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Kinetic theory of the critical behavior of transport coefficients in many-component Bose systems

J. Sak

Department of Physics, Rutgers University, New Brunswick, New Jersey 08903

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The dynamic spherical model introduced by Ma and Senbetu is investigated in the neighborhood of the critical point. Singular parts of the transport coefficients are calculated. The heat conductivity diverges as $\tau^{(d-4)/(d-2)}$ where τ is the reduced temperature and d is the dimension of the space ($2 < d < 4$). The singular part of viscosity is proportional to τ. The volume viscosity is zero.

I. INTRODUCTION

Most of the present theories of dynamical critical properties (transport and kinetic coefficients, frequency-dependent correlation functions) are of phenomenological nature, being based either on mode-mode coupling ideas[1] or on Langevin-type stochastic equations.[2] Although these theories account for the existing experimental data, they have not been derived from the microscopic equations of motion.[3]

Recently, Ma and Senbetu[4] made progress in the direction of constructing a theory based on first principles. They studied a system of m coupled Bose fields in the limit $m\to\infty$ (dynamical spherical model). With the Hamiltonian as the only input they derived a kinetic equation and from this they calculated transport coefficients: heat conductivity, viscosity, and sound attenuation. The results are unexpected. In particular, the heat conductivity and the sound attenuation do not diverge at the critical point. The viscosity is also finite, but this is to be expected.

In this paper the dynamical spherical model is taken up again for the case $T>T_c$ and the transport coefficients are computed. The results are as follows: Let p_c be the inverse correlation length, $p_c \approx \text{const}\,\tau^{\nu}$ for $\tau\to 0$, where $\tau=(T-T_c)/T_c$, $\nu=(d-2)^{-1}$, d is dimensionality of space. Then the heat conductivity $\kappa\approx\text{const}\,p_c^{d-2-z}$, the viscosity $\eta\approx\text{const}+\text{const}'p_c^{d-z}$, and the sound attenuation $\Gamma\approx\text{const}\,p_c^{d-z-2}$, where the dynamical critical exponent $z=2$. The above forms of the temperature dependence of transport coefficients are in agreement with the predictions of dynamical scaling hypothesis[5,6] and are consistent with the mode-mode coupling approach which was applied to this problem by Halperin.[1]

The reason why our results are different from those of Ma and Senbetu is the following. Both papers use approximations which replace the collision operator by a simpler expression. In this paper, in contrast to Ma and Senbetu, the approximation was chosen so as to preserve the conservation laws of the number of particles, momentum, and energy. Another difference, which is of more technical nature, is that our kinetic equation contains the Vlasov term. As a consequence it yields the hydrodynamic modes directly without the necessity of using the two-step procedure in which one solves the kinetic equation without the Vlasov term and subsequently calculates the zeros of the dielectric function in the complex frequency plane.

After formulating the kinetic equation in Sec. II, we derive the kinetic-theoretical formulas for transport coefficients in Sec. III. Approximate eigenvalues and eigenfunctions of the collision operator are found in Sec. IV and are used to find the singular part of transport coefficients in Sec. V.

II. KINETIC EQUATION

Our starting point will be the kinetic equation for the one-particle distribution function. It was derived by Ma and Senbetu[4] by summing diagrams in the leading order in the number of fields, m. Since the limit is performed at constant concentration of particles for each field, the total concentration goes to infinity as $m\to\infty$. By analogy with the theory of electron gas[7] we can argue that in this limit the random-phase approximation (RPA) becomes exact. So we just write down the RPA-Boltzmann equation: Maxwell-Boltzmann statistics is replaced by Bose-Einstein, the interactions are screened by the dielectric constant, and the Vlasov term must be concluded. Let $f_i(\vec{r},\vec{p},t)$ be the one-particle distribution function, $i=1,\ldots,m$. It satisfies the following kinetic equation:

$$\frac{\partial f_i}{\partial t}+\vec{p}\cdot\frac{\partial f_i}{\partial \vec{r}}-u\,\frac{\partial N}{\partial \vec{r}}\cdot\frac{\partial f_i}{\partial \vec{p}}=I_i\,. \tag{1}$$

We put the mass of the particle equal to unity, u

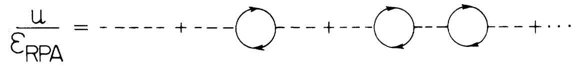

FIG. 1. Effective interaction in the random-phase approximation.

is the coupling constant [the interaction potential is $V(\vec{r}_1, \vec{r}_2) = u\delta(\vec{r}_1 - \vec{r}_2)$, independent of the type of the particle, u is considered of order m^{-1}, as usual], N is the total concentration,

$$N = \sum_{i=1}^{m} \int \frac{d^d p}{(2\pi)^d} f_i(\vec{r}, \vec{p}, t), \tag{2}$$

and I_i is the collision integral,

$$I_i = \frac{1}{2} \sum_{j=1}^{m} \int\int\int \frac{d^d p_1 d^d p' d^d p_1'}{(2\pi)^{3d}} W_{ij} (2\pi)^d \delta(\vec{p} + \vec{p}_1 - \vec{p}' - \vec{p}_1') \delta\left(\frac{p^2}{2} + \frac{p_1^2}{2} - \frac{p'^2}{2} - \frac{p_1'^2}{2}\right)$$
$$\times \{f_i(\vec{p}) f_j(\vec{p}_1)[1 + f_i(\vec{p}')][1 + f_j(\vec{p}_1')] - f_i(\vec{p}') f_j(\vec{p}_1')[1 + f_i(\vec{p})][1 + f_j(\vec{p}_1)]\}. \tag{3}$$

The collision probability (with the δ function factored out) is determined by the first Born approximation (u is of order m^{-1}) in the screened interaction:

$$W_{ij}(\vec{p}, \vec{p}_1, \vec{p}', \vec{p}_1') = w \equiv 2\pi \left(\frac{u}{\epsilon_{\mathrm{RPA}}(\vec{p} - \vec{p}', \frac{1}{2}p^2 - \frac{1}{2}p'^2)} \right)^2 .$$

The effective interaction is given by the sum of the diagrams in Fig. 1. All we shall need to know is its dimensionality in the momentum transfer, which is $4 - d$:

$$\frac{u}{\epsilon_{\mathrm{RPA}}(q, \epsilon)} = \frac{u}{1 - uq^{d-4} g(r/q^2, \epsilon/q^2)} \underset{q \to 0}{\approx} q^{4-d} g^{-1}, \tag{4}$$

where g is some function.

The quantity r, which is the inverse susceptibility (or the effective chemical potential), is a measure of the deviation of the system from the critical point and is used instead of the reduced temperature. In the spherical model it is related to the chemical potential in a simple way:

$$r = -\mu + uN. \tag{5}$$

The temperature dependence of r is given by the exponent γ as $r \sim \tau^\gamma = \tau^{2/(d-2)}$ (in the spherical model). The correlation length is of the order $r^{-1/2}$.

The equilibrium distribution function is of the Bose-Einstein form with $-r$ playing the role of the chemical potential

$$\phi(\vec{p}) = \left[\exp\left(\frac{r + p^2/2}{T} \right) - 1 \right]^{-1} . \tag{6}$$

The first step in the calculation of transport coefficients is to linearize the kinetic equation. Let us write

$$f_i(\vec{r}, \vec{p}, t) = \phi(\vec{p}) + \phi(\vec{p})[1 + \phi(\vec{p})] \psi_i(\vec{r}, \vec{p}, t), \tag{7}$$

substitute this into (1) and (3) and keep only terms of the first order in ψ_i. We get

$$\frac{\partial \psi_i(\vec{r}, \vec{p}, t)}{\partial t} + \vec{p} \cdot \frac{\partial \psi_i(\vec{r}, \vec{p}, t)}{\partial \vec{r}} - u \frac{\partial \phi(\vec{p})}{\partial \vec{p}} \cdot \sum_{j=1}^{m} \int \frac{d^d p'}{(2\pi)^d} \phi(\vec{p}')[1 + \phi(\vec{p}')] \frac{\partial \psi_j(\vec{r}, \vec{p}', t)}{\partial \vec{r}} \frac{1}{\phi(\vec{p})[1 + \phi(\vec{p})]}$$
$$= \frac{1}{\phi(\vec{p})[1 + \phi(\vec{p})]} \sum_{j=1}^{m} \int\int\int \frac{d^d p_1 d^d p' d^d p_1'}{(2\pi)^{2d}} w \delta(\vec{p} + \vec{p}_1 - \vec{p}' - \vec{p}_1') \delta(p^2 + p_1^2 - p'^2 - p_1'^2)$$
$$\times \phi(\vec{p}) \phi(\vec{p}_1)[1 + \phi(\vec{p}')][1 + \phi(\vec{p}_1')][\psi_i(\vec{p}') + \psi_j(\vec{p}_1') - \psi_i(\vec{p}) - \psi_j(\vec{p}_1)]. \tag{8}$$

The many-component Bose system discussed here has some special properties which can be used to simplify the problem. Suppose that a temperature gradient is applied but there are no gradients in the composition of the system. Since all particles have the same mass and same interactions, no particle diffusion will be triggered by the temperature gradient. Therefore the thermodiffusion coefficient is zero. The pressure gradient also cannot cause diffusion in our model. As a result the particle-diffusion modes are decoupled from the sound, viscous, and thermal modes and the latter group can be studied separately.

Let us introduce the function

$$\psi(\vec{r}, \vec{p}, t) = \sum_i \psi_i(\vec{r}, \vec{p}, t) \tag{9}$$

and sum (8) over i. Furthermore, the Fourier transform of ψ is introduced:

$$\chi(\vec{k}, \vec{p}, t) = \int d^d r\, \psi(\vec{r}, \vec{p}, t) e^{-i\vec{k}\cdot\vec{r}} .$$

We get for χ

$$\frac{\partial\chi}{\partial t}+i\vec{k}\cdot\vec{p}\chi+\frac{mu}{T}\,i\vec{k}\cdot\vec{p}\int\frac{d^dp'}{(2\pi)^d}\,\phi(\vec{p}\,')[1+\phi(\vec{p}\,')]\chi(\vec{k},\vec{p}\,',t)$$

$$=\frac{m}{4}\sinh\frac{E}{2T}\int\int\int\frac{d^dp_1d^dp'd^dp_1'}{(2\pi)^{2d}}\,w\delta(\vec{p}+\vec{p}_1-\vec{p}\,'-\vec{p}_1')\delta(p^2+p_1^2-p'^2-p_1'^2)$$

$$\times\left(\sinh\frac{E_1}{2T}\sinh\frac{E'}{2T}\sinh\frac{E_1'}{2T}\right)^{-1}[\chi(\vec{p}\,')+\chi(\vec{p}_1')-\chi(\vec{p})-\chi(\vec{p}_1)]\equiv K\chi\,, \qquad (10)$$

where the explicit form (6) for ϕ was used and we introduced the notation

$$E=r+\tfrac{1}{2}p^2,\quad E_1=r+\tfrac{1}{2}p_1^2,\ \text{etc.} \qquad (11)$$

The collision operator K is defined by the right-hand side of (10).

III. LINEARIZED HYDRODYNAMICS

To solve (10) we shall use the expansion in the uniformity parameter k. This procedure comes in two equivalent forms, the Enskog-Chapman method[8] or the eigenfunctions method.[9] We use the latter formulation. We look for solutions which decay exponentially with time. Such a solution of (10) with time constants $\lambda_n(k)$ leads to the eigenvalue problem

$$(K-ikp_x-ik\Omega)\chi_n(k,\vec{p})=\lambda_n(k)\chi_n(k,\vec{p})\,. \qquad (12)$$

$\vec{k}$ is assumed to point in the x direction, and Ω is the Vlasov operator:

$$\Omega\chi(\vec{p})=\frac{mu}{T}\,p_x\int\frac{d^dp'}{(2\pi)^d}\,\phi(p')[1+\phi(p')]\chi(\vec{p}\,')\,. \qquad (13)$$

One starts by putting $k=0$. The resulting eigenvalue problem

$$K\chi_n(\vec{p})=\lambda_n^{(0)}\chi_n(\vec{p})$$

has a $(d+2)$-fold zero eigenvalue corresponding to collisional invariants (number of particles, momentum, and energy) which form the null subspace and an infinite number of relaxational modes with nonzero decay frequencies. Let us define the scalar product as

$$\langle\chi(\vec{p}),\psi(\vec{p})\rangle=\int\frac{d^dp}{(2\pi)^d}\,\phi(\vec{p})[1+\phi(\vec{p})]\chi(\vec{p})\psi(\vec{p})\,. \qquad (14)$$

With this definition K is a symmetric operator. It has a negative semidefinite spectrum. The following vectors form an orthonormal basis in the null subspace:

$$\begin{aligned}
\chi_1&=1/\sqrt{C_1}\,,\\
\chi_2&=(1/\sqrt{C_2})p_x\,,\\
\chi_3&=(1/\sqrt{C_2})p_y\,, \qquad\qquad (15)\\
&\vdots\\
\chi_{d+2}&=\frac{1}{\sqrt{C_{d+2}}}\left(p^2-\frac{dC_2}{C_1}\right),
\end{aligned}$$

where, from (14),

$$C_1=\langle 1,1\rangle=\int\frac{d^dp}{(2\pi)^d}\,\phi(1+\phi)\,, \qquad (16)$$

$$C_2=\langle p_x,p_x\rangle=\frac{NT}{m}\,, \qquad (17)$$

$$C_{d+2}=T(d+2)Nm^{-1}\langle p^2\rangle_{\text{av}}-\frac{d^2C_2^2}{C_1}\,. \qquad (18)$$

$\langle p^2\rangle_{\text{av}}$ is the average of p^2 with the equilibrium distribution function (6).

In the next step we let k be different from zero and treat the terms linear in k as a perturbation. Since we want to calculate the transport coefficients, we are interested in the modes which converge to the null space of K in the limit $k=0$. We use the degenerate perturbation theory. The Vlasov operator is not symmetric under the definition of the scalar product (14). The adjoint to Ω is

$$\Omega^T\chi(\vec{p})=\frac{mu}{T}\int\frac{d^dp'}{(2\pi)^d}\,p_x'\,\phi(\vec{p}\,')\,[1+\phi(\vec{p}\,')]\chi(\vec{p}\,')\,,$$

which is different from (13). Therefore we must distinguish between the right and left eigenvectors $\bar{\chi}_\alpha$ and $\tilde{\chi}_\alpha$, respectively. The eigenvalues to first order in k will be denoted by $\lambda_\alpha^{(1)}$. The results are

$$\begin{aligned}
\lambda_1^{(1)}&=ikC_S\,,\\
\lambda_2^{(1)}&=-ikC_S\,, \qquad\qquad (19)\\
\lambda_3^{(1)}&=\cdots=\lambda_{d+2}^{(1)}=0\,.
\end{aligned}$$

The velocity of sound C_S is given by

$$C_S^2 = \frac{1}{d^2}\,\frac{C_{d+2}}{C_2} + \frac{C_2}{C_1} + \frac{mu}{T} C_2 \,, \tag{20}$$

which can be checked to be the inverse adiabatic compressibility.

The right eigenvectors are

$$\bar{\chi}_1 = \frac{1}{\sqrt{2}}\left[-\frac{1}{C_S}\left(\frac{C_2}{C_1}\right)^{1/2} \chi_1 + \chi_2 - \frac{1}{C_S d}\left(\frac{C_{d+2}}{C_2}\right)^{1/2} \chi_{d+2} \right], \tag{21}$$

$$\bar{\chi}_2 = \frac{1}{\sqrt{2}}\left[\frac{1}{C_S}\left(\frac{C_2}{C_1}\right)^{1/2} \chi_1 + \chi_2 + \frac{1}{C_S d}\left(\frac{C_{d+2}}{C_2}\right)^{1/2} \chi_{d+2} \right], \tag{22}$$

$$\bar{\chi}_3 = \chi_3, \ldots, \quad \bar{\chi}_{d+1} = \chi_{d+1} \,, \tag{23}$$

$$\bar{\chi}_{d+2} = \frac{1}{C_S}\left\{ \frac{1}{d}\left(\frac{C_{d+2}}{C_2}\right)^{1/2} \chi_1 - \left[\left(\frac{C_2}{C_1}\right)^{1/2} + \frac{mu}{T}(C_1C_2)^{1/2}\right]\chi_{d+2} \right\}. \tag{24}$$

For the left eigenvectors one obtains

$$\tilde{\chi}_1 = \frac{1}{\sqrt{2}}\left\{ -\frac{1}{C_S}\left[\left(\frac{C_2}{C_1}\right)^{1/2} + \frac{mu}{T}(C_1C_2)^{1/2}\right]\chi_1 + \chi_2 - \frac{1}{dC_S}\left(\frac{C_{d+2}}{C_2}\right)^{1/2} \chi_{d+2} \right\}, \tag{25}$$

$$\tilde{\chi}_2 = \frac{1}{\sqrt{2}}\left\{ \frac{1}{C_S}\left[\left(\frac{C_2}{C_1}\right)^{1/2} + \frac{mu}{T}(C_1C_2)^{1/2}\right]\chi_1 + \chi_2 + \frac{1}{dC_S}\left(\frac{C_{d+2}}{C_2}\right)^{1/2} \chi_{d+2} \right\}, \tag{26}$$

$$\tilde{\chi}_3 = \chi_3, \ \ldots, \quad \tilde{\chi}_{d+1} = \chi_{d+1} \,, \tag{27}$$

$$\tilde{\chi}_{d+2} = \frac{1}{C_S}\left[\frac{1}{d}\left(\frac{C_{d+2}}{C_2}\right)^{1/2} \chi_1 - \left(\frac{C_2}{C_1}\right)^{1/2} \chi_{d+2} \right]. \tag{28}$$

The values of 1 and 2 for α correspond to two sound modes, $\alpha = 3, \ldots, d+1$ are viscous modes and $\alpha = d+2$ is the heat mode.

In the next step we evaluate the eigenvalues to the second order in k, and denote by $\lambda_\alpha^{(2)}$ the term in $\lambda_\alpha(k)$ which is quadratic in k. The perturbation theory gives

$$\lambda_\alpha^{(2)} = \sum_{\beta > d+2} \langle \tilde{\chi}_\alpha, k(p_x + \Omega)\bar{\chi}_\beta \rangle \frac{1}{\lambda_\beta^{(0)}} \times \langle \tilde{\chi}_\beta, k(p_x + \Omega)\bar{\chi}_\alpha \rangle \,. \tag{29}$$

To remove the restriction on β, (29) can be rewritten[10] using the identity

$$\langle \tilde{\chi}_{\alpha'}, (kp_x + k\Omega)\bar{\chi}_\alpha \rangle = i\lambda_\alpha^{(1)} \delta_{\alpha'\alpha} \tag{30}$$

and the spectral resolution of $(K-\epsilon)^{-1}$:

$$\frac{1}{K-\epsilon} = \sum_\beta \bar{\chi}_\beta \frac{1}{\lambda_\beta^{(0)} - \epsilon} \tilde{\chi}_\beta \,. \tag{31}$$

We get

$$\lambda_\alpha^{(2)} = \lim_{\epsilon \to 0^+} \left\langle (kp_x + k\Omega^T - i\lambda_\alpha^{(1)})\tilde{\chi}_\alpha, \frac{1}{K-\epsilon}(kp_x + k\Omega - i\lambda_\alpha^{(1)})\bar{\chi}_\alpha \right\rangle \tag{32}$$

To obtain specific formulas for the attenuation of the hydrodynamic motion we substitute into (32) the values (19) of $\lambda_\alpha^{(1)}$ and the expressions (21)–(28) for the functions $\bar{\chi}_\alpha$ and $\tilde{\chi}_\alpha$. Let us start with the viscous mode, $\alpha = 3$. Equation (32) reduces to

$$\lambda_3^{(2)} = \lim_{\epsilon \to 0^+} k^2 \left\langle \chi_3, p_x \frac{1}{K-\epsilon} p_x \chi_3 \right\rangle . \tag{33}$$

This is to be compared to the relaxation frequency of the viscous mode predicted by hydrodynamics[11,12]:

$$\lambda_3^{(2)} = -(\eta/\rho)k^2 \,, \tag{34}$$

where $\rho = N$ is the density and η the viscosity. Using (15) we have an expression for η:

$$\eta = -\lim_{\epsilon \to 0^+} \frac{1}{C_2} \left\langle p_x p_y, \frac{1}{K-\epsilon} p_x p_y \right\rangle . \tag{35}$$

To get the heat conductivity κ set $\alpha = d+2$ in (32) and equate the result to the hydrodynamic expression:

$$\frac{\kappa}{\rho C_p} = -\frac{C_2}{C_1 C_{d+2}} \frac{1}{C_S^2} \left(1 + \frac{C_1 mu}{T}\right) \lim_{\epsilon \to 0^+} \left\langle p_x \left(p^2 - \frac{d+2}{d}\langle p^2 \rangle_{\text{av}}\right), \frac{1}{K-\epsilon} p_x \left(p^2 - \frac{d+2}{d}\langle p^2 \rangle_{\text{av}}\right)\right\rangle . \tag{36}$$

Let us note that the heat flux in (36) $p_x\{p^2 - [(d+2)/p]\langle p^2\rangle_{\text{av}}\}$ is perpendicular to the null space and coincides with the conjugate flux in the sense of Mori.[13] $(d+2)\langle p^2\rangle_{\text{av}}/2d$ is the enthalpy per particle. As in the case of the Boltzmann gas, kinetic theory automatically produces Kubo-type formulas with conjugate fluxes.

Finally, let us get an expression for sound attenuation. The hydrodynamic result is

$$\Gamma = \frac{1}{\rho}\left[\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\kappa + \frac{2(d-1)}{d}\eta + \zeta\right] \equiv \Gamma_\kappa + \Gamma_\eta + \Gamma_\zeta\,, \tag{37}$$

where ζ is the volume viscosity. On the other hand, if we put $\alpha = 1$ in (32), we get after some calculation [using (36)]

$$\Gamma = \Gamma_\kappa - \frac{1}{C_2}\lim_{\epsilon\to 0^+}\left\langle\left(p_x^2 - \frac{1}{d}p^2\right), \frac{1}{K-\epsilon}\left(p_x^2 - \frac{1}{d}p^2\right)\right\rangle. \tag{38}$$

Comparing the angular integrations in the second term of (38) and in (35) it can be shown that

$$\left\langle\left(p_x^2 - \frac{1}{d}p^2\right), \frac{1}{K-\epsilon}\left(p_x^2 - \frac{1}{d}p^2\right)\right\rangle = \frac{2(d-1)}{d}\left\langle p_x p_y, \frac{1}{K-\epsilon}p_x p_y\right\rangle,$$

so that the second term in (38) is equal to Γ_η. The Γ_ζ piece is missing, from which we conclude that

$$\zeta = 0\,, \tag{39}$$

as in the Boltzmann gas.

IV. EIGENVALUE PROBLEM

Our next problem is to calculate the heat conductivity from (36). Actually, all we want to do is to find whether it diverges as $r \to 0$ and what is the exponent of the leading singularity.

Let us first look at the temperature dependence of the factors outside the current-current correlation function in (36) as $r \to 0$. C_p is finite, since the specific-heat exponent $\alpha = (d-4)/(d-2)$ is negative for $2 < d < 4$. C_2 and C_{d+2} are also finite, whereas C_1 diverges as $r^{(d-4)/2}$. C_S remains finite. On the whole, the numerical factor in (36) remains finite at the critical temperature and the singular behavior of κ comes from the correlation function

$$\lim_{\epsilon\to 0^+}\left\langle p_x\left(\frac{p^2}{2} - h\right), \frac{1}{K-\epsilon}p_x\left(\frac{p^2}{2} - h\right)\right\rangle = \lim_{\epsilon\to 0^+}\sum_\beta \frac{\langle p_x(p^2/2 - h), \chi_\beta\rangle^2}{\lambda_\beta^{(0)} - \epsilon}\,, \tag{40}$$

where h is enthalpy per particle.

We need to know the relaxation spectrum $\lambda_\beta^{(0)}$ and the relaxation modes of the system, which means to solve the full eigenvalue problem $K\chi_\beta = \lambda_\beta^{(0)}\chi_\beta$. This is probably impossible to do in closed form and we resort to approximations. An often used argument in the kinetic theory, which was also used by Ma and Senbetu,[4] runs as follows: Write the collision operator $K\chi(\vec{p})$ as a sum of two terms,

$$K\chi(\vec{p}) = \omega(\vec{p})\chi(\vec{p}) + H\chi(\vec{p})\,, \tag{41}$$

where [see (10)]

$$\omega(\vec{p}) = 4m\sinh\frac{p^2/2 + r}{2T}\int\!\!\int\!\!\int \frac{d^dp_1 d^dp' d^dp_1'}{(2\pi)^{2d}}\, w\, \frac{\delta(\vec{p} + \vec{p}_1 - \vec{p}\,' - \vec{p}_1{}')\,\delta(p^2 + p_1^2 - p'^2 - p_{12}')}{\sinh(E_1/2T)\sinh(E'/2T)\sinh(E_1'/2T)}\,, \tag{42}$$

and H is the Hilbert operator consisting of the remaining three terms in K. If we neglected the Hilbert piece of K, the relaxation spectrum would consist of the range of the function $\omega(\vec{p})$. In some cases (when the Hilbert operator is completely continuous[14]) it can be proved that H can at most add discrete values to the spectrum $\omega(\vec{p})$. This is probably not our case, but something else can happen. Look at the integral (42). The collision probability w depends on the momenta as $|\vec{p} - \vec{p}\,'|^{8-2d}$ [see (4)]. We are interested in the low-frequency end of the spectrum and small r and can replace $\sinh x$ by x. Then the integral becomes dimensionless in momenta and we must expect it to depend logarithmically on r or p, whichever cuts if off first in low-momenta regions. However, since the logarithm comes from the integration region where all momenta are small simultaneously it will be suppressed by the corresponding contributions in the Hilbert operator. This can be seen in the full collision operator where the factor $[\chi(\vec{p}_1{}') + \chi(\vec{p}\,') - \chi(\vec{p}_1) - \chi(\vec{p})]$ will be small and there may not be any logarithms in the spectrum after all. In what follows it will be assumed that the spectrum of K is given by the explicit p dependence in front of the integral in (42) and the possibility of logarithms will be ignored. Our results may therefore be wrong by logarithmic factors. We proceed on the assumptions that the relaxation spectrum has the form

$$\omega(\vec{q}) = Cu^2\sinh[(q^2/2 + r)/2T]\,, \tag{43}$$

where C is some constant and q is a d-dimensional vector.

Let us turn to the problem of determining the distribution functions of the relaxation modes which we need in (40). One could try to replace the collision operator by the multiplicative operator $K\chi_{\vec{q}}(\vec{p}) \to \omega(\vec{p})\chi_{\vec{q}}(\vec{p})$. This is a relaxation-time approximation with the difference that a whole spectrum of frequencies is present instead of one. However, this procedure would violate the conservation laws. In particular, the solutions could be taken in the form $\chi_{\vec{q}}(\vec{p}) \propto \delta(\vec{p}-\vec{q})$ but this function is not orthogonal to the null space of K. There exists a generalization of the relaxation-time approximation which is free of this drawback.[15] The preservation of the conservation laws in our treatment accounts for the difference between the results of Ma and Senbetu[4] and ours. Define an operator K_m:

$$K_m\chi(\vec{p}) \equiv \omega(\vec{p})\chi(\vec{p}) - \omega(\vec{p})[\chi_1(\vec{p})\langle\chi_1,\chi\rangle + \chi_2(\vec{p})\langle\chi_2,\chi\rangle + \cdots \chi_{d+2}\langle\chi_{d+2},\chi\rangle] . \tag{44}$$

This operator has the same null space as K:

$$K_m\chi_\alpha(\vec{p}) = 0, \quad \alpha = 1, \ldots, d+2 \tag{45}$$

and the spectrum $\omega(\vec{q})$. Let us solve the eigenvalue problem

$$K_m\chi_{\vec{q}}(\vec{p}) = \omega_{\vec{q}}\chi_{\vec{q}}(\vec{p}) . \tag{46}$$

The eigenvalues are

$$\omega_{\vec{q}} = \omega(\vec{q}) \tag{47}$$

and the corresponding eigenfunctions

$$\chi_{\vec{q}}(\vec{p}) = B_q\delta(\vec{p}-\vec{q}) + P\frac{\omega(p)}{\omega(p)-\omega(q)}[A_1\chi_1(\vec{p}) + A_2\chi_2(\vec{p}) + \cdots A_{d+2}\chi_{d+2}] . \tag{48}$$

P means principal value and A_i's are constants to be determined from the orthogonality conditions $\langle\chi_\alpha, \chi_{\vec{q}}\rangle = 0$:

$$B_q\langle\chi_1, \delta(\vec{q}-\vec{p})\rangle + A_1\left\langle\chi_1, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_1\right\rangle + A_{d+2}\left\langle\chi_1, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_{d+2}\right\rangle = 0 , \tag{49}$$

$$B_q\langle\chi_2, \delta(\vec{q}-\vec{p})\rangle + A_2\left\langle\chi_2, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_2\right\rangle = 0 , \tag{50}$$

$$B_q\langle\chi_{d+2}, \delta(\vec{q}-\vec{p})\rangle + A_1\left\langle\chi_{d+2}, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_1\right\rangle + A_{d+2}\left\langle\chi_{d+2}, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_{d+2}\right\rangle = 0 . \tag{51}$$

We have a system of linear equations for A_i. After solving for A_i one finds B_q from the normalization condition

$$\langle\chi_{\vec{q}}, \chi_{\vec{q}'}\rangle = (2\pi)^d\delta(\vec{q}-\vec{q}') . \tag{52}$$

As we shall see later, we need only solutions for $q^2 \sim r \ll T$. In this limit the equations can be simplified. Consider, for instance, the coefficient of A_2 in (50):

$$\left\langle\chi_2, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_2\right\rangle = 1 + \omega(q)\left\langle\chi_2, \frac{P}{\omega(p)-\omega(q)}\chi_2\right\rangle . \tag{53}$$

The scalar product can be estimated

$$\left\langle\chi_2, \frac{P}{\omega(p)-\omega(q)}\chi_2\right\rangle \approx \frac{T}{C_2}\int\frac{d^dp}{(2\pi)^d}\frac{P}{p^2-q^2} \times \frac{p_x^2}{4\sinh^2[(p^2/2+r)2T]} \approx \text{const}[\max(r, q^2)]^{(d-4)/2} .$$

Since this expression is multiplied by $\omega(q) \sim \max(q^2, r)$ the second term on the right-hand side of (53) behaves as $[\max(q^2, r)]^{(d-2)/2}$, which is small for $d > 2$. Replacing the coefficient of A_2 in (50) by unity we have

$$A_2 = -B_q\langle\chi_2, \delta(\vec{q}-\vec{p})\rangle . \tag{54}$$

The other constants can be estimated by similar arguments, but we shall not need them. The normalization constant B_q has the form

$$B_q = 2\sinh\left(\frac{r+q^2/2}{2T}\right)\beta\left(\frac{q^2}{r}\right) , \tag{55}$$

where β is a dimensionless function. The first factor in (55) normalizes the δ function, and the factor β adjusts the normalization after orthogonalization.

V. TRANSPORT COEFFICIENTS

We are now ready to evaluate the expression (40) giving the singularity in the temperature dependence of the heat conductivity. We need the matrix element (the enthalpy terms do not contribute by orthogonality)

$$\langle p_xp^2, \chi_{\vec{q}}\rangle = B_q\left(\langle p_xp^2, \delta(\vec{p}-\vec{q})\rangle - \left\langle p_xp^2, P\frac{\omega(p)}{\omega(p)-\omega(q)}\chi_2\right\rangle \times \langle\chi_2, \delta(\vec{p}-\vec{q})\rangle\right) . \tag{56}$$

Here only the δ function and the A_2 term in $\chi_{\vec{q}}$ contribute and we also used (54). For small $\vec{q}$ the second term in (56) dominates and gives

$$\langle p_xp^2, \chi_{\vec{q}}\rangle \approx \text{const}\frac{q_x}{\sinh[(r+q^2/2)/2T]} . \tag{57}$$

Thus we get finally for κ

$$\kappa \approx \text{const}\frac{1}{u^2}\int \frac{d^dq}{(2\pi)^d}\frac{q_x^2}{\sinh^3[(r+q^2/2)/2T]}$$

$$\approx \text{const}\frac{1}{u^2}p_c^{d-4}, \tag{58}$$

where p_c is the inverse correlation length, $p_c \sim \sqrt{r}$. Comparing this result with the form expected from scaling

$$\kappa \approx \text{const}p_c^{d-2-z}, \tag{59}$$

we conclude that

$$z = 2. \tag{60}$$

The calculation of viscosity is much simpler and we will just quote the result:

$$\eta \approx \frac{1}{u^2}\text{const} + \text{const}'\frac{1}{u^2}p_c^{d-2}. \tag{61}$$

η remains finite at T_c. This is consistent with scaling, which predicts

$$\eta \approx \frac{1}{u^2}\text{const} + \frac{1}{u^2}\text{const}'p_c^{d-z}. \tag{62}$$

The most singular part of the sound attenuation coefficient is determined by the Γ_κ part and has the same singularity as κ:

$$\Gamma \approx \text{const}p_c^{d-4}. \tag{63}$$

Formulas (39), (59), and (63) are the principal new results of this paper. The result (62) for viscosity is the same as obtained by Ma and Senbetu.[4]

It is gratifying to see that a kinetic-theoretical approach to critical dynamics works and gives sensible results at least for one example. It was crucial that we have a kinetic equation which is valid in the neighborhood of the critical point. Moreover, the equation is quite similar to the Boltzmann equation, so that the machinery of the kinetic theory can be used on it. In contrast to this situation, explicit kinetic equations for realistic systems (e.g., helium) are not known. Their derivation (perhaps by ϵ or $1/n$ expansion of the collision integral) is a major unsolved problem of critical dynamics.

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