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Magnetic susceptibility of the impure classical Heisenberg chain

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An exact site-model calculation of the zero-field magnetic susceptibility in both annealed and quenched limits of an impure one-dimensional classical Heisenberg chain, published recently by Tonegawa, Shiba, and Pincus, is extended to the general case where g factors of different kinds of constituent magnetic ions differ from one another.

I. INTRODUCTION

In a recent paper by Tonegawa, Shiba, and Pincus[1] (to be hereafter referred to as TSP) the thermodynamic properties of an impure one-dimensional classical Heisenberg chain with nearest-neighbor exchange were treated exactly in the thermodynamic limit. They considered both bond and site impurities and considered the annealed and quenched limits for each of these models. Calculating the zero-field magnetic susceptibility for the site model in which two kinds of magnetic ions, I and H, are distributed on the $N+1$ cation sites of the open chain, they assumed that the g factors,[2] g_I and g_H, of the I and H ions are equal. The purpose of the present paper is to extend their calculation of the susceptibility in the site model to the general case where g_I and g_H take independently arbitrary values. It is shown particularly in the cases of diluted magnets where J_{II} (the I-I pair exchange constant) $=J_{IH}$ (the I-H pair exchange constant) $=0$ and J_{HH} (the H-H pair exchange constant) $=\pm J$ ($J>0$) (Ref. 2) that for both the annealed and quenched limits, the difference between the zero-field susceptibility for $g_I=g_H=g$ and that for $g_I=0$ and $g_H=g$ is equal to $2\rho K\chi_0$ in the thermodynamic limit ($N\gg 1$). Here ρ is the concentration of I ions, $K=J/2k_BT$, and $\chi_0=Ng^2\mu_B^2/12J$, where k_B is the Boltzmann constant, T is the absolute temperature, and μ_B is the Bohr magneton.

II. ANNEALED LIMIT

First we discuss the annealed limit. Standard linear response theory gives for the zero-field susceptibility $\chi^{(a)}$ in this limit of the site model

$$\chi^{(a)}=(1/k_BT)\langle M^zM^z\rangle_g, \tag{1}$$

with

$$M^z=\mu_B\sum_{i=0}^{N}\left(\frac{g_I}{2}p_i+\frac{g_H}{2}(1-p_i)\right)S_i^z. \tag{2}$$

In these equations, g_I and g_H are the g factors[2] of the I and H constituent ions, respectively; all other notations are the same as those used in TSP. Note, for example, that $\langle\cdots\rangle_g$ denotes the grand canonical ensemble average associated with the Hamiltonian given by Eq. (3.1) in TSP, while p_i is the occupation variable which is 1 if an I ion occupies the ith site and 0 if an H ion occupies that site.

Employing the method discussed in Sec. III A of TSP, we obtain[3]

$$\phi_{lm}^{II} \equiv \langle p_l p_m \vec{S}_l \cdot \vec{S}_m \rangle_g = 3\langle p_l p_m S_l^z S_m^z \rangle_g = \begin{cases} (1/\Xi)\,\mathrm{Tr}(A_0 A^{l-1} A' B^{m-l-1} B' A^{N-m}) & \text{for } 1 \le l < m \le N, \\ (1/\Xi)\,\mathrm{Tr}(A_0 A^{l-1} A' A^{N-l}) & \text{for } 1 \le l = m \le N, \\ (1/\Xi)\,\mathrm{Tr}(A_0' B^{m-1} B' A^{N-m}) & \text{for } 0 = l < m \le N, \\ (1/\Xi)\,\mathrm{Tr}(A_0' A^N) & \text{for } 0 = l = m, \\ \phi_{ml}^{II} & \text{for } 0 \le m < l \le N; \end{cases} \tag{3}$$

$$\phi_{lm}^{IH} \equiv \langle p_l (1-p_m) \vec{S}_l \cdot \vec{S}_m \rangle_g = 3\langle p_l (1-p_m) S_l^z S_m^z \rangle_g = \begin{cases} (1/\Xi)\mathrm{Tr}(A_0 A^{l-1} A' B^{m-l-1} B'' A^{N-m}) & \text{for } 1 \le l < m \le N, \\ (1/\Xi)\mathrm{Tr}(A_0' B^{m-1} B'' A^{N-m}) & \text{for } 0 = l < m \le N, \\ 0 & \text{for } l = m, \\ \phi_{ml}^{HI} & \text{for } 0 \le m < l \le N; \end{cases} \tag{4}$$

$$\phi_{lm}^{HI} \equiv \langle (1-p_l) p_m \vec{S}_l \cdot \vec{S}_m \rangle_g = 3\langle (1-p_l) p_m S_l^z S_m^z \rangle_g = \begin{cases} (1/\Xi)\mathrm{Tr}(A_0 A^{l-1} A'' B^{m-l-1} B' A^{N-m}) & \text{for } 1 \le l < m \le N, \\ (1/\Xi)\mathrm{Tr}(A_0'' B^{m-1} B' A^{N-m}) & \text{for } 0 = l < m \le N, \\ 0 & \text{for } l = m, \\ \phi_{ml}^{IH} & \text{for } 0 \le m < l \le N; \end{cases} \tag{5}$$

$$\phi_{lm}^{HH} \equiv \langle (1-p_l)(1-p_m) \vec{S}_l \cdot \vec{S}_m \rangle_g = 3\langle (1-p_l)(1-p_m) S_l^z S_m^z \rangle_g = \begin{cases} (1/\Xi)\mathrm{Tr}(A_0 A^{l-1} A'' B^{m-l-1} B'' A^{N-m}) & \text{for } 1 \le l < m \le N, \\ (1/\Xi)\mathrm{Tr}(A_0 A^{l-1} A'' A^{N-l}) & \text{for } 1 \le l = m \le N, \\ (1/\Xi)\mathrm{Tr}(A_0'' B^{m-1} B'' A^{N-m}) & \text{for } 0 = l < m \le N, \\ (1/\Xi)\mathrm{Tr}(A_0'' A^N) & \text{for } 0 = l = m, \\ \phi_{ml}^{HH} & \text{for } 0 \le m < l \le N. \end{cases} \tag{6}$$

Here Ξ, the grand partition function of the system, is given by Eq. (3.4) in TSP, and

$$A_0 = \begin{pmatrix} 1 & \sqrt{\lambda} \\ \sqrt{\lambda} & \lambda \end{pmatrix}, \quad A_0' = \begin{pmatrix} 0 & \sqrt{\lambda} \\ 0 & \lambda \end{pmatrix}, \quad A_0'' = \begin{pmatrix} 1 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}, \tag{7}$$

$$A = \begin{pmatrix} z_{HH} & \sqrt{\lambda} z_{IH} \\ \sqrt{\lambda} z_{IH} & \lambda z_{II} \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & \sqrt{\lambda} z_{IH} \\ 0 & \lambda z_{II} \end{pmatrix}, \quad A'' = \begin{pmatrix} z_{HH} & 0 \\ \sqrt{\lambda} z_{IH} & 0 \end{pmatrix}, \tag{8}$$

$$B = \begin{pmatrix} y_{HH} & \sqrt{\lambda} y_{IH} \\ \sqrt{\lambda} y_{IH} & \lambda y_{II} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & \sqrt{\lambda} y_{IH} \\ 0 & \lambda y_{II} \end{pmatrix}, \quad B'' = \begin{pmatrix} y_{HH} & 0 \\ \sqrt{\lambda} y_{IH} & 0 \end{pmatrix}. \tag{9}$$

In the elements of these matrices, λ, the absolute activity for the I ion, is given by Eq. (3.10) in TSP, and z_{II}, z_{IH}, z_{HH} and y_{II}, y_{IH}, y_{HH} are given, respectively, by Eqs. (3.3) and (3.12) in TSP.

It is easy to show that, in the limit $1 \ll l \le m \ll N$, the quantities ϕ_{lm}^{XY} ($X, Y = I$ or H) become

$$\phi_{lm}^{II} = \begin{cases} \left(\dfrac{1}{a_+}\right)^2 \left[(P^{-1}A'Q)_{11}(Q^{-1}B'P)_{11}\left(\dfrac{b_+}{a_+}\right)^{m-l-1} + (P^{-1}A'Q)_{12}(Q^{-1}B'P)_{21}\left(\dfrac{b_-}{a_+}\right)^{m-l-1} \right] & \text{for } 1 \ll l < m \ll N, \\ (P^{-1}A'P)_{11}/a_+ = \rho & \text{for } 1 \ll l = m \ll N; \end{cases} \tag{10}$$

$$\phi_{lm}^{IH} = \begin{cases} \left(\dfrac{1}{a_+}\right)^2 \left[(P^{-1}A'Q)_{11}(Q^{-1}B''P)_{11}\left(\dfrac{b_+}{a_+}\right)^{m-l-1} + (P^{-1}A'Q)_{12}(Q^{-1}B''P)_{21}\left(\dfrac{b_-}{a_+}\right)^{m-l-1} \right] & \text{for } 1 \ll l < m \ll N, \\ 0 & \text{for } 1 \ll l = m \ll N; \end{cases} \tag{11}$$

$$\phi^{HI}_{lm}=\begin{cases}\left(\frac{1}{a_+}\right)^2\left[(P^{-1}A''Q)_{11}(Q^{-1}B'P)_{11}\left(\frac{b_+}{a_+}\right)^{m-l-1}+(P^{-1}A''Q)_{12}(Q^{-1}B'P)_{21}\left(\frac{b_-}{a_+}\right)^{m-l-1}\right] & \text{for } 1\ll l<m\ll N,\\ 0 & \text{for } 1\ll l=m\ll N;\end{cases} \tag{12}$$

$$\phi^{HH}_{lm}=\begin{cases}\left(\frac{1}{a_+}\right)^2\left[(P^{-1}A''Q)_{11}(Q^{-1}B''P)_{11}\left(\frac{b_+}{a_+}\right)^{m-l-1}+(P^{-1}A''Q)_{12}(Q^{-1}B''P)_{21}\left(\frac{b_-}{a_+}\right)^{m-l-1}\right] & \text{for } 1\ll l<m\ll N,\\ (P^{-1}A''P)_{11}/a_+=1-\rho & \text{for } 1\ll l=m\ll N;\end{cases} \tag{13}$$

where ρ is the concentration of I ions, a_+ is one of the eigenvalues of the matrix A (a_+ is greater than the absolute value of the other eigenvalue a_- of A), $b_\pm$ are the eigenvalues of the matrix B, P and Q are the matrices which diagonalize the matrices A and B, respectively, with inverses P^{-1} and Q^{-1}, and, for example, $(P^{-1}A'Q)_{11}$ is the (1-1) element of the matrix product $P^{-1}A'Q$. The explicit expressions for a_+, $b_\pm$, P, P^{-1}, Q, and Q^{-1} are given in TSP; see Eq. (3.7) for a_+, Eqs. (3.17) and (3.18) for $b_\pm$, Eq. (3.23) for P and P^{-1}, and Eqs. (3.24) and (3.25) for Q and Q^{-1}.

From Eqs. (1), (2), and (10)–(13) we obtain in the thermodynamic limit ($N\gg 1$) the following result for $\chi^{(a)}$:

$$\begin{aligned}\chi^{(a)}=\frac{N\mu_B^2}{12k_BT}\Bigg\{\rho g_I^2+(1-\rho)g_H^2+\frac{2}{a_+}\Bigg[&g_I^2\left((P^{-1}A'Q)_{11}(Q^{-1}B'P)_{11}\frac{1}{a_+-b_+}+(P^{-1}A'Q)_{12}(Q^{-1}B'P)_{21}\frac{1}{a_+-b_-}\right)\\&+g_Ig_H\Big([(P^{-1}A'Q)_{11}(Q^{-1}B''P)_{11}+(P^{-1}A''Q)_{11}(Q^{-1}B'P)_{11}]\frac{1}{a_+-b_+}\\&+[(P^{-1}A'Q)_{12}(Q^{-1}B''P)_{21}+(P^{-1}A''Q)_{12}(Q^{-1}B'P)_{21}]\frac{1}{a_+-b_-}\Big)\\&+g_H^2\left((P^{-1}A''Q)_{11}(Q^{-1}B''P)_{11}\frac{1}{a_+-b_+}+(P^{-1}A''Q)_{12}(Q^{-1}B''P)_{21}\frac{1}{a_+-b_-}\right)\Bigg]\Bigg\}.\end{aligned} \tag{14}$$

It can be easily shown that, when $g_I=g_H(=g)$, Eq. (14) agrees exactly with Eq. (3.26) in TSP.

In the remainder of this section we discuss briefly $\chi^{(a)}$ for the special cases of a diluted ferromagnet where J_{II} (the I-I pair exchange constant) $=J_{IH}$ (the I-H pair exchange constant) $=0$, J_{HH} (the H-H pair exchange constant) $=J(>0)$, $g_I=0$, and $g_H=g$ and of a diluted antiferromagnet where $J_{II}=J_{IH}=0$, $J_{HH}=-J(J>0)$, $g_I=0$, and $g_H=g$. The zero-field susceptibility $\chi^{(a)}$ for these special cases can be expressed as

$$\chi^{(a)}=2(1-\rho)K\chi_0(a_++b_+)/(a_+-b_+), \tag{15}$$

with

$$\chi_0=Ng^2\mu_B^2/12J,\quad K=J/2k_BT \tag{16}$$

and

$$a_+=z-\frac{1-2\rho}{2(1-\rho)}+\frac{1}{2(1-\rho)}[4\rho(1-\rho)z+(1-2\rho)^2]^{1/2}, \tag{17}$$

$$b_+=\pm y, \tag{18}$$

where $z=(\sinh K)/K$ and $y=(K\cosh K-\sinh K)/K^2$. In Eq. (18) we take the + sign for the diluted ferromagnet and the − sign for the diluted antiferromagnet. It should be noted here that the right-hand side of Eq. (15) is equal to the quantity which is obtained by subtracting $2\rho K\chi_0$ from the right-hand side of Eq. (3.26) in TSP for $J_{II}=J_{IH}=0$ and $J_{HH}=\pm J$. The low-temperature expansions of $\chi^{(a)}$ in the cases of the diluted ferromagnet and antiferromagnet are given, respectively, by[4]

$$\chi^{(a)}/\chi_0=(1-\rho)(4K^2-2K+\cdots) \tag{19}$$

and

$$\chi^{(a)}/\chi_0=(1-\rho)(1+1/2K+\cdots). \tag{20}$$

In Figs. 1 and 2 the numerical results for $\chi^{(a)}$ in both of these cases are plotted as functions of the reduced temperature k_BT/J at representative values of the concentration ρ of I ions.

III. QUENCHED LIMIT

We now consider the quenched limit. Let $\tilde{\chi}$ be the zero-field susceptibility for a given configuration of the I and H ions. Again, according to linear response theory, $\tilde{\chi}$ is given by

$$\tilde{\chi}=(1/k_BT)\langle M^zM^z\rangle_c, \tag{21}$$

where, as in TSP, $\langle\cdots\rangle_c$ stands for the canonical ensemble average associated with the Hamiltonian (3.1) in TSP, and M^z is defined by Eq. (2). The zero-field susceptibility $\chi^{(q)}$ in the quenched limit is calculated from the equation

$$\chi^{(a)} = \langle\tilde{\chi}\rangle_{\rm conf} = \frac{\mu_B^2}{12k_BT}\sum_{l=0}^{N}\sum_{m=0}^{N}\{g_I^2\langle p_l p_m \tilde{\omega}_{lm}\rangle_{\rm conf} + g_I g_H[\langle p_l(1-p_m)\tilde{\omega}_{lm}\rangle_{\rm conf} + \langle(1-p_l)p_m\tilde{\omega}_{lm}\rangle_{\rm conf}] + g_H^2\langle(1-p_l)(1-p_m)\tilde{\omega}_{lm}\rangle_{\rm conf}\}. \tag{22}$$

Here $\langle\cdots\rangle_{\rm conf}$ denotes the arithmetic average over all the configurations of the I and H ions for a given concentration ρ of I ions, and $\tilde{\omega}_{lm}$ is the spin-spin correlation function for a given configuration, i.e.,

$$\tilde{\omega}_{lm} = \langle\vec{S}_l\cdot\vec{S}_m\rangle_c = 3\langle S_l^z S_m^z\rangle_c. \tag{23}$$

The quantities $\langle p_l p_m \tilde{\omega}_{lm}\rangle_{\rm conf}$, etc., can be rewritten

$$\langle p_l p_m \tilde{\omega}_{lm}\rangle_{\rm conf} = \begin{cases} \rho^2\langle\tilde{\omega}_{lm}\rangle_{{\rm conf};\ I\in l,\ I\in m} & \text{for } l\neq m, \\ \rho & \text{for } l=m; \end{cases} \tag{24}$$

$$\langle p_l(1-p_m)\tilde{\omega}_{lm}\rangle_{\rm conf} = \begin{cases} \rho(1-\rho)\langle\tilde{\omega}_{lm}\rangle_{{\rm conf};\ I\in l,\ H\in m} & \text{for } l\neq m, \\ 0 & \text{for } l=m; \end{cases} \tag{25}$$

$$\langle(1-p_l)p_m\tilde{\omega}_{lm}\rangle_{\rm conf} = \begin{cases} \rho(1-\rho)\langle\tilde{\omega}_{lm}\rangle_{{\rm conf};\ H\in l,\ I\in m} & \text{for } l\neq m, \\ 0 & \text{for } l=m; \end{cases} \tag{26}$$

$$\langle(1-p_l)(1-p_m)\tilde{\omega}_{lm}\rangle_{\rm conf} = \begin{cases} (1-\rho)^2\langle\tilde{\omega}_{lm}\rangle_{{\rm conf};\ H\in l,\ H\in m} & \text{for } l\neq m, \\ 1-\rho & \text{for } l=m. \end{cases} \tag{27}$$

In these equations $\langle\cdots\rangle_{{\rm conf};\ X\in l,\ Y\in m}(X, Y = I$ or $H)$ represents the arithmetic average over all the configurations of the I and H ions subject to the condition that the lth and mth sites are occupied by the X and Y ions, respectively. For simplicity, we denote $\langle\tilde{\omega}_{lm}\rangle_{{\rm conf};\ X\in l,\ Y\in m}$ by ω_{lm}^{XY}. By the use of

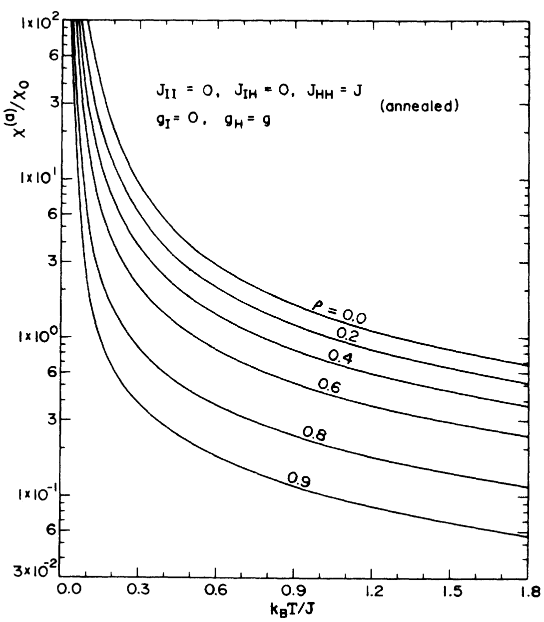

FIG. 1. Zero-field susceptibility $\chi^{(a)}/\chi_0$, where $\chi_0 = Ng^2\mu_B^2/12J$, as a function of the reduced temperature k_BT/J for the case of the diluted ferromagnet $[J_{II}=J_{IH}=0, J_{HH}=J(>0), g_I=0$, and $g_H=g]$ in the annealed limit. Labels on the individual curves denote the values of the concentration ρ of I ions.

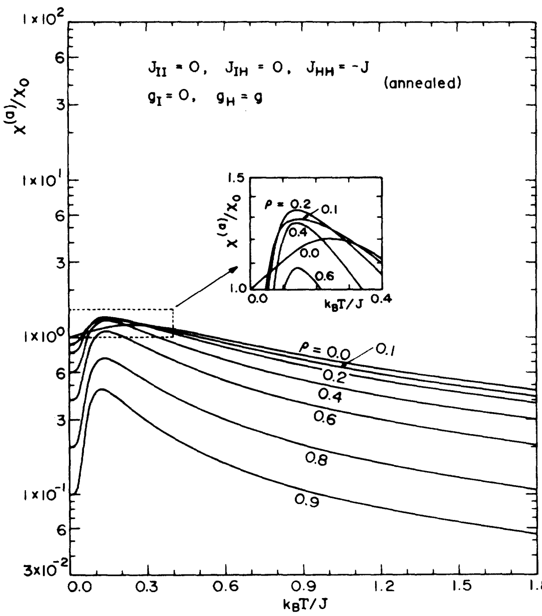

FIG. 2. Same as Fig. 1 but for the case of the diluted antiferromagnet $[J_{II}=J_{IH}=0, J_{HH}=-J(J>0), g_I=0$, and $g_H=g]$. Note that the region surrounded by the dashed line is enlarged in the inset.

the method discussed in Sec. III B of TSP, ω_{lm}^{XY} can be calculated easily.[5] The results for $l \neq m$ are[6]

$$\begin{pmatrix} \omega_{lm}^{II} & \omega_{lm}^{HI} \\ \omega_{lm}^{IH} & \omega_{lm}^{HH} \end{pmatrix} = D^{|l-m|-1} \begin{pmatrix} u_{II} & u_{IH} \\ u_{IH} & u_{HH} \end{pmatrix} \text{for } l \neq m, \tag{28}$$

where

$$D = \begin{pmatrix} \rho u_{II} & (1-\rho)u_{IH} \\ \rho u_{IH} & (1-\rho)u_{HH} \end{pmatrix}, \tag{29}$$

u_{II}, u_{IH}, and u_{HH} being given by Eq. (3.44) in TSP; equivalently, they are

$$\omega_{lm}^{II} = R_{11}(R_{11}^{-1}u_{II} + R_{12}^{-1}u_{IH})d_+^{|l-m|-1} + R_{12}(R_{21}^{-1}u_{II} + R_{22}^{-1}u_{IH})d_-^{|l-m|-1} \quad \text{for } l \neq m, \tag{30}$$

$$\omega_{lm}^{IH} = R_{21}(R_{11}^{-1}u_{II} + R_{12}^{-1}u_{IH})d_+^{|l-m|-1} + R_{22}(R_{21}^{-1}u_{II} + R_{22}^{-1}u_{IH})d_-^{|l-m|-1} \quad \text{for } l \neq m, \tag{31}$$

$$\omega_{lm}^{HI} = R_{11}(R_{11}^{-1}u_{IH} + R_{12}^{-1}u_{HH})d_+^{|l-m|-1} + R_{12}(R_{21}^{-1}u_{IH} + R_{22}^{-1}u_{HH})d_-^{|l-m|-1} \quad \text{for } l \neq m, \tag{32}$$

$$\omega_{lm}^{HH} = R_{21}(R_{11}^{-1}u_{IH} + R_{12}^{-1}u_{HH})d_+^{|l-m|-1} + R_{22}(R_{21}^{-1}u_{IH} + R_{22}^{-1}u_{HH})d_-^{|l-m|-1} \quad \text{for } l \neq m, \tag{33}$$

where $d_\pm$ are the eigenvalues of the matrix D, R is the matrix which diagonalizes D, and R_{11} and R_{11}^{-1} are, for example, the (1-1) elements of R and its inverse matrix R^{-1}, respectively. The explicit expressions for $d_\pm$ are given by Eqs. (3.56) and (3.57) in TSP, and those for R and R^{-1} by Eqs. (3.58) and (3.59) in TSP.

Using Eqs. (22), (24)–(27), and (30)–(33) and the explicit expressions for $d_\pm$, R, and R^{-1}, we can calculate the zero-field susceptibility $\chi^{(q)}$ in the thermodynamic limit ($N \gg 1$), obtaining the result[7]

$$\chi^{(q)} = \frac{N\mu_B^2}{12k_BT}\,\frac{1}{\Delta}\{g_I^2\rho[1+\rho u_{II} - (1-\rho)u_{HH} - \rho(1-\rho)(u_{II}u_{HH} - u_{IH}^2)] + 4g_Ig_H\rho(1-\rho)u_{IH} + g_H^2(1-\rho)[1-\rho u_{II} + (1-\rho)u_{HH} - \rho(1-\rho)(u_{II}u_{HH} - u_{IH}^2)]\}. \tag{34}$$

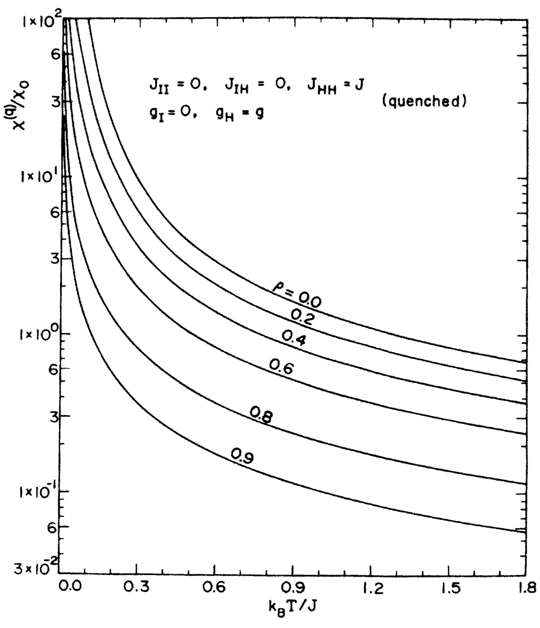

FIG. 3. Zero-field susceptibility $\chi^{(q)}/\chi_0$, where $\chi_0 = Ng^2\mu_B^2/12J$, as a function of the reduced temperature k_BT/J for the case of the diluted ferromagnet [$J_{II} = J_{IH} = 0$, $J_{HH} = J(>0)$, $g_I = 0$, and $g_H = g$] in the quenched limit. Labels on the individual curves denote the values of the concentration ρ of I ions.

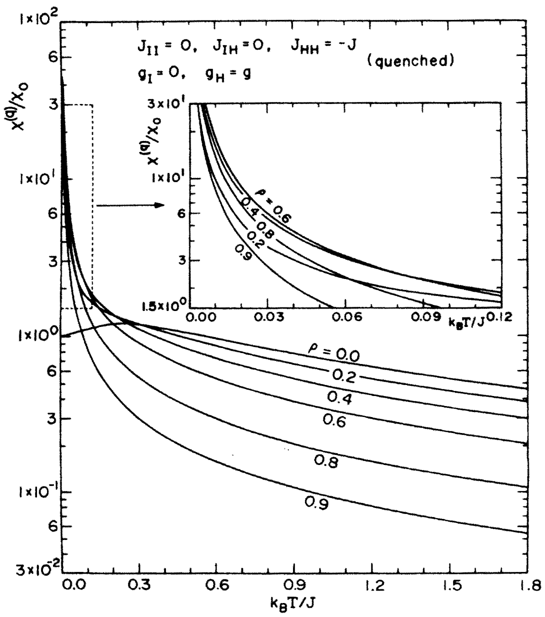

FIG. 4. Same as Fig. 3 but for the case of the diluted antiferromagnet [$J_{II} = J_{IH} = 0$, $J_{HH} = -J(J>0)$, $g_I = 0$, and $g_H = g$]. Note that the region surrounded by the dashed line is enlarged in the inset.

Here the denominator Δ is

$$\Delta = 1 - \rho u_{II} - (1-\rho)u_{HH} + \rho(1-\rho)(u_{II}u_{HH} - u_{IH}^2). \tag{35}$$

When $g_I = g_H (=g)$, Eq. (34) agrees with Eq. (3.61) in TSP.

In the cases of a diluted ferromagnet $[J_{II} = J_{IH} = 0$, $J_{HH} = J(>0)$, $g_I = 0$, and $g_H = g]$ and of a diluted antiferromagnet $[J_{II} = J_{IH} = 0$, $J_{HH} = -J(J>0)$, $g_I = 0$, and $g_H = g]$, the susceptibility $\chi^{(q)}$ becomes

$$\chi^{(q)} = 2(1-\rho)K\chi_0 \frac{1 \pm (1-\rho)u}{1 \mp (1-\rho)u}, \tag{36}$$

where $u = \coth K - K^{-1}$, and the upper and lower signs apply for the diluted ferromagnet and antiferromagnet, respectively. The quantities, χ_0 and K, are defined by Eq. (16). The difference between the right-hand side of Eq. (36) and that of Eq. (3.61) in TSP for $J_{II} = J_{IH} = 0$ and $J_{HH} = \pm J$ is again $-2\rho K\chi_0$. The low-temperature expansion of $\chi^{(q)}$ for the diluted ferromagnet is given by

$$\frac{\chi^{(q)}}{\chi_0} = \begin{cases} (1-\rho)\left(\dfrac{2(2-\rho)}{\rho} K - \dfrac{4(1-\rho)}{\rho^2} + \cdots\right) & \text{for } 0 < \rho \le 1, \\ 4K^2 - 2K + \cdots & \text{for } \rho = 0, \end{cases} \tag{37}$$

and that of $\chi^{(q)}$ for the diluted antiferromagnet is

$$\frac{\chi^{(q)}}{\chi_0} = (1-\rho)\left(\frac{2\rho}{2-\rho} K + \frac{4(1-\rho)}{(2-\rho)^2} + \cdots\right). \tag{38}$$

In Figs. 3 and 4 we summarize the numerical results of $\chi^{(q)}$ for the two cases discussed above.

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*Permanent address.

[1] T. Tonegawa, H. Shiba, and P. Pincus, Phys. Rev. B 11, 4683 (1975).

[2] Note that, as in TSP (Ref. 1), we have used Fisher's definitions of the g factor and the exchange constant [see M. E. Fisher, Am. J. Phys. 32, 343 (1964)].

[3] Since we are concerned with a one-dimensional system with open ends at the 0th and Nth sites, ϕ_{lm}^{II}, etc., depend independently on l and m.

[4] The expressions for the low-temperature expansions of $\chi^{(a)}/\chi_0$ for $J_{II} = 0$, $J_{IH} = 0$, $J_{HH} = J$ and for $J_{II} = 0$, $J_{IH} = 0$, $J_{HH} = -J$ in Table I of TSP (Ref. 1) are misleading and should be changed to $(1-\rho)(4K^2 - 2K + \cdots) + 2\rho K$ and $2\rho K + (1-\rho)(1 + 1/2K + \cdots)$, respectively.

[5] See also the Appendix by D. Hone, P. A. Montano, T. Tonegawa, and Y. Imry [Phys. Rev. B 12, 5141 (1975)].

[6] We note that ω_{lm}^{XY} depends only on $|l-m|$.

[7] This result has also been obtained independently by S. Katsura [Can. J. Phys. 53, 854 (1975)].