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Spontaneous breakdown of continuous symmetries near two dimensions

E. Brézin and J. Zinn-Justin

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, Boîte Postale 2, 91190 Gif-Sur-Yvette, France

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The long-distance properties of classical Heisenberg ferromagnets below the transition point are related to a continuous-field theory, the nonlinear σ model. The renormalizability of this model in two dimensions and its ultraviolet asymptotic freedom are used to derive renormalization-group equations valid above $d = 2$. It is argued that this model is renormalizable up to four dimensions. The scaling properties which incorporate critical and Goldstone singularities follow. Explicit calculations of exponents and of correlation functions in powers of $d - 2$ are given. A technique is proposed to make calculations in the symmetric phase applicable even in two dimensions.

I. INTRODUCTION

The classical Heisenberg model, with an $O(n)$-symmetric interaction, is described by the Hamiltonian

$$\mathcal{H} = -\sum_{ij} V_{ij}\vec{S}_i \cdot \vec{S}_j \tag{1}$$

in which the $\vec{S}_i$ are unit n-component vectors associated with the sites i of a periodic d-dimensional lattice; V_{ij} is a short-ranged positive translationally invariant interaction. The partition function is given as

$$Z = \int \prod_i [\delta(S_i^2 - 1) d^n S_i] e^{-\mathcal{H}/T}. \tag{2}$$

This model has a phase transition above two dimensions and its long-distance behavior may be studied through an expansion around mean-field theory. The result is that the critical properties are given, as first shown by Wilson and Kogut,[1] by a continuous-field theory, namely the linear σ model, whose interaction is

$$\mathcal{H} = \int d^d x \left(\frac{(\nabla\phi)^2}{2} + \frac{m_0^2 \vec{\phi}^2}{2} + \frac{g_0}{4!} (\vec{\phi}^2)^2 \right). \tag{3}$$

This theory has an infrared stable fixed point[2] below four dimensions, with g_0 of order $4 - d$; this leads to the famous Wilson-Fisher[3] $4 - d$ expansion. This model, in which a continuous symmetry is broken, has infrared singularities both in the critical domain and in the ordered phase for any temperature below T_c due to the $n - 1$ massless Goldstone modes. These singularities are not naturally taken into account by this formalism[4] and the aim of this work is to show that another expansion is well suited for the understanding of both the Goldstone and the critical singularities.[5] The first step will be the construction of a low-temperature expansion of the partition function (2). Then it will be shown that, in the long-distance limit, a continuous-field theory,[5] which is the nonlinear σ model,[6] is equivalent to the Heisenberg model. It will be established that this nonlinear σ model has a phase transition above two dimensions,[7] and that apart from the special $n = 2$ Abelian case, the critical temperature is proportional to $d - 2$. This model is asymptotically free in two dimensions and renormalizable above two dimensions within a double series expansion in $d - 2$ and in the temperature. The renormalization-group equations will then be derived; their integration will exhibit scaling properties with both critical and Goldstone singularities.

The set-up of the article is the following: In Sec. II it is shown that the Heisenberg problem below T_c coincides in the long-distance limit with a continuous-field theory. This is done by analyzing the behavior of the low-temperature expansion.

In Sec. III, it is shown that this field theory is renormalizable, within a double expansion in powers of the temperature and of $d - 2$. The corresponding renormalization-group equations are derived.

In Sec. IV, scaling properties are derived from the renormalization-group equations, containing the combined structure of critical and Goldstone singularities.

In Sec. V, the $n = 2$ problem is discussed separately. The continuous version of this model coincides with the quantum sine-Gordon equation. The scaling properties in the low-temperature region follow.

Section VI is devoted to the large-n limit. General arguments support the identification of the nonlinear σ model and of the $(\vec{\phi}^2)^2$ theory in the scaling region. The renormalizability of the nonlinear σ model is thus extended up to four dimensions within the $1/n$ expansion.

Section VII contains calculations of various physical quantities below T_c, up to two-loop order. In particular, critical indices are given at order

$(d-2)^2$.

In Sec. VIII, the problem of the continuation above T_c is discussed. The example of two dimensions reveals the difficulties. An explicit numerical method is proposed, and the amplitude of the low-temperature divergence of the magnetic susceptibility is approximately calculated.

II. LOW-TEMPERATURE EXPANSION AND CONTINUOUS LIMIT

In the ordered phase the vectors $\vec{S}_i$ fluctuate around the direction $\hat{u}$ of spontaneous symmetry breaking and at low temperature these fluctuations are very small. Therefore it is natural to express $\vec{S}_i$ as

$$\sigma_i = \vec{S}_i \cdot \hat{u} = (1-\vec{\Pi}_i^2)^{1/2}, \tag{4a}$$

$$\vec{\Pi}_i = \vec{S}_i - (\vec{S}_i \cdot \hat{u})\hat{u}, \tag{4b}$$

and expand the interaction in powers of $\vec{\Pi}_i$.

In terms of the $(n-1)$ $\vec{\Pi}$ modes the partition function becomes

$$Z = \int \prod_i \frac{d^{n-1}\Pi_i}{(1-\vec{\Pi}_i^2)^{1/2}} \exp\left(\frac{1}{T}\sum_{ij} V_{ij}[(1-\vec{\Pi}^2)^{1/2}(1-\vec{\Pi}_j^2)^{1/2} + \vec{\Pi}_i \cdot \vec{\Pi}_j]\right). \tag{5}$$

The standard loopwise expansion[8] of the functional integral (5) generates an expansion in powers of T. This requires the expansion of $(1-\Pi^2)^{1/2}$ in powers of Π^2 to the appropriate order. The integrations over the Π field are performed from minus to plus infinity, neglecting again exponential corrections in $1/T$. The corresponding Feynman diagrams involve propagators, which are the inverse of the quadratic part of the action, namely

$$G_{\alpha\beta}(q) = \frac{T}{\tilde{V}(0) - \tilde{V}(q)}\delta_{\alpha\beta}, \quad \alpha,\beta = 1,\ldots,n-1 \tag{6}$$

in which

$$\tilde{V}(q) = \sum_j V_{ij} e^{\vec{q}\cdot\vec{r}_{ij}}. \tag{7}$$

This propagator behaves as expected like $1/q^2$ for small q. The interaction is thus obtained from higher-order terms in the expansion of the square roots, and from the integration measure written as

$$\prod_i \frac{1}{(1-\Pi_i^2)^{1/2}} \equiv \exp\left(-\frac{1}{2}\sum_i \ln(1-\Pi_i^2)\right). \tag{8}$$

The problem is now to examine the long-distance limit of this theory. The discussion is very similar here to the one given for the expansion around mean-field theory,[8] namely the propagator may be replaced by its most divergent part,

$$G_{\alpha\beta}(q) = (T/q^2)\delta_{\alpha\beta}.$$

Then, in the same way the interaction terms which involve only $\tilde{V}(0) - \tilde{V}(q)$ may be approximated again by their dominant q^2 part. The diagrams simplified in this way are exactly those of the continuous nonlinear σ model whose Euclidean action is

$$\mathcal{A} = \int d^dx \, \tfrac{1}{2}\{[\nabla(1-\vec{\Pi}^2)^{1/2}]^2 + (\nabla\vec{\Pi})^2\}, \tag{9}$$

with of course the same invariant measure

$$\prod_x \frac{d^{n-1}\Pi(x)}{[1-\vec{\Pi}^2(x)]^{1/2}}.$$

However, this theory suffers of ultraviolet divergences which have to be regularized. Actually the Heisenberg model provides an O(n)-invariant regularization of the continuous-field theory.

III. POWER COUNTING AND RENORMALIZATION

Since we are now interested in the long-distance properties of the nonlinear σ model, we shall establish the renormalization-group equations for this theory. As usual, they will follow the discussion of the renormalization of the theory.[8] In two dimensions the $\vec{\Pi}$ field is dimensionless and the theory is renormalizable by power counting. The counter terms are thus arbitrary local functions of the $\vec{\Pi}$ field with at most two derivatives. Furthermore, since the theory can be regularized, as said above, in an invariant way, the renormalized action will also be invariant. The only invariant involving at most two derivatives is proportional to the action itself up to a rescaling of the fields. There will thus be two renormalization constants, namely one field-strength renormalization and a coupling constant, i.e., a temperature renormalization.[9]

Above two dimensions the field acquires the dimension $\frac{1}{2}(d-2)$ and for d fixed larger than 2 the theory is not renormalizable. However, it may formally be defined in a double expansion in powers of T and $d-2$, and then it can be renormalized with the same two renormalization constants.

In two dimensions, in addition to the standard ultraviolet problem of a renormalizable theory one has the infrared divergences coming from the $1/p^2$

propagators. To disentangle these two sources of divergences we shall break the $O(n)$ symmetry in order to give a mass to the Π field. From the renormalization-group point of view it will appear that the most convenient way of breaking this symmetry is to introduce an external source coupled linearly to the σ field, i.e., a magnetic field H. Indeed, since by symmetry the σ field is like the $\vec{\Pi}$ field multiplicatively renormalized by the same factor the addition of this symmetry-breaking term does not introduce any new renormalization constant. One could choose instead to add other "soft terms" in the sense of power counting (i.e., relevant) such as $m^2\vec{\Pi}^2$, but these terms would lead to more complicated renormalization-group equations.

Renormalization-group equations

They follow from the relation between the renormalized and the bare theory. In terms of the renormalized fields and of the renormalized dimensionless temperature t the action reads

$$\mathfrak{A}=\frac{\mu^{d-2}}{2Z_1 t}\int d^d x\left(Z\partial_\nu\Pi^\alpha\partial_\nu\Pi^\alpha+\partial_\nu(1-Z\vec{\Pi}^2)^{1/2}\partial_\nu(1-Z\vec{\Pi}^2)^{1/2}-2\frac{HZ_1}{\sqrt{Z}}(1-Z\vec{\Pi}^2)^{1/2}\right), \tag{10}$$

in which μ is an arbitrary momentum scale which defines the renormalized theory, and plays a role equivalent to the cutoff in the bare theory. The relation between the bare and renormalized theory for the one-particle irreducible functions of the Π field is

$$\Gamma^{(N)}(\vec{p},t,H,\mu)=Z^{N/2}\Gamma_B^{(N)}(\vec{p},T,H_B), \tag{11}$$

with

$$T=tZ_1\mu^{2-d}, \tag{12}$$

$$\frac{H_B}{T}=\frac{H\mu^{d-2}}{t\sqrt{Z}}. \tag{13}$$

The renormalization-group equations follow from the invariance of the bare theory under a change of μ holding T and H_B fixed. This leads to the differential equation

$$\left[\mu\frac{\partial}{\partial\mu}+W(t)\frac{\partial}{\partial t}-\tfrac{1}{2}N\zeta(t)+\left(\tfrac{1}{2}\zeta(t)+\frac{W(t)}{t}-(d-2)\right)H\frac{\partial}{\partial H}\right]\Gamma^{(N)}(\vec{p},t,H,\mu)=0, \tag{14}$$

in which from (12)

$$W(t)=(d-2)t-t\mu\frac{\partial}{\partial\mu}\bigg|_B \ln Z_1 \tag{15}$$

and

$$\zeta(t)=\mu\frac{\partial}{\partial\mu}\bigg|_B \ln Z. \tag{16}$$

The connected correlation functions of the σ field and of the Π field fulfill the equation

$$\left[\mu\frac{\partial}{\partial\mu}+W(t)\frac{\partial}{\partial t}+\tfrac{1}{2}N\zeta(t)+\left(\tfrac{1}{2}\zeta(t)+\frac{W(t)}{t}-(d-2)\right)H\frac{\partial}{\partial H}\right]G^{(N)}(\vec{p},t,H,\mu)=0. \tag{17}$$

The equation for the magnetization $M(t,H,\mu)$, i.e., the expectation value of the σ field is given by Eq. (17) for $N=1$. The free energy $F(t,H,\mu)$ is obtained from (17) by setting $N=0$.

IV. SCALING BEHAVIOR

The calculation of the renormalization constants Z and Z_1, from which one deduces the coefficients of the renormalization-group equations (14)–(17), will be given up to two-loop order in a subsequent section. Let us simply here use the fact that at leading order

$$Z_1=1+(n-2)\frac{t}{d-2}$$

and therefore from (15)

$$W(t)=(d-2)t-(n-2)t^2+O(t^3). \tag{18}$$

[We have included in all explicit calculations of this article a factor $2\pi^{d/2}/(2\pi)^d\Gamma(d/2)$ in our definition of the temperature t.] This shows, as first established by Polyakov,[7] that there is an ultraviolet stable fixed point t_c for $n>2$ of order $d-2$:

$$t_c=(d-2)/(n-2)+O((d-2)^2). \tag{19}$$

This is indeed the (renormalized) critical temperature since a critical point corresponds to an infrared unstable fixed point in the temperature variable. The situation for $n=2$ will be examined separately.

The integration of the partial differential equations (14)–(17) is simplified by the introduction of the zero-field correlation length $\xi(t)$ and of the spontaneous magnetization $\sigma(t)$ defined, respectively, by

$$\left(\mu\frac{\partial}{\partial\mu}+W(t)\frac{\partial}{\partial t}\right)\xi(t,\mu)=0, \tag{20}$$

$$\left(W(t)\frac{\partial}{\partial t}+\tfrac{1}{2}\zeta(t)\right)\sigma(t)=0, \tag{21}$$

i.e.,

$$\xi(t,\mu)=\mu^{-1}t^{1/(d-2)}\exp\left[\int_0^t dt'\left(\frac{1}{W(t')}-\frac{1}{(d-2)t'}\right)\right] \tag{22}$$

below t_c and

$$\sigma(t)=\exp\left(-\frac{1}{2}\int_0^t\frac{\zeta(t')}{W(t')}dt'\right). \tag{23}$$

This expression of the spontaneous magnetization follows from (17) and from the dimensionless character of $\sigma(t)$. With this definition of the correlation length the integration of (14) in zero-field yields the scaling behavior[5]

$$\Gamma^{(N)}(p,t)=\xi^{-d}(t)\sigma^{-N}(t)\phi^{(N)}(p\xi(t)) \tag{24}$$

if one uses together with the differential equation, the canonical dimension d of $\Gamma^{(N)}$.

On Eq. (24) one sees that the length $\xi(t)$ defined by Eq. (20) characterizes the crossover from the long-distance Goldstone behavior to the critical regime which appears at shorter distances for fixed temperature below t_c. This definition of ξ, first given by Josephson, is consistent with the one of Halperin and Hohenberg.[10] On (22) and (23) one sees the low-temperature and critical singularities displayed at the same time. The correlation length diverges at t_c as

$$\xi(t)\sim(t_c-t)^{-\nu}\ \text{with}\ \nu=-1/W'(t_c); \tag{25}$$

the spontaneous magnetization vanishes as

$$\sigma(t)\sim(t_c-t)^{\beta}\ \text{with}\ \beta=-\zeta(t_c)/2W'(t_c). \tag{26}$$

From the behavior of the two-point function at t_c we find

$$\Gamma^{(2)}(p,t_c)\sim p^{2-\eta}$$

with

$$\zeta(t_c)=d-2+\eta.$$

When the field is nonzero the integration of (14) leads to the scaling properties of the one-particle irreducible functions of the $\vec{\Pi}$ field

$$\Gamma^{(N)}(\vec{p},t,H)=\xi^{-d}(t)\sigma^{-N}(t)\times\phi^{(N)}\left(\vec{p}\xi(t),\ \frac{H\sigma(t)}{t}\xi^{d}(t)\right). \tag{27}$$

For the connected correlation functions of the σ or of the Π fields the same analysis leads to the relations

$$G^{(N)}(\vec{p},t,H)=\xi^{(N-1)d}(t)\sigma^{N}(t)F^{(N)}\left(\vec{p}\xi,\ \frac{H\sigma\xi^{d}}{t}\right). \tag{28}$$

In particular the free energy is

$$F(t,H)=\xi^{-d}(t)F^{(0)}\left(\frac{H\sigma(t)\xi^{d}(t)}{t}\right) \tag{29}$$

and the equation of state follows by differentiation with respect to H.

These scaling properties do coincide near the critical point with those which were obtained from the $4-d$ expansion of the linear model.[1,2] However, they contain additional information on Goldstone modes. The physical interpretation of the correlation length, in this problem in which correlation fall off like powers, is the distance at which one sees the crossover between the critical singularities and the Goldstone behavior. These Goldstone modes yield infrared singularities below t_c, which are governed by the trivial infrared fixed point $t^*=0$ of the renormalization-group equations. Under these conditions, the singularities are given by the first nontrivial order of perturbation theory. For instance, this implies that the longitudinal susceptibility diverges as $H^{(d-4)/2}$ for small H, since the argument given above justifies the one which was used in Ref. 11.

Asymptotic freedom and renormalizability for fixed $d>2$

From the point of view of the ultraviolet (uv) properties of the nonlinear σ model, one sees that there is a nontrivial uv fixed point above two dimensions. The existence of an uv fixed point indicates that this theory is renormalizable above two dimensions, even outside the $d-2$ expansion. (This is similar to the problem of the continuation of a massless ϕ^4 theory below four dimensions[2,12]). Indeed, the large momentum behavior is not given by perturbation theory but by the power behavior at the uv fixed point t_c:

$$\Gamma^{(N)}(\lambda\vec{p},t)\underset{\lambda\to\infty}{\sim}\lambda^{d-N(d-2+\eta)/2}, \tag{30}$$

which is stable by power counting made on skeleton diagrams. The only point which has been used in this analysis is the negative coefficient of t^2 in

$W(t)$, i.e., the asymptotic freedom of the two-dimensional theory. Therefore this argument would apply to other asymptotically free theories such as non-Abelian gauge theories in four dimensions.

V. ABELIAN CASE

For $n=2$ the low-temperature expansion in the long-distance limit simplifies considerably. As for the non-Abelian case, this limit may be reproduced by the nonlinear σ model, but this model is now equivalent to free field theory. Indeed with the change of variable

$$\sigma(x)=\cos\Theta(x), \quad \pi(x)=\sin\Theta(x), \tag{31}$$

the action becomes simply

$$\int d^dx\,\tfrac{1}{2}(\nabla\Theta)^2 \tag{32}$$

and the invariant measure (8) is flat.

Therefore we are left with a free field theory for which we are interested in the correlation functions of the fields $e^{\pm i\Theta(x)}$. The functions $W(t)$ and $\zeta(t)$, as well as the correlation functions in zero field, are thus exactly calculable. The result is

$$W(t)=(d-2)t, \tag{33}$$

$$\zeta(t)=t. \tag{34}$$

One sees that in this low-temperature approximation the critical point has gone to infinity. The spontaneous magnetization is

$$\sigma(t)=e^{-t/2(d-2)} \tag{35}$$

and the $\Pi-\Pi$ correlation function in position space

$$G(x)=\left\{\sinh\left[\frac{t}{x^{d-2}}\frac{1}{2^{3-d}}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)\right]\right\}\exp\left[\frac{t}{2}\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{4-d}{2}\right)\right]. \tag{36}$$

If one adds a magnetic field H the theory is no longer trivial since it becomes the quantum sine-Gordon model. However, the behavior of the correlation functions in the phase with broken symmetry may still be deduced from the renormalization-group equation (17), since it implies

$$G^{(N)}(\vec{p},t,H)=t^{(N-1)d/(d-2)}e^{-Nt/2(d-2)}F^{(N)}(\vec{p}t^{1/(d-2)},He^{-t/2(d-2)}t^{-2/(d-2)}). \tag{37}$$

In two dimensions the critical temperature does not go to zero; the function $W(t)$ vanishes identically and the theory is exactly scale invariant for all temperatures in the ordered phase.[13,14] This ordered phase is not characterized by a broken symmetry because there is no spontaneous magnetization[15] as seen from Eq. (35) (the Π and the σ have identical propagators), but by the fact that the correlations fall off like powers as exhibited in the explicit results of Refs. 13 and 16.

For example, the two-point correlation function is proportional to $1/p^{2-t}$ and the magnetization induced by a field H is[13]

$$M=H^{t/(4-t)}. \tag{38}$$

All these results are in agreement with the correspondence found by Coleman[5] between the sine-Gordon theory and the massive Thirring model, which has also continuous indices like the Baxter model.

Note that the theory is meaningless for $t>4$,[17] but we have neglected terms in this approximation which have at least the effect of modifying the temperature. Furthermore, we have integrated the Θ field without taking into account that its range was limited to 2π. In the non-Abelian case we have also neglected the compactness of the n-dimensional sphere but for $n>2$ the curvature of the sphere may be detected locally, whereas for $n=2$ it is impossible to locally distinguish between a circle and a straight line. This is probably the source of the fact that, in this approximation, we find a scale-invariant theory for any temperature.[13,14]

VI. LARGE-n LIMIT

We have seen that it is possible to describe the long-distance behavior of the Heisenberg ferromagnets in two different ways: the ϕ^4 theory which corresponds to an expansion around mean-field theory and the nonlinear σ model obtained by this low-temperature expansion. This leads to the surprising conclusion that these two field theories have the same long-distance limit. In other words the correlation functions of the $g\phi^4$ model evaluated at the infrared stable fixed point g^* are identical to those of the nonlinear σ model, if the mass of the ϕ^4 theory is replaced by a suitable function of the temperature. As a consequence the nonlinear σ model should be renormalizable from two to four dimensions and not simply in the neighborhood of two dimensions. In particular in four dimensions it becomes a free field theory since the ir fixed point of the ϕ^4 theory moves to the origin.

This may be checked explicitly in the large-n

limit. The ϕ^4 theory has been extensively studied for n large[18]; let us show here that the calculations may be done easily for the nonlinear σ model in the same limit and lead to the same results. In order to generate systematically the $1/n$ expansion it is convenient to add to the action a Lagrange multiplier for the condition $\sigma^2+\vec{\Pi}^2=1$ and to integrate over the $\vec{\Pi}$ field:

$$Z=\int\left(d\sigma\, d\vec{\Pi}\, d\alpha\right)\exp\left(-\frac{1}{T}\int d^dx\{\tfrac{1}{2}[(\partial_\mu\sigma)^2+(\partial_\mu\pi)^2]-H\sigma(x)-\tfrac{1}{2}\alpha(x)[1-\pi^2(x)-\sigma^2(x)]\}\right).$$

The integral over $\vec{\Pi}$ gives, up to an irrelevant constant factor,

$$Z=\int(d\sigma d\alpha)\exp\left(-\frac{1}{T}\int d^dx\{\tfrac{1}{2}(\partial_\mu\sigma)^2-H\sigma(x)+\tfrac{1}{2}\alpha(x)\sigma^2(x)-\tfrac{1}{2}\alpha(x)+\tfrac{1}{2}(n-1)T\,\mathrm{tr}\ln[-\Delta+\alpha(x)]\}\right). \tag{39}$$

The steepest-descent method applied to the functional integral (39) generates systematically the $1/n$ expansion.[8] The saddle point in an homogeneous external field is given as

$$H=\alpha_s\sigma_s\,, \tag{40a}$$

$$\frac{(n-1)T}{2}\,\mathrm{tr}\left(\frac{1}{-\Delta+\alpha_s}\right)=\frac{1-\sigma_s^2}{2}\,. \tag{40b}$$

At lowest order neglecting the fluctuations around the saddle point, one recovers the spherical-model limit. Indeed the free energy is the value of the action in (39) at the saddle point. The derivative with respect to H gives the equation of state $M=M(H,T)$

$$(n-1)T\int\frac{d^dp}{p^2+H/M}=1-M^2. \tag{41}$$

Introducing the renormalized quantities

$$M=Z^{1/2}M_R,\quad H=Z_1Z^{-1/2}H_R,\quad T=tZ_1, \tag{42}$$

the equation (41) has a finite limit, provided one chooses

$$Z^{-1}=Z_1^{-1}=1-\frac{(n-1)}{(d-2)}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)t \tag{43}$$

and one recovers then the equation of state of the spherical model[19]

$$1-M^2=\frac{(n-1)t}{(d-2)}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\left[1-\left(\frac{H}{M}\right)^{(d-2)/2}\right]. \tag{44}$$

The coefficients $W(t)$ and $\zeta(t)$ of the renormalization-group equations are easily deduced from (43) by Eqs. (15)–(16):

$$W(t)=(d-2)t-(n-1)t^2\Gamma(1-\tfrac{1}{2}(d-2))\Gamma(1+\tfrac{1}{2}(d-2)), \tag{45}$$

$$\zeta(t)=(n-1)t\Gamma(1-\tfrac{1}{2}(d-2))\Gamma(1+\tfrac{1}{2}(d-2)). \tag{46}$$

This yields the spherical model exponents

$$1/\nu=-W'(t_c)=d-2+O(1/n), \tag{47}$$

$$\eta=\zeta(t_c)-(d-2)=0+O(1/n), \tag{48}$$

as expected from (44).

An analysis identical to the one performed in Ref. 2 on the large-n limit of the ϕ^4 theory shows that the nonlinear model in the $1/n$ expression is meaningful for fixed d up to $d=4$.

It is instructive to compare the functional (39) to the analogous functional integral which generates the $1/n$ expansion of the linear σ model.

The initial linear model is

$$Z_L=\int(d\sigma d\vec{\Pi})\exp\left[-\int d^dx\left(\tfrac{1}{2}[(\partial_\mu\sigma)^2+(\partial_\mu\vec{\Pi})^2]-H\sigma(x)+\frac{g_0}{4!}(\sigma^2+\vec{\Pi}^2)^2+\tfrac{1}{2}m_0^2(\sigma^2+\vec{\Pi}^2)\right)\right] \tag{49}$$

and if one introduces a Gaussian source $\alpha(x)$ conjugate to $\sigma^2+\vec{\Pi}^2$ one obtains

$$Z_L=\int(d\sigma d\vec{\Pi} d\alpha)\exp\left[-\int d^dx\left(\tfrac{1}{2}[(\partial_\mu\sigma)^2+(\partial_\mu\vec{\Pi})^2+\alpha(x)(\vec{\Pi}^2+\sigma^2)]-H\sigma(x)-\frac{3}{2g_0}[\alpha(x)-m_0^2]^2\right)\right]$$

or, after integration over $\vec{\Pi}$,

$$Z_L=\int(d\alpha d\sigma)\exp\left[-\int dx\left(\tfrac{1}{2}[(\partial_\mu\sigma)^2+\alpha(x)\sigma^2(x)]-H\sigma(x)-\frac{3}{2g_0}[\alpha(x)-m_0^2]^2+\frac{n-1}{2}\,\mathrm{tr}\ln[-\Delta+\alpha(x)]\right)\right]. \tag{50}$$

Up to a normalization of the field variables the linear problem (50) differs from the nonlinear one (39) only by the presence of an α^2 term. In the low-momentum region the α propagator, obtained by taking the inverse of the quadratic form at the saddle point, behaves as p^{4-d}.

Therefore this additional α^2 term, treated as a perturbation over the nonlinear problem (39), has the effect of adding one α propagator on one line of a given diagram, giving an additional power p^{4-d} in the low-p limit. It is thus a perturbation irrelevant to the leading low-momentum behavior of the theory. This justifies, within the $1/n$ expansion, the identification of the long-distance limits of the two models. Furthermore the nonlinear model is exactly scale invariant, whereas the linear model has this property only when the coupling constant is at the infrared stable fixed point. Let us illustrate these remarks by calculating the renormalized equation of state of the linear model at leading order. Keeping for the functional (50) the saddle-point contribution, exactly as was done for the nonlinear model (39), we find the equation for the bare quantities

$$-M^2+\frac{6}{g_0}\left(\frac{H}{M}-m_0^2\right)=(n-1)\int\frac{d^dp}{p^2+H/M}, \tag{51}$$

which is renormalized through the relations

$$\frac{1}{g_0}=\frac{1}{g}-\frac{n-1}{6}\int\frac{d^dp}{(p^2+\mu^2)^2},$$

$$\frac{m_0^2}{g_0}=\frac{m^2}{g}-\frac{n-1}{6}\int\frac{d^dp}{p^2}.$$

The renormalized equation of state is thus

$$-M^2-\frac{6}{g}m^2=(n-1)\left(\frac{H}{M}\right)^{d/2-1}\frac{\Gamma(d/2)\Gamma(2-d/2)}{d-2}+\frac{H}{M}\left[(n-1)\frac{1}{2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)-\frac{6}{g}\right]. \tag{52}$$

This equation differs from (44), the large-n limit of the nonlinear σ model, by the presence of a term linear in H/M which is subleading in the critical region (in which H/M is small), provided the dimension is smaller than four; thus, as expected, (52) and (44) coincide in the critical region provided one rescales the magnetization and the field by a factor $\sqrt{t}$ and that one relates the mass to the temperature by

$$\frac{1}{t}=\frac{n-1}{d-2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)-\frac{6}{g}m^2. \tag{53}$$

The mass m^2 is indeed a linear measure of the temperature near t_c. Furthermore if the coupling constant g takes its infrared fixed point value

$$\frac{1}{g^*}=\frac{n-1}{12}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right), \tag{54}$$

the two equations of state become as expected identical.

VII. EXPLICIT CALCULATIONS AT TWO-LOOP ORDER

For a given correlation function, at each finite order in the temperature, only a finite number of terms generated by the expansion of the action in powers of the Π field are needed. For instance, in order to calculate the propagators at two-loop order (i.e., the orders t, t^2, and t^3) it is sufficient to keep terms up to order $(\vec{\Pi}^2)^3$ in the action.

In order to calculate the renormalization constants the following procedure will be applied: A magnetic field is introduced to avoid all infrared divergences; then dimensional regularization is used, and the renormalization constants are chosen in order to remove all divergences in two dimensions.

The two renormalization constants Z and Z_1 can be deduced from the calculation of the inverse transverse propagator in a field.

A. One-loop order

The calculation is very simple and yields, after the cancellation of the divergent contribution coming from the measure with the quadratic divergence of the one-loop diagram,

$$\Gamma^{(2)}(p,t,H,\mu)=\frac{Z}{Z_1}\frac{\mu^{d-2}}{t}\left(p^2+\frac{HZ_1}{\sqrt{Z}}\right)+\left(p^2+\frac{n-1}{2}H\right)\int\frac{d^dq}{q^2+H}, \tag{55}$$

which is made finite with the choice

$$Z=1-(n-1)t\mu^{2-d}\int\frac{d^dp}{p^2+\mu^2}+O(t^2),$$

$$Z_1=1-(n-2)t\mu^{2-d}\int\frac{d^dp}{p^2+\mu^2}+O(t^2).$$

Within the $d-2$ expansion at this order it is sufficient to keep the pole terms:

$$Z=1+(n-1)t/(d-2), \tag{56}$$

$$Z_1=1+(n-2)t/(d-2), \tag{57}$$

from which follow the formulas

$$W(t)=(d-2)t-(n-2)t^2+O(t^3), \tag{58}$$

$$\zeta(t)=(n-1)t+O(t^2). \tag{59}$$

Thus the renormalized two-point function is

$$\Gamma^{(2)}(\vec{p},t,H,\mu=1)=\frac{1}{t}(p^2+H)-\frac{1}{2}\left(p^2+\frac{n-1}{2}H\right)\ln H+O(t,d-2). \tag{60}$$

The equation of state is recovered from

$$H/Mt=\Gamma^{(2)}(p=0). \tag{61}$$

In order to put these relations under the scaling forms (27) and (29), let us compute the spontaneous magnetization and the correlation length from Eqs. (22) and (23): at this order one finds

$$\xi(t)=\mu^{-1}(1-t/t_c)^{-1/(d-2)}(t/t_c)^{1/(d-2)}, \tag{62}$$

$$\sigma(t)=(1-t/t_c)^{(n-1)/2(n-2)}. \tag{63}$$

The equation of state in scaling form will be given below at two-loop order; the correlation function at this order is

$$t\Gamma^{(2)}=p^2+H-\tfrac{1}{2}t[p^2+\tfrac{1}{2}(n-1)H]\ln H+O(t^2). \tag{64}$$

At the critical point, this may be written in the scaling form

$$t_c\Gamma^{(2)}(t_c,p,H)=H/M+p^2H^{-2\eta/(d+2-\eta)}.$$

It is not difficult to calculate other correlation functions at the same order. For instance the longitudinal propagator

$$G_L(p)=\int dx\, e^{ipx}\langle[1/Z-\Pi^2(x)]^{1/2}[1/Z-\Pi^2(0)]^{1/2}\rangle_{\text{connected}}$$

is found to be

$$\frac{2}{(n-1)t^2}G_L(p)=I_1(p,H)+\frac{d-2}{2}I_2(p,H)$$
$$+t[(p^2+2H)-(n-1)(p^2+H)]I_1(p,H)$$
$$-\tfrac{1}{2}(n-3)t\ln H-\tfrac{1}{4}(n-3)t\ln H\left(H\frac{\partial I_1}{\partial H}\right),$$

in which we have defined

$$I_1(p,H)=\int_0^1 dx[p^2x(1-x)+H]^{-1}$$
$$=\frac{2}{p(4H+p^2)^{1/2}}\ln\frac{(4H+p^2)^{1/2}+p}{(4H+p^2)^{1/2}-p},$$

$$I_2(p,H)=\int_0^1 dx[p^2x(1-x)+H]^{-1}\ln[p^2x(1-x)+H].$$

B. Two-loop order

An analogous calculation of the transverse two-point correlation function of the Π field gives at two-loop order, after cancellation of the quadratic divergences,

$$\Gamma^{(2)}(p)=\frac{Z}{Z_1t}\left(p^2+\frac{HZ_1}{\sqrt{Z}}\right)+\left(p^2Z+\frac{n-1}{2}HZ_1\sqrt{Z}\right)\int\frac{dq}{q^2+HZ_1/\sqrt{Z}}+t\left(\frac{3n-5}{2}\right)\left(p^2+\frac{n-1}{4}H\right)\int\frac{dq_1dq_2}{(q_1^2+H)(q_2^2+H)}$$
$$-\frac{t}{2}(n-3)[p^2+\tfrac{1}{2}(n-1)H]\int\frac{dq_1dq_2}{(q_1^2+H)^2(q_2^2+H)}-\frac{t}{2}(n-1)\int dq_1dq_2\frac{[(p+q_1+q_2)^2-(q_1+q_2)^2]^2}{(q_1^2+H)(q_2^2+H)[(p+q_1+q_2)^2+H]}$$
$$-t\int dq_1dq_2\frac{[(p+q_1)^2-q_1^2][(p+q_2)^2-q_2^2]}{(q_1^2+H)(q_2^2+H)(p+q_1+q_2)^2+H}. \tag{65}$$

The renormalization constants which make this express finite are

$$Z=1+\frac{(n-1)t}{d-2}+(n-1)(n-\tfrac{3}{2})\frac{t^2}{(d-2)^2}+O(t^3), \tag{66}$$

$$Z_1=1+\frac{(n-2)t}{d-2}+\frac{t^2}{(d-2)^2}(n-2)[n-2+\tfrac{1}{2}(d-2)]+O(t^3), \tag{67}$$

from which one deduces by Eqs. (15) and (16),

$$W(t)=(d-2)t-(n-2)t^2-(n-2)t^3+O(t^4), \tag{68}$$

$$\zeta(t)=(n-1)t+O(t^3). \tag{69}$$

Thus the (renormalized) critical temperature t_c defined by $W(t_c)=0$ is

$$t_c=\frac{d-2}{n-2}-\frac{(d-2)^2}{(n-2)^2}+O((d-2)^3). \tag{70}$$

This second-order contribution to t_c depends on the renormalization conditions and is therefore nonuniversal. However the exponents $\nu=-1/W'(t_c)$ and $\eta=\zeta(t_c)-(d-2)$ are indeed universal and are given from (68) and (69) as[4]

$$\frac{1}{\nu}=(d-2)+\frac{(d-2)^2}{n-2}+O((d-2)^3), \tag{71}$$

$$\eta = \frac{d-2}{n-2} - \frac{(n-1)}{(n-2)^2}(d-2)^2 + O((d-2)^3). \tag{72}$$

With the choice (66) and (67) of Z and Z_1 the inverse propagator $\Gamma^{(2)}$ becomes cutoff independent and it may be written

$$\begin{aligned}\Gamma^{(2)} = {} & \frac{1}{t}(p^2+H) - \tfrac{1}{2}[p^2+\tfrac{1}{2}(n-1)H]\left(\ln H + \frac{d-2}{4}\ln^2 H\right) + \frac{n-1}{32}(3n-5)tH\ln^2 H + \frac{t}{8}(n-3)\left(p^2+\frac{n-1}{2}H\right)\ln H \\ & + \frac{(n-1)t}{2}\int \frac{[(p+q_1+q_2)^2-(q_1+q_2)^2]^3}{(q_1^2+H)(q_2^2+H)[(q_1+q_2)^2+H][(p+q_1+q_2)^2+H]}\,d^dq_1\,d^dq_2 \\ & - p^2t\left((p^2+H)+\frac{n-1}{2}(p^2-2H)\right)\int \frac{1}{(q_1^2+H)(q_2^2+H)[(q_1+q_2)^2+H]}\,d^dq_1\,d^dq_2 \\ & + t\int \frac{[(p+q_1)^2-q_1^2][(p+q_2)^2-q_2^2][(p+q_1+q_2)^2-(q_1+q_2)^2]}{(p_1^2+H)(q_2^2+H)[(q_1+q_2)^2+H][(p+q_1+q_2)^2+H]} + tp^2[\tfrac{1}{8}(n-1)\ln^2 H - \tfrac{1}{2}(n-2)\ln H].\end{aligned} \tag{73}$$

This transverse propagator contains the equation of state, as seen from Eq. (61).

The explicit calculation from Eq. (73), gives this equation under the form

$$\frac{H}{M^\delta} = \left(\frac{t}{t_c}\right)^{-2/(d-2)}\left(1+t-t_c-\frac{1-t/t_c}{1/M^\beta}\right)^{2/(d-2)}.$$

In this equation the only universal features are the low-temperature singularity $t^{-2/(d-2)}$ and the critical behavior for t near t_c; and neglecting corrections to scaling it may be replaced by

$$\frac{H}{M^\delta} = \left(\frac{t}{t_c}\right)^{-2/(d-2)}\left(1+\frac{t/t_c-1}{M^{1/\beta}}\right)^{2/(d-2)}. \tag{74}$$

This equation, as such, does not fulfill Griffith's analyticity[20]: It is only in the sense of an expansion in powers of $d-2$ that it is satisfied since at lowest order the exponent γ is equal to $2/(d-2)$.

VIII. BEHAVIOR ABOVE T_c, LIMIT OF DIMENSION TWO

The theory which has been developed until now is restricted in zero field to $t \leq t_c$, the phase of spontaneous symmetry breaking. In particular in two dimensions this phase disappears since the transition takes place at zero temperature. In the presence of an external field both the Goldstone and the critical singularities disappear and this allows one to extend the low-temperature expansion above t_c. From the renormalization-group equations also, whose coefficients are regular at t_c, one can obtain information about the high-temperature phase. At t_c, in zero field, the $\vec{\Pi}$ and the σ propagators are proportional to the same power of momentum; the coefficients in front of these powers are identical as a consequence of the Ward-Takahashi identities. The symmetry is thus explicit at the critical point. This suggests, and we know it for the Heisenberg model, that above T_c the symmetry between the $\vec{\Pi}$ and the σ is restored, and that they all become massive with a mass $m = \xi^{-1}$, which can be deduced from the renormalization-group equations since from (20) we have

$$\xi(t) = \xi_0 \exp\left(-\int_t^{t_0}\frac{dt'}{W(t')}\right). \tag{75}$$

Application of this formula to the two-dimensional case gives a mass scale related to the coupling constant t by

$$m(t) = \mu\, t^{-1/(n-2)} e^{-1/t(n-2)} f(t) \tag{76}$$

in which $f(t)$ is regular around $t=0$, an expression that perturbation theory could not give. The exponential term of Eq. (76) had been obtained by Polyakov.[7] This gives the solution to the problem of infrared slavery in this asymptotically free model. This calculation, however, cannot give the normalization of this mass. In order to obtain a quantitative result about the absolute normalization of the mass, it is necessary to calculate the correlation function in the symmetric phase. As said above, it is thus necessary to first add a magnetic field, use then perturbation theory in the coupling constant, extrapolate this perturbation series above t_c, with the help of the renormalization-group equations, and then take the zero-field limit. A possible procedure of this nature will be presented now.

Explicit calculations in the high-temperature phase rely on the following procedure, which has been applied here for the equation of state. One first calculates the low-t expansion; then this polynomial in t is put in a form dictated by the renormalization-group equation, which involves the scaling variable

$$z = 1-(1-t/t_c)/M^{1/\beta},$$

small in the ordered phase. This variable goes to plus infinity in the disordered phase when the applied field vanishes. The asymptotic behavior in z is known from scaling. If one is interested in a quantity such as the amplitude of the susceptibility above t_c, the problem consists in finding the coefficient in front of the large-z power law from the knowledge of the Taylor series near $z=0$. This may be done by various numerical techniques such as Padé approximants. A more precise illustration of the difficulties and of a possible solution is provided by the example of the two-dimensional problem for which there is no ordered phase. The low-temperature expansion of the magnetization is found to be

$$\frac{1}{M} = 1-(n-1)\frac{t}{4}\ln H + \frac{(n-1)(3n-5)}{32}t^2\ln^2 H + \frac{(n-1)(n-3)}{16}t^2\ln H + O(t^3). \tag{77}$$

The renormalization-group equation (29) implies the scaling property

$$H\frac{\xi^2(t)\sigma(t)}{t} = f\left(\frac{M}{\sigma(t)}\right) \tag{78}$$

in which ξ and σ are defined by integration of (20) and (21) up to a multiplicative factor.

The expansion (77) determines $f(x)$ for large x:

$$f(x) \underset{x\to\infty}{\sim} x^{(n-7)/(n-1)} \exp\left(\frac{2}{n-2}(x^{2(n-2)/(n-1)}-1)\right). \tag{79}$$

Two facts may be added to the previous calculations:

(i) From the structure of the low-t expansion it is straightforward to verify that

$$x\frac{d\ln f(x)}{dx} = x^{2(n-2)/(n-1)} \sum_k a_k x^{-2k(n-2)/(n-1)} \tag{80}$$

for x large.

(ii) Griffiths's analyticity, i.e., the fact that since there is no transition, H is analytic and odd in M for small M, implies

$$f(x) = \sum_k b_k x^{2k+1} \tag{81}$$

for x small.

These two constraints can be incorporated through the following standard parametrization[21]

$$x = \frac{\theta}{(1-\theta^2)^{(n-1)/2(n-2)}}, \quad -1<\theta<1 \tag{82}$$

$$\frac{x\,d\ln f(x)}{dx} = \frac{1}{1-\theta^2}\left(1+\sum_{k=1} C_k\theta^{2k}\right). \tag{83}$$

From the low-temperature expansion we know a few terms of the expansion near $\theta=1$. It is thus possible to apply various extrapolation methods, the most naive being the truncation of the series $\sum C_k\theta^{2k}$. This is what has been applied here. It yields

$$\frac{x\,d\ln f(x)}{dx} = \frac{1}{1-\theta^2}\left(1-\frac{n^2-5}{(n-1)^2}\theta^2 + \frac{6n-10}{(n-1)^2}\theta^4\right),$$

from which we obtain

$$f(\theta) = \exp\left(\frac{-1+7n-4n^2}{(n-1)^2(n-2)}\right)\theta(1-\theta^2)^{-(n-7)/2(n-2)} \times \exp\left(\frac{2}{n-2}\frac{1}{1-\theta^2} + \frac{\theta^2(3n-5)}{(n-2)(n-1)^2}\right). \tag{84}$$

This gives for instance an estimate of the magnetic susceptibility

$$\frac{M}{H}\underset{H\to 0}{=} \frac{\xi^2(t)\sigma^2(t)}{t}\exp\left(\frac{1+3n-2n^2}{(n-1)^2(n-2)}\right) \tag{85}$$

and in the low-temperature region, our renormalizations are such that

$$\frac{\xi^2\sigma^2}{t} \underset{t\to 0}{\sim} e^{2/(n-2)t}t^{3/(n-2)}. \tag{86}$$

It is thus possible to make explicit calculations in the high-temperature phase by this method.

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[1]K. G. Wilson and J. B. Kogut, Phys. Rep. 12C, 75 (1974).

[2]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, to be published), Vol. VI.

[3]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[4]Recently D. R. Nelson has exponentiated the first-order coexistence-curve singularities which occur in the 4-d expansion through a parquet approximation [Phys. Rev. B 13, 2222 (1976)].

[5]E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976).

[6]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[7]A. M. Polyakov, Phys. Lett. B 59, 79 (1975); this has also been shown through recursion methods by A. A. Migdal {Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys.–JETP (to be published)]}.

[8]See, for instance, Ref. 2.

[9]It is the absence of mass renormalization in this model which enables us to interpret the coupling constant as

a temperature; in the $g_0\phi^4$ theory it is also possible to change the scale of ϕ in such a way that the Lagrangian becomes $(1/g_0)\,\mathcal{L}(\phi)$ with the coefficient of ϕ^4 in $\mathcal{L}(\phi)$ equal to unity. However, the theory has an additional dependence in g_0 through the bare mass.

[10]B. D. Josephson, Phys. Lett. 21, 608 (1966); B. Halperin and P. Hohenberg, Phys. Rev. 177, 952 (1969).

[11]E. Brézin and D. J. Wallace, Phys. Rev. B 7, 1967 (1973); D. J. Wallace and R. K. P. Zia, Phys. Rev. B 12, 5340 (1975).

[12]K. Symanzik, Desy report No. 73/39 (unpublished).

[13]V. Berezinskii, Zh. Eksp. Teor. Fiz. 59, 907 (1970); 61, 1144 (1971) [Sov. Phys.-JETP 32, 493 (1971) ; 34, 610 (1972)] .

[14]This may be an artifact of this simplified continuous theory. In an XY model solved by A. Luther and D. Scalapino (unpublished), there is also a finite T_c. On the other hand, K. Wilson (private communication), through a more-detailed analysis of Migdal's recursion formula, has found contributions to $W(T)$ of the form $-e^{-a/T}$ and a vanishing critical temperature.

[15]N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1960); P. C. Hohenberg, Phys. Rev. 158, 383 (1967). In field theory this theorem has been derived by S. Coleman [Commun. Math. Phys. 31, 259 (1973)] .

[16]S. Coleman, Phys. Rev. D 11, 2088 (1975).

[17]In our normalizations $t=4$ corresponds to the maximum value 8π derived by S. Coleman (Ref. 15).

[18]S. K. Ma, Rev. Mod. Phys. 45, 589 (1973); K. G. Wilson, Phys. Rev. D 7, 2911 (1973). See also Ref. 2.

[19]T. H. Berlin and M. Kac, Phys. Rev. 86, 821 (1952).

[20]R. B. Griffiths, Phys. Rev. 158, 176 (1967).

[21]B. D. Josephson, J. Phys. C 2, 1113 (1969).