Hydrodynamics of vortex generation in flowing superfluid helium

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The hydrodynamic energy for a system of vortices in a flowing superfluid is considered. We obtain the conditions in which the energy of vortices is reduced due to superfluid flow under a specified set of constraints, and the amount of this reduction. The deceleration of the superfluid (or the amount of extra pressure required to maintain constant superfluid flux) on vortex generation is calculated. The general theory is applied to two simple situations: superfluid flow in a smooth channel and superfluid flow through an orifice.

I. INTRODUCTION

Onsager[1] and Feynman[2] proposed the existence of quantized vortices to explain the results of experiments on the moment of inertia and meniscus of rotating superfluid helium. Quantized vortices have since been observed in helium in flow through channels[3,4] and orifices,[5] and in the transport of ions.[6,7] In these latter experiments vortices form as rings. The significance of vortices in producing phase slippage and deceleration of the superfluid[4,8] has also since been recognized.

Some nagging questions about production of vortices have, however, largely remained unanswered. One concerns the critical velocity of flow to produce vortices. This question has only been answered by a dimensional argument by Feynman.[2] Several later authors have sought to use the Landau criteria for phonon and roton generation for generation of vortices. Since the momentum of a vortex is not a well-defined quantity owing to the long-range nature of the velocity fields associated with a vortex, they have replaced momentum with impulse in the Landau criterion. The appropriateness of this substitution is, however, far from clear, particularly in bounded media. (This question is discussed at length in Sec. III D.)

In this situation a hydrodynamic calculation of the energetics of vortices in a flowing superfluid is worth doing. We have sought to answer the questions: What are the conditions in which the energy of vortices is reduced because of superfluid flow under a specified set of constraints, and what is the amount of this reduction? Besides the constraints imposed by the geometry of superfluid flow we have found it important to specify whether the flow is with total flux held constant or the total circulation along a given path held constant. With this information in hand, we have obtained the conditions for vortex generation in a given geometry and the amount of deceleration of the superfluid on vortex generation for the case of constant circulation (or the extra pressure required to maintain constant flux). Unambiguous answers to these questions are obtained from purely hydrodynamic considerations and without introducing the concept of impulse. In turn we are then able to clarify the use of the Landau criterion for vortex generation.

There are other more fundamental questions concerning the process of vortex nucleation. As emphasized especially by Vinen,[9] superfluid helium and vortices are described by macroscopic wave functions, which in effect make them "classical" objects. The tunneling barrier from one state to the other involves changes in the wave function over macroscopic distances and is well-nigh impenetrable. The way out of this difficulty is to invoke the presence of surfaces from which vorticity may be introduced into the liquid. This gets one into questions of coherence near a surface and the microscopic nature of the core of a vortex. Such things are very imperfectly understood. We will not be able to say much about this basic question.

The plan of this paper is as follows: We consider, in Sec. II, general properties of the hydrodynamic interaction energy of superfluid flow and vortices. In Sec. III, we specialize this discussion to a superfluid flow in a single channel with a vortex. We derive the condition for vortex generation with specified constraints and clarify the connection between the results here and the interaction energy obtained from a naive use of impulse. For detailed applications, we consider in Sec. IV and V, respectively, two cases: a vortex ring in flowing superfluid in a circular tube and a vortex in front of an aperture through which helium is flowing. The energies of the vortex ring in these two geometries, with superfluid flow specified to be zero, have been considered by Van Vijfeijken, Walraven, and Staas[10] and by Walraven,[11] respectively. One of the results obtained in Sec. IV and V is that the interaction energy for the same flow is larger for the orifice geometry than for the smooth channel. This puts on a quantitative footing the preconception that vortices preferentially form near sharp protuberances.

II. HYDRODYNAMIC ENERGY FOR ARBITRARY FLOW PATTERNS

Our approach is as follows. We suppose that the energy of a container of superfluid with or without vortices—except perhaps when the cores are close to the walls—is given by classical hydrodynamics. The only question about this assumption seems to involve small corrections to the energy of a vortex, due to proper treatment of the core. Our treatment is sufficiently general that these could produce only minor numerical effects.

Next we suppose that we can discuss vortex formation under conditions of no outside interference; that is, we start with a flow pattern in a multiply connected region and ask whether vortices can form and whether they will result in a decrease of the flow. Under these assumptions it is clear that they can form only if the energy with the vortex is no greater than that without—generally we might imagine that with the vortices present some of the pure hydrodynamic energy will be converted into heat.

So far, the subject is quite simple, but then we run into the question of rates and mechanisms of vortex formation. In common with previous work, we find this to be difficult and will not be able to say very much that is useful. A significant advantage of our method is that the question of defining the proper momentum or impulse is entirely avoided.

A. Circulations and fluxes

We now concern ourselves with the classical hydrodynamics of irrotational flow in a multiply connected region. The multiple connection has two sources. First, the vessel may be multiply connected, as in the case of a torus. Second, the presence of vortex rings requires that regions about their cores be excluded from the volume in which the flow is irrotational. Flow in such a case is discussed by Lamb,[12] whose results we now adapt.

In an n-connected region we can, by definition, draw $n-1$ independent closed circuits equipped with arrows to indicate the positive direction. There are, in general, many ways the circuits can be chosen but we shall adopt the requirement that for each vortex there is a circuit which goes just around its core so that it could be shrunk to vanishing size in the absence of the vortex. The remaining circuits, which can still be drawn in more than one way, will be called container circuits.

Now, to make the region simply-connected, we insert mathematical barriers or cuts, each of which is completely bounded by the walls and/or vortices and intersects exactly one circuit, whose name it takes. Such a cut has a front and a back, such that the circuit goes in its positive direction from the front surface, through the fluid, and returns to the back. The integral

$$C=\int \vec{v}\cdot d\vec{l} \tag{2.1}$$

over the circuit in this forward direction is called the circulation and is equal to the velocity-potential discontinuity ($\varphi_{\text{back}}-\varphi_{\text{front}}$) which must exist at the barrier for the given flow pattern. Through each cut there will be a flux of fluid Φ which is positive if it goes through from back to front. We shall adopt the convention of labeling container circulations and container fluxes by greek subscripts (thus C_α, Φ_α, etc.) and vortex circulations and vortex fluxes by latin subscripts (thus C_i, Φ_i, etc.).

There is one subtlety. For each vortex, there are two ways of inserting the cut which satisfy all our conditions. One, usually considered natural, is a cut whose entire boundary is the vortex. The other is an annulus bounded by the vortex and the wall. Physically, it makes no difference. Either method is perfectly proper. On the other hand, for a given flow pattern, the container circuits must not pass through vortex cuts. Thus at least one container circuit will have to go inside the vortex if the cut is outside and outside if the cut is inside. Its circulation in the two cases will differ by the circulation of the vortex.

This is still no problem for a given flow. What we shall do, however, is compare two different flows, one with a vortex added, the other without. It will be appropriate to compare these flows for fixed values of the container circulations when the circuits go through regions unaffected by the nucleation and growth of the vortex. Thus, if we envision a vortex ring nucleating in the interior of the fluid and growing in area, we will want to keep fixed the circulation over the circuit avoiding the vortex. Accordingly, the cut would be made with the vortex as its entire boundary. On the other hand, if we envision the vortex ring nucleating at the wall and growing in from the wall, the circulation along a path through the ring should be fixed; the cut should be made from the ring to the wall.

B. Energy with specified circulations

We now express the energy of the flow pattern in terms of the circulation. Experimentally the condition of constant circulation arises, for example, for superfluid helium rotating in a torus (without any external forces). The energy is then given by

$$2E = Y_{ij}C_iC_j + 2Y_{i\alpha}C_iC_\alpha + Y_{\alpha\beta}C_\alpha C_\beta . \tag{2.2}$$

The fluxes through the cuts i, α, etc., are obtained by

$$\Phi_i = \frac{\partial E}{\partial C_i} = Y_{ij}C_j + Y_{i\beta}C_\beta , \tag{2.3}$$

$$\Phi_\alpha = \frac{\partial E}{\partial C_\alpha} = Y_{\alpha j}C_j + Y_{\alpha\beta}C_\beta . \tag{2.4}$$

Therefore, equivalently,

$$2E = \Phi_i C_i + \Phi_\alpha C_\alpha . \tag{2.5}$$

Also

$$Y_{mn} = Y_{nm} \quad (m, n = i, j \ldots, \ \alpha, \beta \ldots) . \tag{2.6}$$

These equations can be deduced from Lamb, Ref. 12, pp. 54–56, as well as the requirement that Y be a positive definite matrix.

The reader will not fail to notice a close parallel with electric circuits, with voltages corresponding to circulations, currents to fluxes, and Y_{mn} to the admittance matrix, which we shall call it.

C. Energy and enthalpy with some circulations and some flows specified

Since the Φ's are completely determined by the C's we can replace any number of C's with Φ's as independent variables. We shall show the form only for the case where we specify C_i's and Φ_α's. This situation arises, for example, if a piston is pushed at a constant rate through a tube containing superfluid helium so that the flux through the container cut is kept constant. From Eqs. (2.2)–(2.6) one can obtain

$$2E = Z_{\alpha\beta}\Phi_\alpha\Phi_\beta + \overline{Y}_{ij}C_iC_j , \tag{2.7}$$

$$C_\alpha = Z_{\alpha\beta}\Phi_\beta - Z_{\alpha\beta}Y_{\beta j}C_j , \tag{2.8}$$

$$\Phi_i = Y_{i\alpha}Z_{\alpha\beta}\Phi_\beta + \overline{Y}_{ij}C_j , \tag{2.9}$$

where

$$\overline{Y}_{ij} = Y_{ij} - Y_{i\alpha}Z_{\alpha\beta}Y_{\beta j} \tag{2.10}$$

and Z is the inverse of the greek-letter block of Y and plays the role of impedance matrix for irrotational flow in the container. We note that there are no cross terms in the energy between Φ's and C's. (This is a purely mathematical feature of quadratic forms and would be true for any set of C's and Φ's we chose as independent variables.) Thus for a given set of Φ's, the energy is a minimum when no vortices are present. This is a very slightly special case of the generalized Kelvin theorem (Lamb, Ref. 12, article 55) that the energy is a minimum for fixed Φ's when the flow is irrotational.

There is an analogy here with the expression of the energy of a mechanical system as a center-of-mass kinetic energy plus the energy in the center-of-mass frame. This analogy will be exact in a special case considered in Sec. III.

While we have seen that E is not a "potential" for obtaining Φ_i and C_α by differentiation, when Φ_α and C_i are given (as it is for the case where all C's are given), it is easy to obtain such a potential by Legendre transformation.

Consider

$$2H = Y_{ij}C_iC_j + 2Y_{i\alpha}C_iC_\alpha + Y_{\alpha\beta}C_\alpha C_\beta - 2\Phi_\alpha C_\alpha \tag{2.11}$$

to be evaluated when minimized with respect to C_α. By appropriate addition and subtraction (2.11) can be rewritten as

$$\begin{aligned} 2H = Y_{ij}C_iC_j &+ Y_{\alpha\beta}[C_\alpha + Z_{\alpha\gamma}(Y_{\gamma i}C_i - \Phi_\gamma)] \\ &\times [C_\beta + Z_{\beta\delta}(Y_{\delta j}C_j - \Phi_\delta)] \\ &- Z_{\alpha\beta}(Y_{\beta i}C_i - \Phi_\beta)(Y_{\alpha j}C_j - \Phi_\alpha) . \end{aligned} \tag{2.12}$$

Now the second term in (2.12) is a positive definite quadratic form. Therefore H is minimized with respect to C_α when the second term in (2.12) is zero. Thus, using (2.10),

$$2H = \overline{Y}_{ij}C_iC_j + 2Z_{\alpha\beta}Y_{\beta i}C_i\Phi_\alpha - Z_{\alpha\beta}\Phi_\alpha\Phi_\beta . \tag{2.13}$$

Clearly we now have

$$\Phi_i = \frac{\partial H}{\partial C_i}, \quad C_\alpha = -\frac{\partial H}{\partial \Phi_\alpha} . \tag{2.14}$$

We may by analogy call H the enthalpy of a hydrodynamic system.

Now if the flow starts with all $C_i = 0$, and a set of C_α's, we are interested in whether at a later time some C_i's may be nonzero. The total energy T must be unchanged, but this consists of not only E, attributable to the flow pattern of interest, but also of energy which may have gone into heat. This must be at least as great at the later time; therefore E at the later time must be no greater than at the earlier.

Now let us relax the condition of fixed C_α's, supposing that the barriers become physical with possible pressure differences across them. Then the rate of change of T is

$$\frac{dT}{dt} = -\int d\vec{S} \cdot \vec{v} p , \tag{2.15}$$

where p is the pressure difference across the barrier, $\vec{v}$ is the velocity field, and we integrate over the barrier. For the conditions we consider, however, p is just $-\rho(\dot{\phi} + \frac{1}{2}v^2)$, where ϕ is the velocity potential. Under some circumstances, it is a good approximation to neglect differences of ρv^2 across the barrier and set the discontinuity in φ equal to the constant C_i, even when the barriers are physical rather than mathematical. (This is the case, for instance, if the barrier represents a piston far from the ends of a uniform tube.) Then dT/dt becomes $\Phi_\alpha \dot{C}_\alpha$. When Φ_α is constant, then $T-\Phi_\alpha C_\alpha$ is a constant, as T was for constant C_α, and differs from H by the heat energy, as T did from E. Thus for conditions of constant Φ_α, H plays the same role E did for constant C_α.

III. SINGLE VORTEX RING IN A CHANNEL

A. Definitions

Now let us consider a container topologically equivalent to a torus. Its one container circuit will be given $\alpha=0$. We also add a vortex ring, with $i=1$. We eliminate some subscripts by replacing Y_{00}, Y_{10}, Y_{11} by Y, fY, $\overline{Y}_v$, respectively. Thus $\frac{1}{2}\overline{Y}_v C_1^2$ is the energy of a vortex with $\Phi_0=0$ (not $C_0=0$)—a choice made simply because this quantity is a bit more directly calculated and has been calculated for a number of geometries.[10,11] Both Y's and f depended on the geometry of the tube and the vortex. f is Φ_1/Φ_0 for $C_1=0$. Recalling that there are various ways of making the cut, we find some relations between the corresponding f's.

First, consider a full cut across the channel, with front and back chosen in the same sense as for the container cut, which contains the vortex ring. Now remove the part between the ring and the wall. The f for the remaining cut will be f_v. This choice then determines the positive direction for the vortex circuit. If we make the other choice for the vortex cut, namely, the part we just discarded between the ring and the wall, we must reverse back and front if we wish to maintain the same sense for the vortex circuit. Thus, for this case, f is f_v-1. For the other two choices of back and front we would reverse the signs of the f's and of the circulation. Recall also, for the two choices, C_0 differs by C_1, since the container circuit goes on different sides of the ring.

B. Criteria for generating vortices with constant circulation

With these preliminaries out of the way, we can specialize the results of Sec. II as follows:
The energy is givey by

$$2E=Z\Phi_0^2+\overline{Y}_v C_1^2 \tag{3.1a}$$

$$=YC_0^2+2fYC_0C_1+\left|\overline{Y}_v+f^2Y\right|C_1^2, \tag{3.1b}$$

while the flux through the container cut and the vortex cut are given, respectively, by

$$\Phi_0=Y(C_0+fC_1), \tag{3.2}$$

$$\Phi_1=\overline{Y}_v C_1+f\Phi_0. \tag{3.3}$$

The enthalpy (with flux through the container cut specified) is given by

$$2H=\overline{Y}_v C_1^2+2\Phi_0 fC_1-Z\Phi_0^2. \tag{3.4}$$

We note that if C_1 has the right sign to reduce $|\Phi_0|$ it also reduces E linearly, and conversely. This is an immediate consequence of (3.2), since the latter implies that if a vortex is formed with no increase in energy, Φ_0 must be reduced. It is easily checked that while C_0 and f are individually different for the two cuts, C_0+fC_1 is the same—Φ_0 is a physical quantity.

The criterion for forming a vortex without an increase in energy can be written

$$-2fC_1\Phi_0^0\geq(\overline{Y}_v+f^2Y)C_1^2, \tag{3.5}$$

when $\Phi_0^0\equiv YC_0$.

The deceleration produced by the addition of a vortex is clearly

$$\Delta\Phi_0=-fYC_1. \tag{3.6}$$

From (3.6) it is straightforward to derive an equation for the change in superfluid velocity in a given geometry. We do this for two simple geometries in Secs. IV and V.

C. Criteria for generating vortices with constant flux through the container

If we consider the matter at constant flux Φ_0, instead of constant C_0, and assume as suggested above that the appropriate criterion is that H should not increase, we find instead

$$-2fC_1\Phi_0\geq\overline{Y}_v C_1^2, \tag{3.7}$$

and the decrease of circulation through the container cut by addition of a vortex is

$$\Delta C_0=fC_1. \tag{3.8}$$

The first of these represents an easing of the requirement for vortex production compared to (3.5), which can be significant, as will be seen below. The second can be regarded as an impulsive pressure which must be applied to the barrier to maintain the flow in a quasi-steady-state. With ν vortices being produced per unit time, there would be an average pressure of νfC_1 required to maintain the flow.

D. Comparison with previous work

In the past, people have tried to obtain results of this type by following the example of Landau's[13] work on critical velocities for phonon and roton creation. This argument makes essential use of the momentum of an excitation. A vortex, however, has a long-range velocity field, which is not absolutely integrable. The total momentum is therefore undefined in an unbounded region, and highly sensitive to boundaries in bounded regions. Kelvin showed long ago that for many purposes one could use a quantity he called impulse, which is well defined for a vortex and obeys a conservation law like that for momentum, in an unbounded incompressible fluid. It has therefore seemed natural to use impulse in the Landau criterion instead of momentum.

The objections to this procedure are as follows:

(i) The essential property of momentum used by Landau is that the Hamiltonian H of a system in a frame moving with velocity $\vec{V}$ is related to that, H_0, in a frame at rest by $H = H_0 - \vec{P}\cdot\vec{V}$. This was never established for impulse, as far as we know.

(ii) Since the use of impulse in lieu of momentum is based on their shared conservation properties, there is no indication of what to do in situations where neither is conserved, as in a channel with varying cross section.

It could be argued that these objections could be circumvented at least in some cases. We would respond that our method removes the point of doing this. The key advantage of the present method is that within the domain of irrotational hydrodynamics, conservation of circulation is much more powerful and more generally applicable than conservation of either momentum or impulse. Moreover, the energy appears in a form which does not require the use of a moving frame of reference as in the Landau argument.

The relation between our point of view and those of previous authors may be seen most clearly by a further specialization to an ideal torus of cross section A and length L with $A/L^2 \ll 1$. Then $Y = A/L$, $f_v = A_v/A$, and $f^2 Y \ll Y_v$. (No difference arises in this simple geometry for the case of constant circulation and constant flux through the container cut.) Here A_v is the area of the projection of the vortex ring on a cross section of the torus. Equally well, we can think of a straight tube of area A and length L, imposing periodic boundary condition on the velocity. Now, defining $v \equiv \Phi_0^0/A$, (3.5) and (3.7) become, if we make the cut across the vortex ring,

$$-2vC_1A_v \le \overline{Y}_v C_1^2 \tag{3.9}$$

and, if the cut is made between the ring and the wall,

$$2vC_1(A - A_v) \ge \overline{Y}_v C_1^2 . \tag{3.10}$$

The first of these is just what one gets from a Landau argument, using the impulse $A_v C_1$ as the momentum. The second form would be obtained if, instead, Huggins[14] suggestion for a modified impulse were used.

Referring back to the discussion of cuts, we can say unambiguously that if the vortex is nucleated in the interior and grows out, the first form is appropriate, while the second is correct if it grows in from the wall. This correspondence was also made by Huggins[14] on the basis, we feel, of somewhat less rigorous arguments about impulse.

IV. CALCULATIONS FOR A VORTEX RING IN A UNIFORM CHANNEL

Consider the case of a uniform channel with a vortex ring that starts with a radius nearly that of the channel. The cut due to the vortex, S_1 (see Fig. 1), is then the surface between the channel and the ring. The energy in this situation is given by (3.1a) or (3.2), where for a uniform channel of cross section A and length L

$$Y = \Phi_0/C_0 = A/L , \tag{4.1}$$

and f is simply the fraction of cross section occupied by S_1, with appropriate sign. If C_1 is as shown in Fig. 1, i.e., if the flow due to C_1 is opposed to that of C_0 on S_1, f is negative. In this case the interaction term $2fYC_0C_1$ leads to a reduction in energy. The vortex tends to shrink if it can dissipate the energy.

Now we shall consider the energetics of the situation in detail for a circular channel of radius R_t and a vortex of radius R. In (3.1b), the additional term $f^2YC_1^2$ coming from writing the vortex energy in terms of $\overline{Y}_v$ is unimportant if the channel is long enough. By comparison of (3.5) and (3.7) it follows that the condition for vortex generation in the present geometry is the same for the case of specified total circulation and specified total flow.

The interaction term in (3.1b) can be written in terms of the flux Φ_0 through the channel:

$$2fYC_0C_1 = (R_t^2 - R^2)/R_t^2 \Phi_0 C_1 . \tag{4.2}$$

The term $\overline{Y}_v C_1^2$ in (3.1b) has been calculated in this

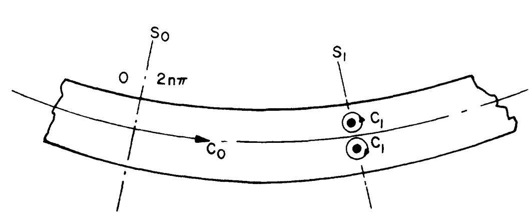

FIG. 1. Geometry of circulation due to flow in a tube and circulation due to a vortex ring.

geometry by Van Vijfeijken, Walraven, and Staas. In Fig. 2, we reproduce their results. For $R \ll R_t$,

$$E_v = \tfrac{1}{2}\rho C_1^2 R[\ln(8R/a) - 2], \quad R \ll R_t \tag{4.3}$$

(a is the core radius), the customary value for energy of a vortex far from any surface. For

$$\delta \equiv (1 - R/R_t) \ll 1, \quad E_v = \tfrac{1}{2}\rho C_1^2 \ln(\delta/\delta_0), \tag{4.4}$$

where $\delta_0 = a/2R_t$. The position of the maximum in E_v, R_c, depends weakly on δ_0 and is around $0.9R_t$. The decrease in E_v from that given by (4.3) is due to the interference of the flow pattern due to the vortex in the channel with the flow pattern due to image vorticity introduced to satisfy the boundary conditions on the surface of the channel. This hydrodynamic calculation of the energy cannot be trusted for $\delta/a \to 1$, but is adequate near $R \sim R_c$.

For vortex generation and deceleration of the superfluid at $T = 0$, conditions (3.5) or (3.7) must be satisfied for all positions (radii) of the vortex from its creation to its eventual disappearance or, if it exists, to the position of a local minima of energy, where it will have a stable radius. The latter is not found in the present geometry. From the foregoing discussion of the energetics, a vortex formed at the walls can contribute to the deceleration only if the interaction energy (4.2) is equal to or larger than that plotted in Fig. 2 for all values of R. In view of the maxima at $R \simeq R_c$, Φ must therefore be larger than the characteristic value Φ_c at which

$$C_1\Phi_c(R_t^2 - R_c^2)/R_t^2 \simeq E_v(R_c). \tag{4.5}$$

For $\Phi \gtrsim \Phi_c$ the vortex nucleated at the wall will ultimately shrink to zero radius. Noting from Fig. 2 that $R_c \simeq 0.9R_t$ and $E_v(R_c) \simeq 3\rho C_1^2 R_t$ for $\delta_0 = 10^{-4}$, we estimate from (4.5) that the velocity to generate vortices is about $0.2C_1/R_t$.

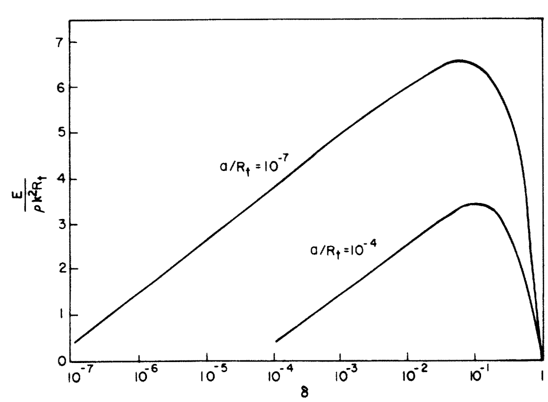

FIG. 2. Normalized energy of a vortex ring of radius R in a circular tube of radius R_t as a function of $\delta = (R_t - R)/R_t$ for zero flow in the tube, after Van Vijfeijken, Staas, and Walraven, Ref. 10.

If we consider vortex nucleation due to thermal fluctuations at finite temperature, the probability of nucleation is proportional to $\exp[-E_c(\Phi_0)/kT]$, where

$$E_c(\Phi_0) = E_v(R_c) - C_1\Phi_c(R_t^2 - R_c^2)/R_t^2, \tag{4.6}$$

where R_c, the position of the maximum in E_c, will depend weakly on Φ_c.

We reiterate that the barrier reduction in (4.6) comes about only for vortices of the proper sign for the circulation, so that the flow pattern of the vortex opposes that of the channel in S_1. The velocity of the vortex along the channel is given by

$$V_z = (2\pi R\rho C_1)^{-1}\frac{dE}{dR} \tag{4.7}$$

for constant container circulation, and

$$V_z = (2\pi R\rho C_1)^{-1}\frac{dH}{dR} \tag{4.8}$$

for constant flux through the container. From (4.7) and (4.8) one can conclude that the vortices that penetrate the barrier and contribute to the deceleration of the superfluid move along the channel in the same direction as the flow through the channel. The deceleration of the superfluid due to a vortex generated at the surface and shrinking to zero radius is, from (3.6),

$$\Delta\Phi = -YC_1 = -AC_1/L. \tag{4.9}$$

If ν is the rate of generation of vortices per unit length of the channel, the force communicated to the fluid is $\rho(\Delta\Phi)\nu L$ and the pressure W is $\rho(\Delta\Phi)\nu L/A$. The superfluid velocity then decreases according to

$$\frac{dv_s}{dt} = \nabla\mu = \nabla\left(\frac{W}{\rho}\right) = -C_1\nu, \tag{4.10}$$

where μ is the chemical potential. Equation (4.9) was written down on dimensional grounds by Langer and Reppy.[4] It was derived, independently, by Huggins.[15] (The present derivation is considerably simpler.)

If we imagine, as has been implicitly assumed in the literature,[4] that a small vortex is nucleated and *grows* to the radius of the tube, a similar analysis shows that such a vortex will move along the channel in a direction opposite to that of the flow through the channel. For the reasons stated in the Introduction we consider such a nucleation hard to envisage and favor nucleation at the walls. An experiment capable of measuring the velocity of a vortex along the channel could distinguish between the two possibilities. Our remarks on the direction of vortex motion are equivalent to the statement that vortices nucleated at the wall have the sense of smoke rings, while those nucleated from vanishing size in the interior have the sense of vortices in the wake behind a sphere.

V. VORTEX RING IN FRONT OF AN ORIFICE

A. Zero net flux through the orifice

Walraven[11] has considered the energy for the geometry sketched in Fig. 3: an orifice of radius R_0 in an infinite plane, with a vortex ring of core radius a and radius R situated axially at a distance z from the orifice. The boundary condition imposed is that there is no net flux through the orifice. In our notation, Walraven has calculated the coefficient $\bar{Y}_v$ in Eq. (3.1a). At $z = 0$, the result[11] is

$$\frac{E_v}{\rho C_1^2 R_0} \equiv \frac{1}{2}\frac{\bar{Y}_v}{\rho R_0} = \frac{R}{2R_0}\left[\ln\left(\frac{8R}{a}\right) - 2\right] + \left(\frac{R}{R_0}\right)^2 + \frac{R}{2R_0}\ln\left|\frac{R-R_0}{R+R_0}\right| . \tag{5.1}$$

For $R \ll R_0$, (5.1) reduces to (4.3). For $R \to R_0$, the energy in terms of $\delta = (R_0 - R)/R_0$ is

$$E_v/\rho C_1^2 R_0 = \tfrac{1}{2}\ln|\delta/\delta_0| \quad \text{for } \delta \to 0 , \tag{5.2}$$

where $\delta_0 = a/4R_0$. In between the two limits $R \to R_0$ and $R \to 0$, we have a maximum of the energy, E_{max}. Walraven's[11] evaluation of Eq. (5.1) is reproduced in Fig. 4, where the curves are stopped at $R_0 - R = a$. The maximum arises at $R_c \simeq 0.9R_0$. Between $R = R_0$ and $R = R_c$, the vortices move in the opposite direction for the same sense of circulation to those outside this radius, as can be discerned by Eq. (4.7).

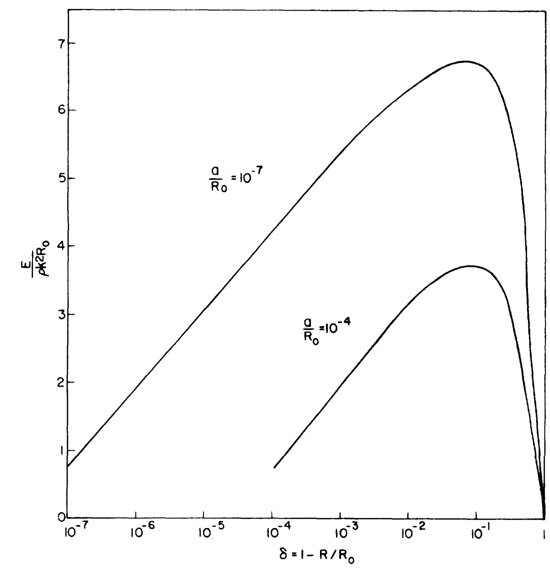

FIG. 4. Normalized energy of a vortex ring of radius R coaxially in front of an orifice of radius R_0, as a function of $\delta = (R_0 - R)/R_0$, with flux through the orifice $\Phi_0 = 0$, after Walraven, Ref. 11.

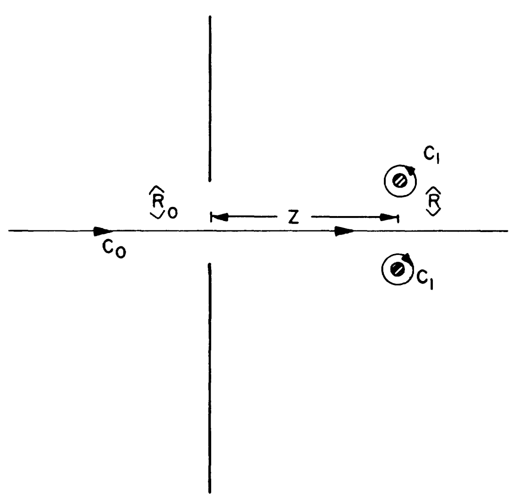

FIG. 3. Geometry of a vortex ring coaxially in front of an orifice in an infinite plane.

For completeness, we present Walraven's result for $E(R, Z)$ in the Appendix. In Fig. 5 we plot constant-energy contours in the R-Z plane based on this expression. Three distinct regimes are seen in Fig. 5. (i) $E > E_{max}$: for these, $R > R_c$ for all Z. (ii) $E < E_{max}$: for these, $R < R_c$ for all Z. This is the region of vortices which can pass through the orifice. (iii) $E < E_{max}$: for which $R > R_c$. This is the region in which vortices exist only near the plane. For a given value of E there is a maximum value $Z_{max}(E)$ and for each $Z < Z_{max}(E)$ there are two values of R. For the upper value, the drift velocity of the vortex is as given by the orientation of C_1 for a vortex in an unbounded fluid; for the smaller value it is in the opposite direction. This region may be called the region of trapped vortices. For every curve in the trapped region, there is one in the untrapped region (ii) of the same energy. The two regions meet at $R = R_c$, $Z = 0$ for $E = E_{max}$. The other limit of these regions is $E = 0$, for which $R \to \infty$ in region (iii) and $R \to 0$ in region (ii). Note that under the specified condition of constant total current, there is a maximum radius R_c for vortices to go through the orifice.

In the absence of the orifice, all loci have the shape of those in region (i) displaying the increase in the radii of vortex rings as they approach a sur-

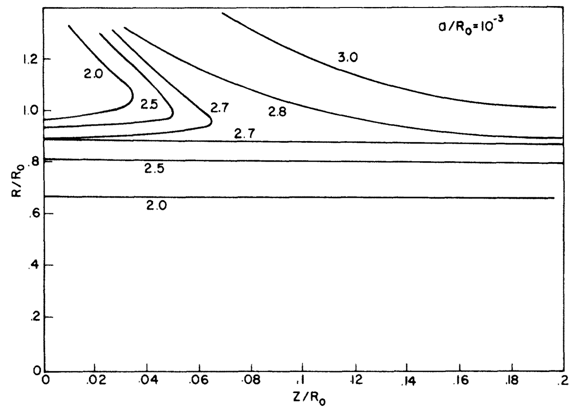

FIG. 5. Locus of constant-energy curves for a vortex ring in front of an orifice with the flux due to superfluid flow equal to zero. The lines are labeled by the normalized energy $E/\rho k^2 R_0$.

face due to the interaction with their image. In the orifice geometry, image effects are negligible for small enough vortices and they go through the orifice as in region (ii). Region (iii) occurs as intermediate between regions (i) and (ii).

B. Specified net circulation due to superfluid flow

1. Energetics

We now consider the case where the plane containing the orifice is placed in a closed channel of cross section much larger than the orifice and the circulation C_0 around the channel is specified. In the absence of a vortex, the flux through the orifice is

$$\Phi_0 = 2\rho R_0 C_0 , \tag{5.3}$$

so that Y defined by Eq. (3.2) is $2\rho/R_0$. We can calculate the energy using Eq. (3.2) noting that $\overline{Y}_v$ has already been given by Walraven.[11] The only new quantity to be calculated is the interaction function $f(R,Z)$, which is simply the fraction of the flux not going through the vortex ring. In the Appendix, we calculate $f(R,Z)$. We note here some simple limits of $f(R,Z)$. At $Z=0$,

$$f(R,0)=\begin{cases} 0 \quad \text{for } R>R_0 & (5.4\text{a}) \\ \left[\dfrac{(R_0^2-R^2)}{R_0^2}\right]^{1/2} \quad \text{for } R<R_0 . & (5.4\text{b}) \end{cases}$$

As $Z/R \gg 1$, $f(R,Z) \rightarrow 1$. From (5.4b) we observe that in the orifice geometry, the term $Yf^2C_1^2$ in Eq. (3.2) is very important for small vortices. Also, owing to the interaction with superfluid flow, the energy of smaller vortices is reduced.

In Fig. 6, we have plotted the energy vs the radius of the vortex for $C_0 = 0$ (linear scale this time) for several values of Z. The remarkable difference between this energy at $Z=0$ with constant circulation ($C_0=0$) and the energy with constant flow ($\Phi=0$), Fig. 4, is at once apparent. Only a weak maximum occurs now at $R/R_0 \simeq 0.4$. Between $R/R_0 \simeq 0.4$ and 1, V_Z of the vortex is in the opposite direction to that for other regions. Note that there is now both a minimum permissible energy and a maximum permissible energy to go through the orifice since as $R \rightarrow 0$, the normalized energy at $Z=0$ tends to 2π. The difference arises owing to the term $Yf^2C_1^2$ introduced to keep the circulation constant. The singularity at $Z=0$ near $R \simeq R_0$ is unaltered from the previous case.

In Fig. 7 we plot the constant-energy contours in the R-Z plane for $C_0=0$. Again there are three regions to be distinguished: (i) $2\pi\rho C_1^2 R_0 < E < E_{max}$: these have a radius between 0 and $\overline{R} \simeq 0.4 R_0$ and they go through the orifice traveling in the "normal" direction. (ii) $E > E_{max}$, $R > \overline{R}$: these are strongly affected by the image force, do not go through the orifice, and also travel normally. (iii) $E < E_{max}$: this is the region of the trapped vortices and has properties similar to the trapped region discussed earlier in the case of constant specified flux.

In Fig. 8 we plot the normalized energy as a function of R for various Z and for $C_0 = 2\pi$, and with the vortex oriented so that the interaction energy with the flow is attractive. Comparing with Fig. 6, we note that at $Z=0$, R_c has moved from 0.4 to about 0.9, by changing C_0 from 0 to 2π. Region (i) has therefore grown and the trapping region (iii) has shrunk.

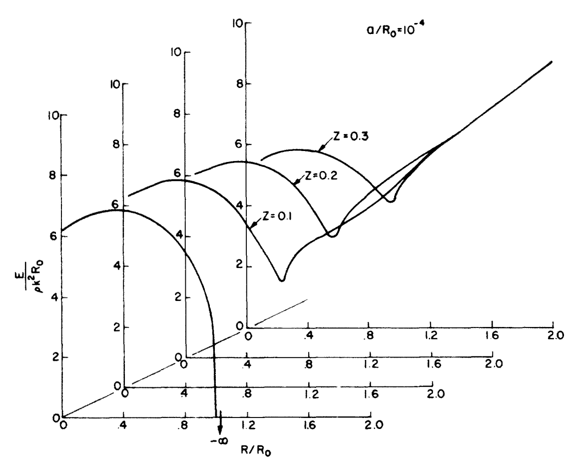

FIG. 6. Normalized energy of a vortex ring in front of an orifice as a function of its radius for various values of Z, the distance in front of the orifice. The circulation C_0 is specified to be zero.

2. *Condition for vortex generation*

Let us consider the condition (3.5) for vortex generation for a vortex ring close to the orifice at $T=0$. Let $\epsilon = z/R_0$. A straightforward expansion of Eqs. (A10) and (A9) for positive $\delta \ll 1$ and $\epsilon \ll 1$ yields

$$\tfrac{1}{2}Y_v C^2 \simeq \rho C_1^2 R_0 \left\{ \left(\frac{1}{2} - \frac{83}{128}\epsilon^2\right) \ln\frac{\delta}{\delta_0} - \frac{83}{128}\epsilon^2 \ln\frac{\delta_0}{2} + \frac{1}{32}\frac{\epsilon^2}{\delta^2 - \epsilon^2 + O(\epsilon^2\delta)} \right\}, \tag{5.5}$$

$$f^2 Y C_1^2 \simeq (2\pi\delta)\rho C_1^2 R_0 , \tag{5.6}$$

$$f C_1 \Phi_0 = \rho R_0 2^{1/2}[(\delta^2+\epsilon^2)^{1/2}+\delta]^{1/2} C_0 C_1 . \tag{5.7}$$

It is noteworthy that at $\epsilon = 0$, the interaction energy is proportional to $\delta^{1/2}$ (and $\epsilon^{1/2}$ at $\delta = 0$), whereas the interaction energy is proportional to δ for a vortex in a smooth tube. Thus sharp protuberances tend to favor generation of vortices by superfluid flow. This is due to the larger velocity of the fluid near such protuberances.

At $\epsilon = 0$, the energy condition (3.5) is satisfied for

$$\delta^{1/2} C_0 = \tfrac{1}{2}\pi\delta C_1 + \tfrac{1}{4}C_1 \ln(\delta/\delta_0) . \tag{5.8}$$

Because the interaction energy goes as $\delta^{1/2}$, for small δ the flow for vortex creation is very small indeed. However, if one looks at the trajectories in Fig. 8, it is at once apparent that for small flow these vortices are trapped at the walls and do not contribute to deceleration of the superfluid.

The condition for deceleration of the superfluid is that the vortices untrap and appear downstream. The trapping regime is characterized by the z component of the velocity of a vortex opposite to that of a vortex of the same sense of circulation far from any surface. We may put the condition for appearance downstream of a vortex nucleated at the orifice with positive circulation and of a radius nearly equal to that of the orifice as $V_z > 0$ for the entire trajectory of the vortex in the R-Z plane. This means that $dE/dR \gtrsim 0$ for all z, for vortex generation at constant circulation through the container cut. This statement is equivalent to the statement that condition (3.5) be satisfied at all times from the creation of a vortex to its eventual annihilation.

From the foregoing discussion of the energetics, one can easily conclude that the characteristic value of flux at which this condition is satisfied is

$$E((R_c, 0), \Phi) \lesssim 0 . \tag{5.9}$$

Equation (5.9) yields

$$(1 - R_c^2/R_0^2)^{1/2}\rho C_1 \Phi \gtrsim E_v(R_c) + (1 - R_c^2/R_0^2)\rho R_0 C_1^2 \tag{5.10}$$

for vortex generation.

At a finite temperature, the thermal activation energy E_c to generate vortices is reduced, owing to the flow, and is approximately given by

$$E_c(\Phi) \simeq E_v(R_c) + (1 - R_c^2/R_0^2)\rho R_0 C_1^2 - (1 - R_c^2/R_0^2)^{1/2}\rho C_1 \Phi . \tag{5.11}$$

For the rate of change of superfluid velocity in the present geometry one can again obtain Eq. (4.9). Comparing Eq. (5.10) and Eq. (4.5), and noting that $R_c/R_0 \simeq R_c/R_t \simeq 0.9$ and that $E_v(R_c)$ is about the same for the case of a smooth channel (see Fig. 2) and for the case of an orifice (see Fig. 4) (with $R_0 = R_t$), we can conclude that the flux required to generate vortices that contribute to superfluid deceleration in the orifice geometry is about half that in the smooth-channel geometry.

C. Specified net flux

As discussed in Secs. II and III, we must consider H instead of E in this case. We have not plotted diagrams analogous to Fig. 5 to Fig. 8 in this case, but we know that the diagrams will be analogous to the case of zero total flux, Fig. 5, rather than Fig. 6. This is because of the absence of the term $f^2 Y C_1^2$ in Eq. (3.4). The region of trapped vortices will, of course, be reduced, owing to flow, compared to Fig. 5, and now there is only a minimum radius to go through the orifice. From (4.8) and the discussion above, the

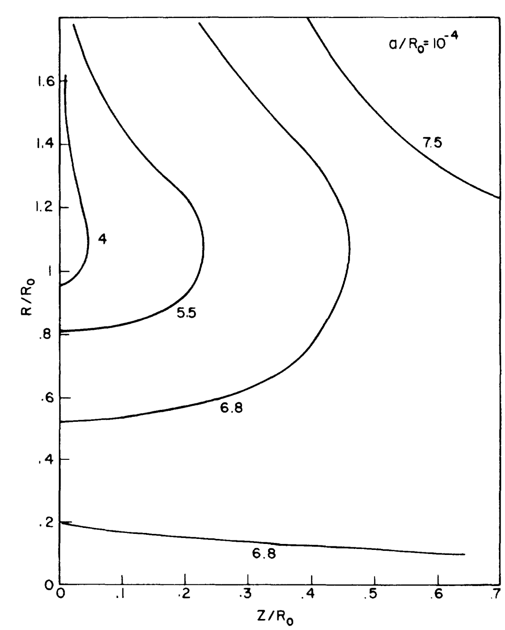

FIG. 7. Locus of constant-energy curves for a vortex ring in front of an orifice with $C_0 = 0$. The lines are labeled by the normalized energy $E/\rho k^2 R_0$.

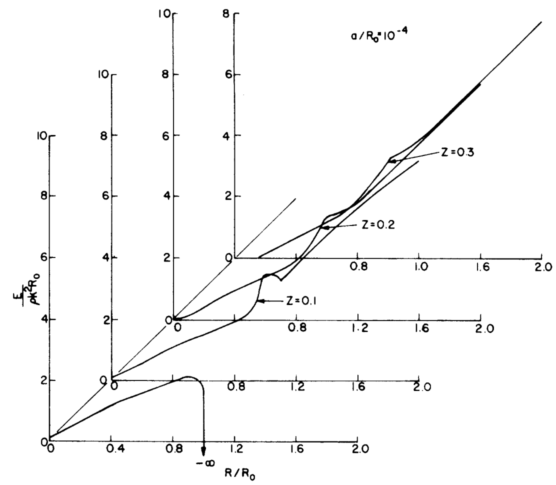

FIG. 8. Normalized energy of a vortex ring in front of an orifice as a function of its radius for various values of Z, the distance in front of the orifice. The circulation due to superfluid flow is kept constant, equal to $\hbar/m$. The (constant) energy due to the flow alone is not included.

characteristic condition for appearance downstream of a vortex nucleated at $T=0$ at the orifice and of a radius nearly equal to the orifice is that

$$H((R_c, 0), \Phi) \lesssim 0 . \tag{5.12}$$

This means that

$$(1 - R_c^2/R_0^2)^{1/2}\rho C_1 \Phi \gtrsim E_v(R_c) . \tag{5.13}$$

At finite temperatures thermal activation energy E_c to generate vortices is reduced owing to the flow,

$$E_c(\Phi) \simeq E_v(R_c) - (1 - R_c^2/R_0^2)^{1/2}\rho C_1 \Phi . \tag{5.14}$$

Characteristically (see Fig. 4), $R_c \simeq 0.9R_0$ and $E_v \simeq 3\rho R_0^2 C_1^2$ for $\delta \simeq 10^{-4}$. Comparing Eqs. (5.10) and (5.12), one finds that the flux required to generate vortices at constant circulation that cause superfluid deceleration is about 7% smaller for the case of constant total flux than for the case of constant circulation for this value of δ. Thus only a quite sensitive experiment can test this prediction.

VI. CONCLUDING REMARKS

The development in Secs. II and III is completely general, so that one can in principle study the process of vortex generation in more complicated geometries than those treated in Secs. IV and V. Here we only make some qualitative remarks for other geometries.

A straightforward question to ask is whether a steady generation of vortices aids or impedes the generation of the next vortex in, say, the channel geometry or the orifice geometry. For vortices nucleated near the walls, the velocity field of vortices downstream is in the same direction at the vortex cut as the vortex being generated. Thus the interaction energy of vortices of the same sense of circulation is repulsive and vortex production is impeded.

A stationary disk in a superfluid is the dual of the orifice geometry we have considered in Sec. V. The results for the critical velocity will be the same as obtained there, but the vortex will have a circulation opposite to that in the orifice geometry; it will grow in radius and travel in the direction opposite to superfluid flow. Qualitatively similar results will hold for a stationary sphere in a moving superfluid, but the interaction energy for a vortex nearly touching the sphere will be smaller than the disk geometry because the protuberance is not so sharp.

The case of a sphere of finite mass traveling in a stationary superfluid and decelerating due to vortex generation is interesting because it is realized experimentally in the motion of ions through helium. If the sphere is far away from any other surface, it is perfectly alright to use consideration of total impulse conservation of the sphere and the velocity field of the fluid. However, if the velocity of the sphere is specified (or, equivalently, the flux of superfluid), we must consider conservation of H, not E, to obtain another condition for vortex generation, as explained in Sec. II. This will modify somewhat the results of Schwarz and Jang,[16] who have looked at this problem from the point of view of energy and impulse conservation.

Finally let us consider the situation illustrated in Fig. 9(a), which shows the cross section of a channel at which there are two protuberances which are spanned by a vortex line with the direction of circulation shown and the circulation due to flux in the channel, C_0, pointed inwards. From the discussion in Sec. III the interaction energy is

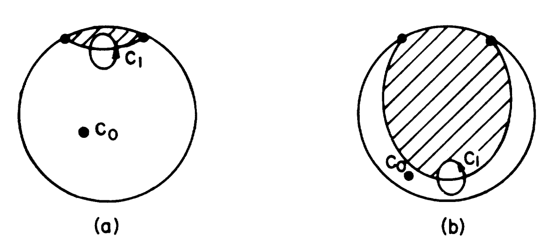

FIG. 9. A vortex line of shown circulation pinned to two points on the walls of the channel grows because of circulation due to superfluid flow C_0 pointed into the paper.

attractive and proportional to the hatched area. The vortex would therefore tend to grow as in Fig. 9(b) and ultimately annihilate at the walls. This process will contribute to the deceleration of the superfluid.

APPENDIX: INTERACTION ENERGY IN THE ORIFICE GEOMETRY

The interaction energy for the orifice geometry is from Eq. (3.2) for a vortex oriented as in Fig. 3, given by

$$E_{\text{int}} = -2\rho C_0 C_1 R_0 f(R, Z) , \tag{A1}$$

where $f(R, Z)$ is the fraction of the superfluid flux not going through the orifice. The superfluid velocity is given by

$$\nabla \cdot \vec{V} = 0$$

or, introducing $V = \nabla\psi$,

$$\nabla^2 \psi = 0 . \tag{A2}$$

The solution of the Laplace equation (A2) in the orifice geometry is usually given (see, for example, Ref. 8) in terms of oblate spheroidal coordinates (v, u, θ). The velocity is in the u direction and given by

$$V = (kC_0/R_0)[\cosh u(\sinh^2 u + \cos^2 v)^{1/2}]^{-1} ; \tag{A3a}$$

u, v, are defined in terms of cylindrical coordinates r, z through the axis of the orifice by

$$r = R_0 \cosh u \sin v, \quad z = R_0 \sinh u \cos v. \tag{A3b}$$

$f(R, Z)$ is given by

$$f(R, Z) = 2\pi \int_R^\infty dr\, r v(r, z). \tag{A4}$$

To express v in terms of r and z note that

$$\left.\begin{matrix} u \\ v \end{matrix}\right\} = \tfrac{1}{2}[\sinh^{-1}(z + ir) \pm \sinh^{-1}(z - ir)] . \tag{A5}$$

Using

$$\sinh^{-1}(z + ir) = (-1)^n \cosh^{-1}\tfrac{1}{2}(s + t) + i(-1)^n \sin^{-1}[2y/(s + t)] , \tag{A6}$$

where n is zero or an integer and

$$s + [(1 + r)^2 + z^2]^{1/2}, \quad t = [(1 - r)^2 + z^2]^{1/2} ,$$

we get

$$u = (-1)^n \cosh^{-1}\tfrac{1}{2}(s + t) , \tag{A7}$$

$$v = (-1)^n \sin^{-1}[2y/(s + t)] + in\pi . \tag{A8}$$

Using (A7) and (A8) in (A3) the integration in (A4) can be performed to yield

$$f(R', Z') = (\tfrac{1}{2})^{1/2}\{[(Z'^2 + R'^2 - 1)^2 + 4Z'^2]^{1/2} \pm (Z'^2 + R'^2 - 1)\}^{1/2} , \tag{A9}$$

where $R' = R/R_0$ and $Z' = Z/Z_0$, and the upper sign is to be chosen for $R' > 1$ and the lower for $R' < 1$. The choice of sign is determined by requiring that at $z = 0$, $f(R, 0) = 0$ for $R' > 1$, and $f(0, 0) = 1$.

The total energy of the vortex with zero flux through the orifice is equal to $\frac{1}{2} Y_v C_1^2$, [Eq. (3.1)]. Walraven's result[11] is reproduced here for completeness:

$$\frac{E}{\rho k^2 R_0} = \frac{R}{2R_0}\left[\ln\left(\frac{8R}{a}\right) - 2\right] + \tfrac{1}{2}\tan^2\left(\frac{\psi}{2}\right)[1 - (1 - q^2 \sin^2\psi)^{1/2}] + \tfrac{1}{8} q \tan\left(\frac{\psi}{2}\right) \ln\left|\frac{1 - q\sin\psi}{1 + q\sin\psi}\right| + \tfrac{1}{2}\tan\left(\frac{\psi}{2}\right)[E(\psi, q) - (1 - \tfrac{1}{2}q^2)F(\psi, q)] , \tag{A10}$$

where

$$q^2 = R^2/(R^2 + Z^2) , \tag{A11}$$

$$\cot\Psi = R/(R^2 + Z^2)^{1/2} , \tag{A12}$$

and $F(\psi, q)$ and $E(\psi, q)$ are the incomplete elliptic integrals of the first and second kind, respectively.

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