BBP potential and the quadratic form which agrees with it for $q \rightarrow 0$. The values of $2p_F$, the maximum momentum transfer in scattering processes at low temperatures, are indicated. The range parameter β is taken to be 3.16 \AA , the value which gives the best fit to the transport coefficient data (see Table I, column C). A much better fit to the BBP potential is obtained over the range of momentum transfers of interest if the quadratic form passes through zero for the same momentum transfer as the BBP potential. This choice is equivalent to taking

$$f = (8/\pi^2) \beta^2 p_F^2 / \hbar^2, \quad (30)$$

and a plot of the quadratic form with this value of f is also shown in Fig. 2. In our calculations we have used this value for f . Throughout we have assumed $V_0 = V_B(0)$.

The spin diffusion coefficient, thermal conductivity and viscosity for 1.3- and 5-at.% solu-

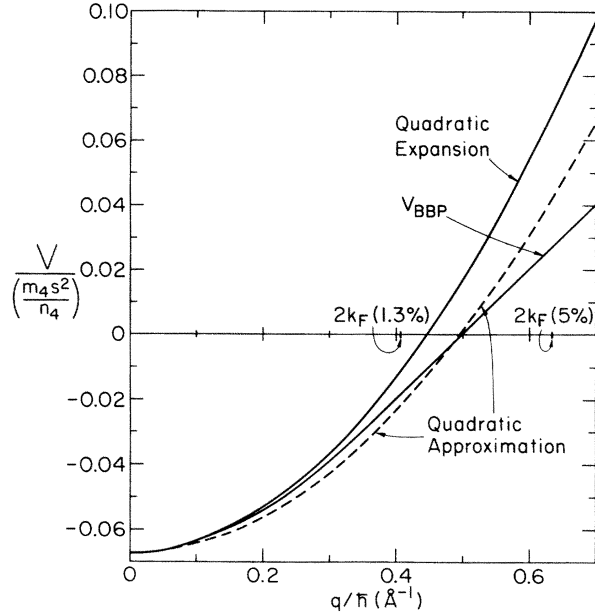


FIG. 2. Comparison between the BBP potential, the quadratic expansion of V_{BBP} about $q=0$, and the quadratic approximation [see Eqs. (7) and (30)]. The parameters chosen are $V_0 = -0.067 m_4 s^2 / n_4$ and $\beta = 3.16 \text{ \AA}$.

tions have been calculated from Eqs. (16)–(28) for various values of V_0 and β . The integrals over the angles θ and ϕ were carried out numerically. For m^* we used the experimentally measured values at the two concentrations.⁹ In the calculations reported below we put $V_B(\eta)$ in the expressions for V_a , V_b , and V_c [Eqs. (8) and (9)] equal to the full BBP potential. One might argue that for consistency one should replace V_B by the quadratic approximation used in calculating the integrals; however, the two prescriptions give results for the transport coefficients which differ from each other by less than or of the order of 1%, and thus the difference is unimportant.

First to see how important the concentration-dependent terms are we have calculated the transport coefficients using the BBP potential that fits the data best when the concentration-dependent terms are neglected. To determine the best values of V_0 and β one should, in principle, minimize χ^2 . However, since the errors on the experimental values are not well determined, and the errors on the various measurements are probably not too different from each other we determined the parameters by minimizing the mean-square deviation of the theoretical values of D and K for 1.3- and 5-at.% solutions from the experimental values. In the set of results in part A of Table I we show the results for this potential. The first column of part A shows the values of the transport coefficients calculated

TABLE I. Transport coefficients of the dilute solutions. T is in kelvin, K is in $\text{erg cm}^{-1} \text{sec}^{-1} \text{K}^{-1}$, D is in units of $10^{-6} \text{ cm}^2 \text{sec}^{-1}$, and η is in micropoise. The set of results marked A is for the potential which gives the best fit if concentration-dependent terms are neglected. The column labeled $|V_{\text{eff}}|^2$ gives results calculated using for W the square of V_{eff} , and the column marked "Second and third order" gives results using only the contributions to W which are second and third order in V_B (see text). The set of results B are for the interaction which gives the best fit to the experimental data when W is taken to be proportional to $|V_{\text{eff}}|^2$ and the set of results C are for the interaction that gives the best fit using only 2nd and 3rd order terms. Note, in particular, the large effect the concentration-dependent terms have on the thermal conductivity and viscosity of the 5-at.% solutions. The experimental value for the viscosity of the 5-at.% solutions is taken from Ref. 14 (see text). It was not included in the fitting of theoretical results to experiment.

	A. $V_0 = -0.072 m_4 s^2 / n_4$ $\beta = 3.3 \text{ \AA}$			B. $V_0 = -0.066 m_4 s^2 / n_4$ $\beta = 3.11 \text{ \AA}$			C. $V_0 = -0.067 m_4 s^2 / n_4$ $\beta = 3.16 \text{ \AA}$		
	Expt.	$ V_{\text{eff}} ^2$	Second and third order	$ V_B ^2$	$ V_{\text{eff}} ^2$	Second and third order	$ V_B ^2$	$ V_{\text{eff}} ^2$	Second and third order
$KT(1.3 \text{ at.}\%)$	11.0	9.0	9.1	10.2	10.7	10.9	11.9	10.4	10.5
$KT(5 \text{ at.}\%)$	24.0	19.0	19.9	28.1	25.3	26.3	35.9	24.0	24.9
$DT^2(1.3 \text{ at.}\%)$	17.2	17.1	17.4	16.9	18.0	18.2	17.6	18.0	18.3
$DT^2(5 \text{ at.}\%)$	90.0	69.7	74.9	72.0	85.7	89.4	93.5	83.0	87.2
$\eta T^2(1.3 \text{ at.}\%)$...	0.028	0.0287	0.0317	0.0324	0.0328	0.0356	0.0317	0.0322
$\eta T^2(5 \text{ at.}\%)$	0.28	0.229	0.237	0.303	0.288	0.295	0.374	0.276	0.284
rms deviation	13.6%	4.52%	4.48%

using V_{eff} calculated to second order in V_B , while the third column of part A shows the results for the same V_B , neglecting concentration-dependent corrections. The difference between the spin-diffusion coefficients calculated with and without concentration-dependent terms is less than 5% for both concentrations. However, the thermal conductivity changes by 11% for a 1.3-at.% solution and by 32% for a 5-at.% solution. The large change in the thermal conductivity shows the importance of the concentration-dependent terms.

In calculating the results in the first column of part A, marked $|V_{\text{eff}}|^2$, we have retained all contributions to V_{eff} of second order in V_B . This is equivalent to retaining in W all terms of second and third order in V_B and some, but not all, of the fourth-order terms. To calculate all the fourth-order terms in W one would have to calculate V_{eff} to third order in V_B . Some idea of the importance of the fourth-order terms in W may be found by comparing the results given above, which include some of the fourth-order terms, with those obtained using only the second and third-order contributions to W . The latter are given in the second column of part A. As may be seen from Table I, the fourth-order terms give a very small contribution, never greater than 5%, which gives one confidence in the neglect of higher-order terms.

Next one may ask if the inclusion of the concentration-dependent terms enables one to fit the experimental data better. In part B of Table I we show results for the potential which gives the best fit if one calculates W using the square

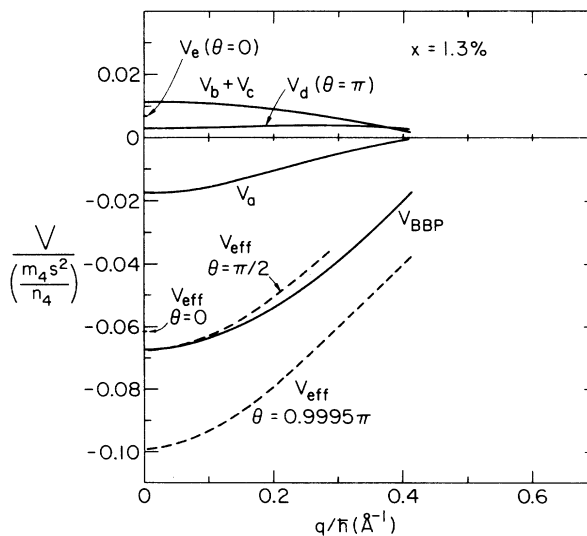


FIG. 3. V_{eff} and V_{BBP} for a 1.3-at.% solution for $V_0 = -0.67 m_4 s^2 / n_4$ and $\beta = 3.16 \text{ \AA}$. The contributions from various second-order terms are also shown.

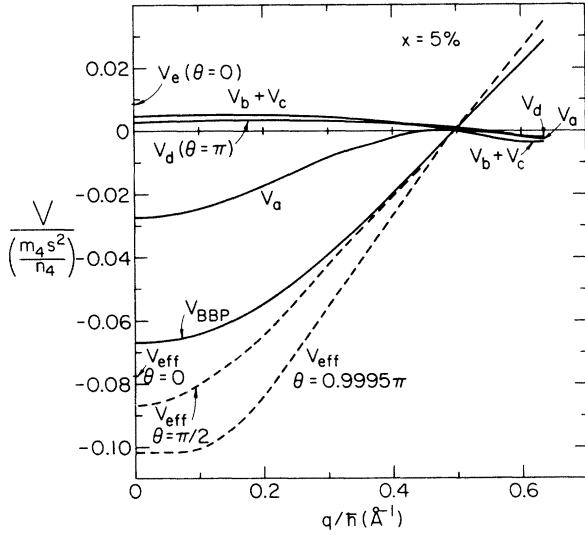


FIG. 4. V_{eff} and V_{BBP} for a 5-at.% solution for $V_0 = -0.067m_4s^2/n_4$ and $\beta = 3.16 \text{ \AA}$. The contributions from various second-order terms are also shown.

of V_{eff} , and in part C are the results for the potential which gives the best fit if one includes only the second- and third-order terms in W . First one notes that the concentration-dependent terms are just as important as for the potential considered in part A. Second, the rms deviation between theory and experiment is significantly better than in part A. By way of comparison we remark that for Ebner's¹³ best concentration-independent potential, which has five parameters, the rms deviation between theory and experiment is 11.3%.

The viscosity of dilute solutions has been measured by Kuenhold, Crum, and Sarwinski.¹⁴ The data do not define a T^{-2} dependence, but by fitting the data assuming the relaxation time to have T^{-2} and constant terms they find $\eta T^2 = 0.28 \mu\text{P K}^2$ at low temperatures. This agrees quite well with the values calculated using the concentration-dependent interaction.

We note that the quadratic approximation used, with f given by (30), agrees quite well with V_{BBP} for small q , but for 5-at.% solutions the difference between these two potentials when $q = 2p_F$ is about

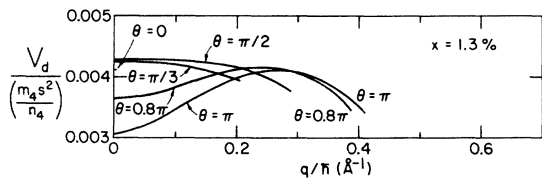


FIG. 5. V_d vs q/\hbar for a 1.3-at.% solution for various angles θ between the two incoming quasiparticles. The parameters used are $V_0 = -0.067m_4s^2/n_4$ and $\beta = 3.16 \text{ \AA}$.

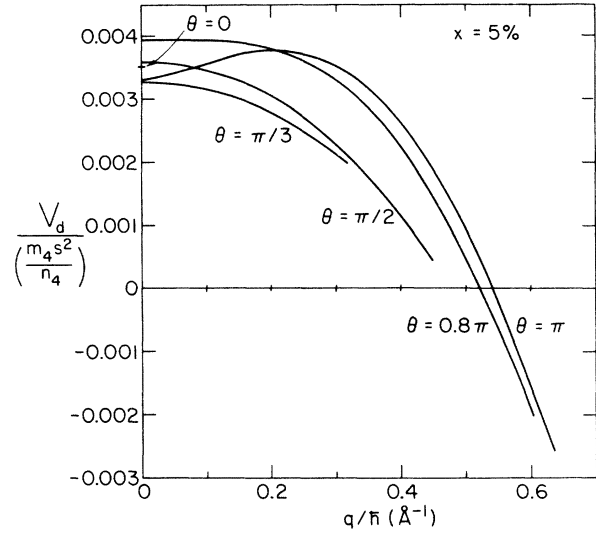


FIG. 6. V_d vs q/\hbar for a 5-at.% solution for various angles θ between the two incoming quasiparticles. The parameters used are $V_0 = -0.067m_4s^2/n_4$ and $\beta = 3.16 \text{ \AA}$.

30%. Other choices of f which give quadratic forms which are reasonable approximations to the BBP potential in the range of momentum transfers of interest here give rise to changes in the transport coefficients of the order of a few percent at most.

The rms deviation between theory and experiment doubles if β is changed from its best value by 6% or V_0 is changed from its best value by 4%. The change in the rms deviation produced by changing both V_0 and β simultaneously by 3% is of a comparable size. For future reference, we quote the values of λ obtained from (26)–(28) using V_{eff} as in Table I, column B. For a 1.3-at.% solution, $\lambda_D = 0.36$, $\lambda_K = 0.33$, and $\lambda_\eta = 0.33$; and for a 5-at.% solution $\lambda_D = 0.81$, $\lambda_K = 0.30$, and $\lambda_\eta = 0.58$. If concentration-dependent terms are ignored, $\lambda_D = 0.32$, $\lambda_K = 0.52$, and $\lambda_\eta = 0.37$ for a 1.3-at.% solution, while $\lambda_D = 0.75$, $\lambda_K = 0.32$, and $\lambda_\eta = 0.54$

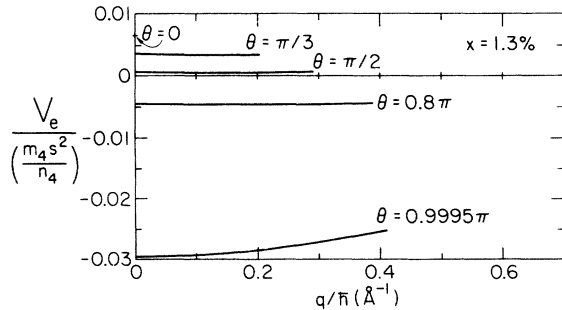


FIG. 7. V_e vs q/\hbar for a 1.3-at.% solution for various angles θ between the two incoming quasiparticles. The parameters used are $V_0 = -0.067m_4s^2/n_4$ and $\beta = 3.16 \text{ \AA}$.

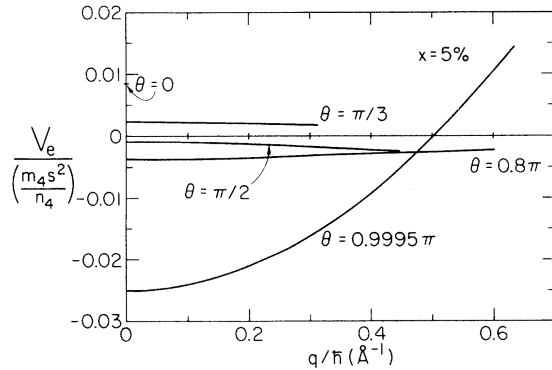


FIG. 8. V_e vs q/\hbar for a 5-at.% solution for various angles θ between the incoming quasiparticles. The parameters used are $V_0 = -0.067 m_4 s^2 / n_4$ and $\beta = 3.16 \text{ \AA}$.

for a 5-at.% solution.

It is interesting to compare V_{eff} and V_B . This is done in Figs. 3 and 4, where we show these potentials for concentrations of 1.3 and 5 at.% as a function of momentum transfer, and for various angles θ . For $q=0$ and $\theta=0$, V_{eff}/V_B is 0.92 for 1.3-at.% solutions and 1.15 for 5-at.% solutions. The large concentration-dependent term for $\theta=0.9995\pi$ is due to the proximity of the singularity of V_e at $\theta=\pi$ referred to earlier. The contributions from various terms in V_{eff} are also shown in the same figures for comparison. V_d is always negative, as may be seen from (2). The sign of contributions to V_b and V_c depends on the relative signs of $V_B(q)$ and $V_B(|\vec{p}_1 - \vec{p}|)$. For 1.3-at.% solutions and for the values of f of interest here V_B is always negative for momentum transfers less than $2p_F$, and therefore V_b and V_c are positive. For 5-at.% solutions V_b and V_c are positive for small q , but negative for larger q . V_d is positive for $q=0$, as is obvious from (5), and for the 1.3-at.% solution is always positive, again because for the potentials we use $V_B(q)$ is negative for $q \leq 2p_F$. V_d can be negative for a 5% solution at large q . Both V_d and V_e depend not only on the momentum transfer but also on the angle θ . We plot these two contributions to V_{eff} in Figs. 5–8 for the two concentrations. V_d is relatively small for all angles. For $q=0$, V_e is positive for small values of θ but becomes negative for large values of θ . V_e decreases by a substantial amount as θ approaches π due to the singularity mentioned before.

IV. DISCUSSION

The main conclusion of our work is that the concentration-dependent contributions to the effective

interaction are large, and must be taken into account in making a detailed comparison between theory and experiment. A second point is that the low-temperature transport-coefficient data can be accounted for better if one takes into account the concentration dependence of the effective interaction than if one neglects it. However one must bear in mind that the uncertainties in the experimental measurements of D and K are probably comparable with the rms deviations between the experimental values and the theoretical ones calculated using the BBP potentials that give the best fits. Consequently, the improvement of the fit when concentration-dependent contributions are taken into account is a weaker argument for the importance of these contributions than is the direct estimation of their magnitude.

It is interesting to compare the values we obtain for $V(0)$ by fitting the transport coefficient data, with the theoretically calculated values. Baym¹⁵ has recently calculated the correction to the BBP result (1) to second order in the ^3He - ^4He mass difference, and finds

$$V(0) = -(\alpha^2 + \Gamma)m_4 s^2 / n_4, \quad (31)$$

where the correction term Γ is estimated to be ≈ 0.01 at zero pressure. Since experimentally $\alpha = 0.285$,¹⁶ one expects $V(0) \approx -0.091 (m_4 s^2 / n_4)$. This is considerably more attractive than the values we find by fitting transport coefficient data ($V(0) \approx -0.066 m_4 s^2 / n_4$). The reason for the discrepancy between the $V(0)$ values is unclear. Conceivably it could be due to the simple way in which we have parametrized the bare potential. One should note, however, that since small momentum-transfer processes are relatively ineffective in degrading fluxes, transport coefficient measurements are not particularly good for pinning down $V(0)$. We did carry out a fit to the transport coefficient data, forcing $V(0)$ to be $-0.091 m_4 s^2 / n_4$. The best fit was obtained for $\beta = 3.27 \text{ \AA}$, and the rms deviation between theory and experiment was very large, about 50%. More experimental measurements and more detailed calculations should be helpful in resolving this discrepancy.

The concentration dependence of the effective interaction will affect other properties of the dilute solutions, such as the osmotic pressure and the superfluid transition temperatures. These effects are now being estimated, and results will be reported elsewhere.

ACKNOWLEDGMENT

We are grateful to Professor Gordon Baym for a number of helpful conversations.

APPENDIX

To evaluate the second-order terms, we rewrite (2)–(6) in the following form by changing the variable of integration and using the step-function property of $n_{\vec{p}}$:

$$V_a = 2V_B^2(q) \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{1}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}} - \frac{1}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}} \right), \quad (\text{A1})$$

$$V_b = -V_B(q) \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{V_B(|\vec{p}_1 - \vec{p}|)}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}} - \frac{V_B(|\vec{p}_1 - \vec{p} - \vec{q}|)}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}} \right), \quad (\text{A2})$$

$$V_c = -V_B(q) \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{V_B(|\vec{p}_2 - \vec{p} + \vec{q}|)}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}} - \frac{V_B(|\vec{p}_2 - \vec{p}|)}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}} \right), \quad (\text{A3})$$

$$V_d = - \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{V_B(|\vec{p}_1 - \vec{p}|)V_B(|\vec{p} - \vec{p}_2 - \vec{k}|)}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{k}}} - \frac{V_B(|\vec{p} - \vec{p}_2|)V_B(|\vec{p}_1 - \vec{p} - \vec{k}|)}{\epsilon_{\vec{p}+\vec{k}} - \epsilon_{\vec{p}}} \right), \quad (\text{A4})$$

$$V_e = - \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{V_B(|\vec{p}_1 - \vec{p}|)V_B(|\vec{p} + \vec{q} - \vec{p}_1|) + V_B(|\vec{p} - \vec{p}_2|)V_B(|\vec{p}_2 - \vec{p} + \vec{q}|)}{\epsilon_{\vec{p}_1} + \epsilon_{\vec{p}_2} - \epsilon_{\vec{p}} - \epsilon_{\vec{p}_1 + \vec{p}_2 - \vec{p}}} \right), \quad (\text{A5})$$

where $\epsilon_{\vec{p}} = p^2/2m^*$. The integral in (A1) is very easy to calculate and is proportional to the Lindhard function defined in (13). Therefore, we have $V_a(\eta) = -N(0)V_B^2(\eta)\chi(\eta)$, where $\eta \equiv q/2p_F$ is a dimensionless momentum-transfer variable.

The other four integrals involve $V_B(y)$ or $V_B^2(y)$ in the integrands. In order to obtain analytical forms for these integrals, we substitute the quadratic form $V_0(1 - 2fy^2)$ for $V_B(y)$ wherever it appears in the integrands of (A2)–(A5), as discussed in the text.

In the vertex correction terms V_b and V_c , we separate the integrands into two parts, the first of which involves $1/(\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}) - 1/(\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}})$, which is an even function of $\hat{p} \cdot \hat{q}$, and the second of which is proportional to $1/(\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}) + 1/(\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}})$, which is an odd function of $\hat{p} \cdot \hat{q}$. Using the symmetry properties and the fact that $\vec{p}_1 \cdot \vec{q} = -\vec{p}_2 \cdot \vec{q} = \frac{1}{2}q^2$ for ^3He quasiparticles on the Fermi surface, the integrals may be simplified and we find

$$V_b = V_c = -V_B(q) \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} V_0 \left(1 - \frac{f}{2p_F^2} (p^2 + p_F^2) \right) \left(\frac{1}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}} - \frac{1}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}} \right) \\ - V_B(q) \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \frac{V_0 f}{2p_F^2} (\vec{p} \cdot \vec{q}) \left(\frac{1}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{q}}} + \frac{1}{\epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}}} \right). \quad (\text{A6})$$

That V_b and V_c are equal follows from the fact that all quasiparticles in initial and final states are on the Fermi surface. The angular part of the integral is easily performed if one chooses the \hat{z} axis to be along \vec{q} . We are then left with only a single integral over the magnitude of \vec{p} :

$$V_b = V_c = \frac{V_B(\eta)N(0)V_0}{4\eta} \int_0^1 d\rho \left(1 - \frac{f}{2}(1 + \rho^2) \right) \rho \ln \left| \frac{\eta + \rho}{\eta - \rho} \right| + \frac{V_B(\eta)N(0)V_0 f}{4} \int_0^1 d\rho \left(-2\rho^2 + \rho\eta \ln \left| \frac{\eta + \rho}{\eta - \rho} \right| \right), \quad (\text{A7})$$

where the dimensionless variable ρ is p/p_F .

To evaluate V_d , we follow the same procedure as we used in calculating the vertex correction terms, since the denominator in (A4) has the same form as the denominator of the previously calculated integrals, except that the momentum transfer \vec{q} is replaced by \vec{k} , the momentum transfer in the exchange channel. Again we decompose the energy denominators in V_d into parts which are even and odd functions of $\hat{p} \cdot \hat{k}$. Then we use the symmetry properties of the integrands and the facts that $\vec{q} \cdot \vec{k} = 0$ and $\vec{p}_1 \cdot \vec{k} = -\vec{p}_2 \cdot \vec{k} = \frac{1}{2}k^2$ for ^3He quasiparticles on the Fermi surface. We obtain

$$V_d = -V_0^2 \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(1 - \frac{f}{p_F^2} (p_F^2 + p^2) + \frac{f^2}{4p_F^4} [(p^2 + p_F^2)^2 + 2(\vec{k} \cdot \vec{p})^2 + 4(\vec{p} \cdot \vec{p}_1)(\vec{p} \cdot \vec{p}_2)] \right) \\ \times \left(\frac{1}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{k}}} - \frac{1}{\epsilon_{\vec{p}+\vec{k}} - \epsilon_{\vec{p}}} \right) \\ + V_0^2 \int_{|\vec{p}| < p_F} \frac{d^3p}{(2\pi\hbar)^3} \left(-\frac{f}{2p_F^2} + \frac{f^2}{4p_F^4} (p^2 + p_F^2) \right) (2\vec{p} \cdot \vec{k}) \left(\frac{1}{\epsilon_{\vec{p}} - \epsilon_{\vec{p}-\vec{k}}} + \frac{1}{\epsilon_{\vec{p}+\vec{k}} - \epsilon_{\vec{p}}} \right). \quad (\text{A8})$$

The angular part of the integral is performed by choosing the \hat{z} axis to lie along \vec{k} and the radial part is expressed in terms of the dimensionless momentum variable $\rho = p/p_F$. The result is

$$\begin{aligned}
 V_d = & \frac{N(0)V_0^2}{4\xi} \int_0^1 d\rho \left(1 - f(1 + \rho^2) + \frac{f^2}{4}(1 + \rho^2)^2 \right) \rho \ln \left| \frac{\xi + \rho}{\xi - \rho} \right| \\
 & + \frac{N(0)V_0^2 f^2}{8} \int_0^1 d\rho \left[\left(2\xi^2 + \eta^2 - \cos^2 \frac{\theta}{2} \right) \left(-2\rho^2 + \xi \rho \ln \left| \frac{\xi + \rho}{\xi - \rho} \right| \right) + \frac{\cos^2 \frac{1}{2}\theta - \eta^2}{\xi} \rho^3 \ln \left| \frac{\xi + \rho}{\xi - \rho} \right| \right] \\
 & + \frac{N(0)V_0^2}{2} \int_0^1 d\rho \left(-f + \frac{f^2}{2}(1 + \rho^2) \right) \left(2\rho^2 - \xi \rho \ln \left| \frac{\xi + \rho}{\xi - \rho} \right| \right). \tag{A9}
 \end{aligned}$$

The integrals over ρ in Eqs. (A7) and (A9) for V_b , V_c , and V_d are of the form

$$\int_0^1 d\rho \rho^{2n+1} \ln \left| \frac{a + \rho}{a - \rho} \right| = \frac{1}{n+1} \left[\frac{a}{2n+1} + \frac{a^3}{2n-1} + \dots + a^{2n+1} + \frac{1}{2}(1 - a^{2n+2}) \ln \left(\frac{1+a}{1-a} \right) \right]. \tag{A10}$$

Substituting (A10) into (A7) and (A9) with appropriate values of n and a , we obtain the expressions shown in (9) and (10).

The denominator in the expression for V_e has a different form from all the others. It is convenient to calculate the integral by working in terms of the total momentum of the incoming pair of quasiparticles $\vec{P} = \vec{p}_1 + \vec{p}_2$. The integral can be simplified by again using the fact that $\vec{p}_1 \cdot \vec{q} = -\vec{p}_2 \cdot \vec{q} = \frac{1}{2}q^2$ for ^3He quasiparticles on the Fermi surface, and is given by

$$\begin{aligned}
 V_e = & -\frac{N(0)V_0^2}{4\pi p_F} \int_{|\vec{p}| < p_F} d^3p \frac{1}{2p_F^2 - P^2 - 2\vec{p} \cdot \vec{P}} \\
 & \times \left(2 - \frac{2f}{p_F^2} (p_F^2 + p^2 - \vec{p} \cdot \vec{P}) + \frac{f^2}{2p_F^4} \{ (p^2 + p_F^2)^2 - 2(p^2 + p_F^2)(\vec{p} \cdot \vec{P}) + 2(\vec{p} \cdot \vec{p}_1)^2 + 2(\vec{p} \cdot \vec{p}_2)^2 \right. \\
 & \left. - 2(\vec{p} \cdot \vec{q})(\vec{p} \cdot (\vec{p}_1 - \vec{p}_2)) \right\}. \tag{A11}
 \end{aligned}$$

After one chooses \vec{P} as the \hat{z} axis the angular integration is lengthy but straightforward and we are left with

$$\begin{aligned}
 V_e = & \frac{N(0)V_0^2}{4 \cos \frac{1}{2}\theta} \int_0^1 d\rho \left(1 - f(1 + \rho^2) + \frac{f^2}{4}(1 + \rho^2)^2 \right) \rho \ln \left| \frac{\rho^2 + 2\rho \cos \frac{1}{2}\theta + \cos \theta}{\rho^2 - 2\rho \cos \frac{1}{2}\theta + \cos \theta} \right| \\
 & + N(0)V_0^2 \int_0^1 d\rho \left(f - \frac{f^2}{2}(1 + \rho^2) \right) \left(-\rho^2 + \frac{\rho^2 + \cos \theta}{4 \cos \frac{1}{2}\theta} \rho \ln \left| \frac{\rho^2 + 2\rho \cos \frac{1}{2}\theta + \cos \theta}{\rho^2 - 2\rho \cos \frac{1}{2}\theta + \cos \theta} \right| \right) \\
 & + \frac{N(0)V_0^2 f^2}{8 \cos^2 \frac{1}{2}\theta} (\sin^2 \frac{1}{2}\theta - 2\eta^2) \int_0^1 d\rho \rho^3 \ln \left| \frac{\rho^2 + 2\rho \cos \frac{1}{2}\theta + \cos \theta}{\rho^2 - 2\rho \cos \frac{1}{2}\theta + \cos \theta} \right| \\
 & + \frac{N(0)V_0^2 f^2}{8 \cos^2 \frac{1}{2}\theta} \left(2 \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} + 2\eta^2 \right) \int_0^1 d\rho \left(-\rho^2(\rho^2 + \cos \theta) + \frac{(\rho^2 + \cos \theta)^2}{4 \cos \frac{1}{2}\theta} \rho \ln \left| \frac{\rho^2 + 2\rho \cos \frac{1}{2}\theta + \cos \theta}{\rho^2 - 2\rho \cos \frac{1}{2}\theta + \cos \theta} \right| \right), \tag{A12}
 \end{aligned}$$

which contains integrals of the form

$$I_n = \int_0^1 d\rho \rho^{2n+1} \ln \left| \frac{\rho^2 + 2\rho \cos \frac{1}{2}\theta + \cos \theta}{\rho^2 - 2\rho \cos \frac{1}{2}\theta + \cos \theta} \right|.$$

The I_n we require here are

$$I_0 = 2 \cos \frac{\theta}{2} \left(1 - \frac{\sin \frac{1}{2}\theta}{2} \ln \frac{1 + \sin \frac{1}{2}\theta}{1 - \sin \frac{1}{2}\theta} \right), \tag{A13}$$

$$I_1 = 2 \cos \frac{\theta}{2} \left(\frac{5}{3} - \cos^2 \frac{\theta}{2} - \frac{\sin \frac{1}{2}\theta}{2} \ln \frac{1 + \sin \frac{1}{2}\theta}{1 - \sin \frac{1}{2}\theta} - \frac{\cos \frac{1}{2}\theta \sin^2 \frac{1}{2}\theta}{2} \ln \frac{1 + \cos \frac{1}{2}\theta}{1 - \cos \frac{1}{2}\theta} \right), \tag{A14}$$

and

$$\begin{aligned}
 I_2 = & 2 \cos \frac{\theta}{2} \left(\frac{31}{15} - \frac{2}{9} \cos^2 \frac{\theta}{2} - \frac{4}{3} \cos^4 \frac{\theta}{2} - \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2} \ln \frac{1 + \cos \frac{1}{2}\theta}{1 - \cos \frac{1}{2}\theta} \right. \\
 & \left. - \left(\frac{1}{2} - \frac{2}{3} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} \ln \frac{1 + \sin \frac{1}{2}\theta}{1 - \sin \frac{1}{2}\theta} \right). \tag{A15}
 \end{aligned}$$

With the help of (A13)–(A15), we obtain the final form for V_e shown in (11).

Note added in proof. The viscosity of 5-at.% solutions has also been measured by D. J. Fisk and H. E. Hall [*Proceedings of the Thirteenth International Conference on Low Temperature Physics*, edited by K. D. Timmerhaus, W. J. O'Sullivan,

and E. F. Hammel (Plenum, New York, 1974), Vol. I, p. 568]; and D. J. Fisk [thesis (University of Manchester) (unpublished)]. The measurements are in good agreement with those of Ref. 14.

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