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Eigenvalue spectrum of interacting massive fermions in one dimension

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The excitation spectrum of massive fermions with self-interaction moving in one space dimension is calculated, using exact solutions for the spin-1/2 chain problem. This spectrum is shown to be the same as in the massive Thirring model, sine-Gordon equation, and Tomonaga-Luttinger with gap, with suitable relations between coupling constants. Solitons and bound solitons occur, corresponding to magnons and bound magnons in the spin chain. The soliton mass is calculated as a function of lattice spacing, and the ratio of binding energy to soliton mass is shown to be given correctly by the WKB approximation.

I. INTRODUCTION

This paper reports a calculation of the excitation spectrum of a model of massive fermions in one space dimension, the massive Thirring model.[1] It has been recognized that this model is equivalent to many others in one space and one time dimension, such as the sine-Gordon equation,[2] and the backward scattering model[3] familiar in solid-state physics. I discuss here the relation of these models to yet another, which is interesting because it has been solved.[4,5] This model, the spin-$\frac{1}{2}$ x-y-z chain, gives the eigenvalues for these problems on lattice. The treatment reported here applies directly to these other problems as well.

The continuum limit of the lattice theory can be taken, which requires renormalizations of the parameters in the theory. The spectrum contains massive particle states and bound states, and is found to be identical to the WKB approximate results given by Dashen, Hasslacher, and Neveu.[6] This provides confirmation that their result is indeed exact, and proves solitons to be ubiquitous.

It is now well known that the continuum theories are unbounded from below for certain values of coupling constant. I suggest that these instability problems can be understood and resolved with the lattice theory. The lattice theory for the massive Thirring model is the x-y-z spin-$\frac{1}{2}$ chain problem solved by Baxter[4] and by Johnson, Krinsky, and McCoy.[5] The instabilities of the continuum theory occur at special symmetry points of the spin problem. But, on the lattice, these points are simply crossover points to another ground-state symmetry. After observing this crossover, the continuum limit of the new theory can be taken. In this way, the Thirring model in an unstable region maps back onto a new massive Thirring model in a stable region. The new mass gap removes the instability of the old theory, and the field operators become redefined in the new continuum limit.

An important application of these results is in the area of statistical mechanics and solid-state physics. The results are cast in a form directly applicable to solve models[3] of the one-dimensional electron gas. In a subsequent paper, the details of this solution are presented. I believe that these areas will provide physical realization of these one-space-dimension models, and that measurement of the soliton and bound soliton spectrum can provide experimental confirmation of the ideas discussed here.

The appearance of equivalent models in these different fields of theoretical physics makes it imperative that some knowledge of the results from one field also penetrate into others. Until now, the same developments have occurred independently. I try to collect these models and relate them, and hope that this will be of some use to subsequent investigations, as well as to this one.

II. SPIN MODELS AND FIELD EQUATIONS

In this section, the relations between continuum and lattice field equations are discussed. The procedure I use is backwards, for rather than putting the continuum field theory on a lattice, the spin chain problem on the lattice is shown to have the continuum field theory as its continuum limit. In any case, there results a consistent prescription for introducing the proper theory on a lattice. In a subsequent section, this equivalence is exploited to discuss the questions concerning stability of these field theories.

The starting point is the Hamiltonian for the lattice problem, the spin-$\frac{1}{2}$ x-y-z chain:

$$\mathcal{H}_s = -\sum_{i,\alpha} J_\alpha S_i^\alpha S_{i+1}^\alpha , \qquad (1)$$

where $\alpha = x, y,$ or z, S_i^α is a spin-$\frac{1}{2}$ operator, and the sum runs over the N sites on a chain. Using the Jordan-Wigner transformation to fermion operators, $S_i^+ = a_i^\dagger \exp i\pi \sum_0^{i-1} n_j$, etc., where $S_i^+ = S_i^x + iS_i^y$, a_i is a Fermi operator, and $n_j = a_j^\dagger a_j$, this becomes a simple Fermi Hamiltonian. Under the further transformation $a_n \to (i)^n a_n$, the result is

$$\mathcal{H} = \sum_i \mathcal{H}(i)\,, \tag{2}$$

$$\mathcal{H}(i) = -\tfrac{1}{2} iv(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + \tfrac{1}{2} iJ_\perp (a_i^\dagger a_{i+1}^\dagger + a_i a_{i+1})(-)^i + J_z a_i^\dagger a_i a_{i+1}^\dagger a_{i+1}\,,$$

where $2v = J_x + J_y$, $2J_\perp = J_x - J_y$, and terms linear in n_i have been dropped, since the ground state has zero magnetization. Together with translational invariance, this requires the ground state average of n_i to be $\frac{1}{2}$, and "new" operators $n_i = n_i - \frac{1}{2}$ can be used. This replacement is correct to all orders in $J_\perp$. In many-body theory, this corresponds to choosing the chemical potential as the single-particle reference energy.

The definition of the continuum limit used here requires converting the chain of N discrete fermion states, one at each site, to a string of length L involving an infinite number of fermion states. That limit is best understood by considering the Fourier-series transformation for operators at a lattice site. As the lattice constant $s = L/N$ tends to zero for fixed L, this discrete sum becomes an integral, with cutoff. Calculations are to be performed with these equations of motion and the limit $s \to 0$ taken afterwards, which defines our cutoff prescription. There are renormalizations of the parameters in the lattice theory which follow from the requirement that the observable quantities be finite in this limit, $s \to 0$.

It is helpful to define the Fourier transform of two fields $\psi_U(k)$ and $\psi_L(k)$ by the relations

$$\psi_U(k) = (2/L)^{1/2} \sum_n a_n e^{ikns}\,, \qquad \psi_L(k) = (2/L)^{1/2} \sum_n a_{n+1} e^{ik(n+1)s}\,, \tag{3}$$

and use the density operators, $\rho_U = a_n^\dagger a_n s^{-1}$ and $\rho_L = a_{n+1}^\dagger a_{n+1} s^{-1}$, where n is an even integer which is summed over the $\frac{1}{2}N$ even sites. The k vectors lie within the Brillouin zone $-\pi/2s < k < \pi/2s$. Written out by components, ψ_U and ψ_L satisfy the field equations

$$\begin{aligned} i\dot\psi_U(k) &= v \sin ks\, \psi_L(k) - iJ_\perp \cos ks\, \psi_L^\dagger(-k) - 4J_Z N^{-1} \sum_{k'} \cos k's\, \psi_U(k-k')\rho_L(k')\,, \\ i\dot\psi_L(k) &= v \sin ks\, \psi_U(k) + iJ_\perp \cos ks\, \psi_L^\dagger(-k) - 4J_Z N^{-1} \sum_{k'} \cos k's\, \psi_L(k-k')\rho_U(k')\,. \end{aligned} \tag{4}$$

In order to construct the continuum limit of these equations, we note that the prescription for determining the Hamiltonian density $\mathcal{H}_c(x)$, from the lattice Hamiltonian $\mathcal{H}_s(i)$, is given by

$$\sum_{i=1}^{N} \mathcal{H}_s(i) \to \int_0^L dx\, \mathcal{H}_c(x)\,, \tag{5}$$

where $x = si$ is a fixed distance in the continuum theory, and $\mathcal{H}_c(x) \to s^{-1}\mathcal{H}(i)$. As $s \to 0$, $i \to \infty$ such that this distance $(is) \to x$ remains fixed in the ratio $i/N = x/L$. The field equations then become

$$\begin{aligned} i\dot\psi_L(k) &= vk\psi_U(k) - im_0\psi_U^\dagger(-k) - 4J_Z L^{-1} \sum_{k'} \psi_U(k-k')\rho_L(k')\,, \\ i\dot\psi_U(k) &= vk\psi_L(k) + im_0\psi_L^\dagger(-k) - 4J_Z L^{-1} \sum_{k'} \psi_L(k-k')\rho_U(k')\,, \end{aligned} \tag{6}$$

where $m_0 = J_\perp/s$, and these equations are understood to be supplemented by a cutoff at s^{-1} arising from the inherent restriction of k to the first Brillouin zone. These are recognized as similar to the field equations for the massive Thirring model, except for the $\psi^\dagger$ appearing instead of the usual ψ. We return to this difference below. The solution of the spin-$\frac{1}{2}$ x-y-z model on a lattice therefore provides a solution to this cutoff field theory.

Relation of this to the usual massive Thirring model can be established by a formal mass perturbation expansion, as given by Coleman. The comparison is best accomplished by expanding the partition function, $Z = \mathrm{Tr}\, e^{-\beta\mathcal{H}}$. This expansion will involve n-point functions of the form

$$\prod_{i,j}^{n} \langle 0 | \psi_L^\dagger(x_i)\psi_U^\dagger(x_i)\psi_L(x_j)\psi_U(x_j) | 0 \rangle$$

with $x_i = (x, t)$; for it the temperature β is assumed large, and $\langle 0|$ is the massless vacuum. I now repeat Coleman's argument[2] that equality of the n-point functions, which occur in expansions of Z, with those of the massive Thirring model proves that the theories have the same eigenvalue spectrum. The form of n-point functions is found using standard methods. First, define $\sqrt{2}\psi_1(k) = \psi_L(k) + \psi_U(k)$ and $\sqrt{2}\psi_2(k) = \psi_U(k) - \psi_L(k)$. The field equations become

$$\begin{aligned} i\dot\psi_1(k) &= v_0 k\, \psi_1(k) - im_0\psi_2^\dagger(-k) - 4J_z L^{-1} \sum_{k'} \psi_1(k-k')\rho_2(k')\,, \\ i\dot\psi_2(k) &= -v_0 k\psi_2(k) + im_0\psi_1^\dagger(-k) - 4J_Z L^{-1} \sum_{k'} \psi_2(k-k')\rho_1(k')\,. \end{aligned} \tag{7}$$

Here $v_0 = v - (2\pi)^{-1}J_Z$, using a known result that interactions of the form $\rho_1\rho_1$ only renormalize the velocity.[7] The n-point functions in the mass perturbation theory are evaluated using methods developed[8] in solving the Luttinger model:

$$\prod_{j,i=1}^{n} \langle 0 | T\psi_1^\dagger(x_i)\psi_2^\dagger(x_i)\psi_2(x_j)\psi_1(x_j) | 0\rangle = \exp\left(2\theta \sum_{i<j}^{n} (-)^{i-j} \ln[(x_i - x_j)^2/s^2]\right), \quad (8)$$

where $2\theta = (\pi v_0 - 2J_Z)^{1/2}(\pi v_0 + 2J_Z)^{-1/2}$, and v is chosen such that the renormalized velocity[9] $v_0[\theta + (4\theta)^{-1}]^{-1} = 1$. This is identical to the result found for the massive Thirring model,[2]

$$\prod_{i=1}^{n} \langle 0 | T\sigma^+(x_i)\sigma^-(y_i) | 0\rangle = \frac{\Pi_{i<j}[(x_i - x_j)^2(y_i - y_j)^2 M^4]^{\beta^2/4\pi}}{\Pi_{i,j}[(x_i - y_j)^2 M^2]^{\beta^2/4\pi}}, \quad (9)$$

provided we identify $\beta^2 = 8\pi\theta$.

It is also possible to perform a direct canonical transformation on the fields to arrive at the usual massive theory. The transformation $\psi_2(k) \to \psi_2^\dagger(k)$ leads to the new field equations, for $ks \ll 1$,

$$\begin{aligned} i\dot{\psi}_1 &= k\psi_1 - im_0\psi_2 + 4J_Z L^{-1}\sum_{k'} \psi_1(k-k')\rho_2(k') \\ i\dot{\psi}_2 &= -k\psi_2 + im_0\psi_1 + 4J_Z L^{-1}\sum_{k'} \psi_2(k-k')\rho_1(k'). \end{aligned} \quad (10)$$

These equations provide the basis for calculating the excitation spectrum of these various equivalent problems, for they all relate to the basic x-y-z spin chain in the continuum limit. The calculation of the excitation spectrum for this model follows.

III. SOLUTION OF LATTICE PROBLEM AND THE CONTINUUM LIMIT

The excitation spectrum of the spin chain, Eq. (1), has been given by Baxter[4] and by Johnson, Krinsky, and McCoy.[5] The pertinent results are summarized by the "free"-state solutions at zero momentum

$$\Delta = 2J_x \,\mathrm{sn}(2\xi, l)[K(k_1)k_1'/K(l')], \quad (11)$$

and the "bound" states, to be identified with bound solitons,

$$\Delta_n = \Delta\,\mathrm{sn}(y, k_1')\,\mathrm{sn}(2K(l) - 2\xi, l)[\mathrm{sn}(2\xi, l)]^{-1}, \quad (12)$$

where $\mathrm{sn}(a, b)$ is the Jacobi elliptic function. The quantities ξ and l are defined by

$$\begin{aligned} l^2 &= (J_x^2 - J_y^2)(J_x^2 - J_Z^2)^{-1}, \\ 2\xi &= \int_0^c dx(1 - l^2\sin^2 x)^{-1/2}, \end{aligned} \quad (13)$$

with $c = \arccos(-J_Z/J_x)$. The function $K(l)$ is given by

$$K(l) = \int_0^{\pi/2} dx(1 - l^2\sin^2 x)^{-1/2} \quad (14)$$

and $l' = (1 - l^2)^{1/2}$. The quantity k_1 is to be evaluated by solving the equation

$$K(k'_1)K(l') = \xi K(k_1), \quad (15)$$

with $k_1' = (1 - k_1^2)^{1/2}$, and finally the object y is defined by

$$yK(l') = nK(k_1)[K(l) - \xi]. \quad (16)$$

It is helpful to introduce the variable μ by $\mu K(l) = \pi\xi$ such that y can be written as

$$y = n(\pi/\mu - 1)K(k_1'), \quad (17)$$

and any value of the positive integer n is a solution provided that the left-hand side of Eq. (17) is less than $\frac{1}{2}\pi$. This variable μ will shortly be related to the exponent θ of a certain correlation function. Solutions to Eq. (17) for $n \neq 0$ correspond to the soliton bound states.

In the limit of weak anisotropy, $J_x \approx J_y$ or $l \approx 0$, these equations simplify, for then $k_1' \approx 0$. The ratio of the bound-state energy to the free-state solution, A_n, is then given by

$$A_n = \sin\left[\frac{n\pi}{2}\left(\frac{\pi}{\mu} - 1\right)\right] = \sin\left[\frac{n\pi}{2}\left(\frac{\theta}{1-\theta}\right)\right]. \quad (18)$$

Provided the x-y anisotropy remains small, $l \approx 0$, the quantity μ is related to the exponent of a transverse spin correlation function, θ, in the spin chain. This relation follows from the definition above Eq. (17), $\mu = \arccos(-J_Z/J_x)$, and the result from the appendix for the exponent $2\theta = 1 + (2/\pi)\arcsin(-J_z/J_x)$, giving $\mu = \pi(1 - \theta)$, which I have used in Eq. (18). This equation for A_n, expressed as a function of exponents only, should not depend on convention, such as continuum or lattice cutoff. This is discussed in the appendix.

The constraint that y be less than $\frac{1}{2}\pi$ in Eq. (17) provides a limit on the largest integer n for a given exponent θ. This exponent is a function which decreases from $\frac{1}{2}$ to 0 as the interaction is made ferromagnetic, $J_z > 0$, while it increases towards 1 as the antiferromagnetic point is approached, $J_Z \to -1$. The allowed values of n are

$$n = 0, 1, 2, \ldots, \theta^{-1} - 1, \quad (19)$$

and bound-state solutions occur whenever $\theta \geq \frac{1}{2}$, which is the ferromagnetic region. The $n=0$ solution always occurs, that is, the free particle (free in sense of unbound) solution is stable.

In the small-l limit, $\mathrm{sn}(2\xi, l) \to \sin 2\xi$ and $\mu = 2\xi$. Using Eq. (15) to solve for k_1' in terms of ξ gives

$$k_1' = 4(l/4)^{\pi/\mu}, \tag{20}$$

where we have used the small-q expansion for $K(q') \approx \ln(4/q)$, with $q' = (1-q^2)^{1/2}$. Substituting into Eq. (11), using $l \approx 0$ and $\mu = 2\xi$, leads to the result

$$\Delta = 8\pi\left(\frac{\sin\mu}{\mu}\right) |J_x| \left(\frac{l}{4}\right)^{\pi/\mu} \tag{21}$$

for the lattice theory, $\mathcal{H}_s(i)$. In the continuum theory, $\mathcal{H}_c(x) = s^{-1}\mathcal{H}_s(i)$, the bare mass is given by $J_\perp s^{-1} \sim l^2 s^{-1}$ for small $J_\perp$. Changing the bare-mass scale from l^2 to $(l')^2$ changes the renormalized-mass scale by $(l'/l)^{\pi/\mu}$.

It is now possible to take the continuum limit and justify the above assumption of small anisotropy, $l \approx 0$. The convention for this limit was discussed with Eq. (5). The ground state of the chain problem is of the form $E_G = \Phi_G N$, for large N, where Φ_G is independent of N (or s). As $s \to 0$, this becomes $E_G = (\Phi_G S^{-1})L$, and diverges. Excitation energies above the ground state are finite as $N \to \infty$, and it is these we wish to calculate in the continuum-limit theory. Consequently, we subtract this infinity from the problem, which is trivial since Φ_G is known for the lattice model.[4] We then keep the length of the string L finite while performing the calculations, and finally take the $L \to \infty$ limit last.

In this limit, the J_x coupling constant in Eq. (21) becomes replaced by $J_x s^{-1}$, and Δ would diverge unless the bare coupling constant, the anisotropy parameter l, is renormalized to zero. That requires $l \to l_r s^{\mu/\pi}$, where l_r is the renormalized coupling constant which is finite as $s \to 0$. This shows that $l \to 0$, provided $\mu > 0$, in the continuum limit, as assumed in the derivation of Eq. (21).

The other coupling constant in the thoery, J_z, enters through the exponent μ, or $\theta = 1 - \mu/\pi$. In the appendix, this exponent, θ, is shown to be identical to the parameter $\beta^2/8\pi$ of the sine-Gordon theory, which in turn is related to the usual Thirring model[2] through $\beta^2 = 4\pi(1 + g/\pi)^{-1}$.

Substituting these relations into the formula for the bound-state ratios A_n, Eq. (18), gives the result

$$A_n = \sin(n\gamma'/16), \tag{22}$$

where $\gamma' = \beta^2(1 - \beta^2/8\pi)^{-1}$. This is the WKB expression[6] for A_n, and the derivation of this result from the continuum limit of the lattice theory verifies that the WKB approximation is exact for this quantity.

Together with the formula relating the observed mass to the bare mass,

$$\Delta = 8\pi\left(\frac{\sin\mu}{\mu}\right)\left(\frac{l_r}{4}\right)^{\pi/\mu}, \tag{23}$$

where we have taken $J_x = 1$ as the energy unit, and $l_r = l s^{-\mu/\pi}$. The result of Eq. (22) completes the determination of the mass gaps in the excitation spectrum. Since the theory is Lorentz invariant in the limit $s \to 0$, it follows directly that the momentum dependence of these excitations is given by

$$\Delta_n^2(k) = \Delta_n^2 + k^2, \tag{24}$$

which completely determines the excitation spectrum.

The result here does not give any direct information about the wave functions which would be needed to prove the theory has interesting scattering states. It is possible to construct these scattering states from the corresponding states in the lattice theory, and perform the continuum limit. I do not yet know if this produces a nontrivial result.

IV. INSTABILITIES OF THE CONTINUUM THEORY

As shown by Coleman,[2] the ground-state energy in the massive Thirring model is unbounded from below for the coupling constant $\beta^2 > 8\pi$, in addition to the instability occurring for $\beta^2 < 0$. Similar stability restrictions are known for the Luttinger model, when the parameters are defined as in the appendix. I propose a method in this section for understanding the meaning of this instabilities and circumventing them.

It is suggestive to consider the relation between exponents in the continuum theory and the lattice theory, discussed above and in the appendix. This relation identifies the exponent $\beta^2/8\pi$ of the massless Thirring model theory with the exponent $\theta = \pi^{-1}\arccos(J_z/J_x)$ in the isotropic ($J_x = J_y$, or $l = 0$) spin chain theory. The point $\beta^2 = 0$ is seen to correspond to $J_z = J_x$ ($= J_y$), that is, the ferromagnetic point, while $\beta^2 = 8\pi$ is the antiferromagnetic point, $J_z = -J_x$. (To see this, reflect every other spin about the $x-y$ plane.) It is interesting to note that $\beta^2 = 4\pi$ is the isotropic $x-y$ model, $J_z = 0$, which is known to be a free-field theory.

This correspondence is made precise in the appendix, where the equivalence of the massive theories to the spin chain problem is proven to all orders in perturbation theory. It suggests a resolution of the instability question, by studying the behavior of the corresponding spin chain theory at the special values of the coupling constants. It turns out that the spin chain has a crossover from one type of ground state to another as $|J_z|$ goes through J_x. A simple transformation reduces the $|J_z|> J_x$ problem to a new problem with $|J_z'|< J_x$, and it is then possible to take the continuum limit of this new theory according to the prescriptions of Sec. III.

Consider the spin chain problem of Eq. (1),

$$\mathcal{H}_s = -\sum_{i,\alpha} J_\alpha S_i^\alpha S_{i+1}^\alpha \,, \tag{1}$$

for the anisotropic case $J_x \gtrsim J_y$ as in Sec. III, but now with $J_z > J_x$ as well. A rotation about the y axis interchanges the x and z spin operators. Follow this by a rotation about the z axis to arrive at

$$\mathcal{H}_s' = -\sum_i (J_z S_i'^x S_{i+1}'^x + J_x S_i'^y S_{i+1}'^y + J_y S_i'^z S_{i+1}'^z) \,. \tag{25}$$

The coupling constants are now in the order $J_x' > J_y' > J_z'$, as in Sec. III, where primes denote transformed quantities. This new problem has a mass term, $J_\perp' \sim J_z - J_x$, and the exponent θ' is in the stability region $0 \leq \theta' \leq 1$. The theory on the lattice is, of course, always well defined. These transformations simply reduce the problem to an equivalent form whose continuum limit has been discussed.

Passing to the continuum limit, leads to the renormalization of the bare mass, $l \sim J_z - J_x \rightarrow 0$, while the J_z' coupling constant, which equals J_y, is unrenormalized. The results of Sec. III can be applied to solve for the eigenvalues of this transformed problem.

From this perspective, we are able to discuss the massive Thirring model as the coupling constant β^2 increases through the instability at $\beta^2 = 8\pi$. Clearly, on the lattice, no instability is ever encountered since the ground-state energy is always finite. Infinities are only possible in the continuum limit. In the region $0 < \beta^2 < 8\pi$ this limit can be taken, since the eigenvalue spectrum as calculated in Sec. II is well defined.

At the point $\beta^2 = 8\pi$, $\theta = 1$, the soliton gap just does vanish, and although not explicitly discussed here, the excitation spectrum has massless particles corresponding to the spin-wave excitations in the Heisenberg antiferromagnet, and a continuum limit is possible.

For $\beta^2 > 8\pi$, the continuum limit cannot be taken, and the theory must be put on the lattice. Identify the coupling constant β^2 with the J_z spin coupling constant, the bare mass with $J_\perp$, and perform the rotations in the spin space to arrive at a new theory in the stable region, with $\theta' < 1$, and $\beta'^2 < 8\pi$, where the primes refer to transformed quantities. The new continuum limit is now permissible because this problem is of the type discussed in Sec. II.

It is interesting to observe the differences in coupling constant renormalization required by this procedure. For the original bare coupling constant $\beta^2 < 8\pi$, the bare mass is renormalized to zero as $s \rightarrow 0$, but β^2 (J_z on the lattice) is not renormalized. Solving for $\beta^2(J_z)$, we can define the theory for $\beta^2 > 8\pi$, where this statement means the bare coupling constant $J_z > J_x$. After performing the rotations to transform the equations into the form suitable for the continuum limit, we see that the new coupling constant J_z' is not renormalized and the new mass term, corresponding to $J_\perp'$, is. This is an interchange of the situation for $\beta^2 < 8\pi$, for the new J_z' was the old bare mass, and the new $J_\perp'$ was the old bare coupling constant. In order to make sense of these transformations, it is obviously necessary to know the relations between bare and renormalized parameters.

It is probably worth noting that this procedure to go around $\beta^2 = 8\pi$ is not necessarily unique, because it is not possible to circumnavigate this point knowing only the continuum-limit theory. If there were several lattice theories with this continuum limit, it might be possible to define another continuation. I have been able to construct one other such theory, which however gives an equivalent continuation, and have the opinion that it should be possible to prove uniqueness.

V. DISCUSSION

Construction and solution of the massive field theory on a lattice and a discussion of the continuum limit is helpful for understanding many problems in solid-state physics as well as model field theories. An example is the one-dimensional electron gas, a problem treated by many workers. The field equations of Sec. II are also encountered in that problem, and the eigenvalue spectrum can be calculated using these methods. This will be discussed in a subsequent paper.

An interesting application of this work is to more-complicated problems of field theory, such as models of particles with internal degrees of freedom. The electron-gas problem is one such example, for the electron spin constitutes an internal SU(2) symmetry. As mentioned above, the massive-field-theory model appears in this problem, and, consequently, that solution is a solution to the SU(2) Thirring model. It is now interesting to consider application of the methods used to solve the SU(2) problem to the more interesting SU(N) models.

The application of these SU(N) continuum field theories to statistical mechanics is a further direction worth pursuing. Many authors have studied the relationship between the Ising model in two dimensions, and the free-field theory, and the relationship between the Baxter model and the interacting-field theory has also been discussed.[9] The field theories with internal symmetries correspond to more-complicated critical phenomena problems in two dimensions, such as the $x-y$ or Heisenberg models. I believe solution of the SU(N) field theory also solves the N-component theory of statistical mechanics. A solution of the $N=2$ problem using these methods has already been constructed.[10]

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APPENDIX: SCALING, CONVENTIONS, AND RELATIONS BETWEEN MODELS IN ONE SPACE DIMENSION

It is important to discuss further the nature of the continuum limit, for there is obvious concern about the meaning of continuum-field equations. Some of the questions are simple matters of definition. I will show here that the results for the exponents of correlation functions, e.g., θ of Sec. IV, depend on the definition of the limit. There are, however, unique field equations, and the specification of limit procedure corresponds to selecting a convention for the coupling constant.

There is clearly a mathematical ambiquity in the continuum-limit field equations, but there is some physical content which is not. That physics takes the form of "scaling laws," familiar from statistical-mechanics problems, and involves relations between the exponent of different correlation functions. These relations are argued to be independent of coupling constant definition.

First, consider the exponent θ, which has a simple meaning. It is the anomalous dimensions of the single fermion operator appearing in those n-point correlation functions written down by Coleman. In the continuum limit, with his convention, $\theta=\beta^2/8\pi$. The exponent π/μ relates the gap in the excitation spectrum Δ to the bare gap, or mass, or $J_\perp$ term, $\Delta\sim J_\perp^{\pi/\mu}$. It is conventional in the phase-transition problem to relate Δ to the temperature scale, using $J_\perp\sim(\delta T)^2$, where δT is the temperature away from the critical point. This defines the exponent ν, $\Delta\sim(\delta T)^\nu$, where $2\nu\equiv\pi/\mu$. I will argue that $2\nu=1-\theta$ is the additional "scaling law" which relates these two exponents, and is independent of convention.

For the spin chain problem, the equivalent of the n-point functions have not yet been calculated, and it is necessary to derive θ indirectly from other results. I show below this also gives $2\nu=(1-\theta)^{-1}$. The relation $2\nu=(1-\theta)^{-1}$ has already been derived for the "massive" Tomonaga-Luttinger model,[8,9] which corresponds to a continuum limit of the spin chain problem.[9] That derivation, and the following derivation for the massive Thirring model, require a homogeneity assumption about a certain vacuum expectation value, which is proven to all orders in the mass perturbation theory. This concludes the listing of results needed to understand the different conventions for the continuum limit of the spin chain, Thirring, and Tomonaga-Luttinger models. The results are summarized, in order, by the equations

$$\mu=\arccos(-J_z/J_x)\ ,\quad \theta=\beta^2(8\pi)^{-1}\ ,\quad \theta=(1+V)^{1/2}(1-V)^{-1/2}(2)^{-1}\ , \tag{26}$$

which together with $\mu=\pi(1-\theta)$, enable use of the mass-gap formula with other coupling-constant conventions.

The calculations of θ for the spin chain involves Baxter's result for the singular part of the ground-state energy, $E_G\sim J_\perp^{1-\alpha/2}$, which is stated in the notation of the Baxter-model phase-transition problem with $\alpha=2-\pi/\mu$. Recognizing $J_\perp$ as $(\delta T)^2$ in that model, identifies the $J_\perp$ term in the spin chain problem as the energy density operator in the Baxter model. The energy-density–energy-density correlation function $\langle\delta\epsilon(x)\delta\epsilon\rangle$ is then the mass-mass correlation function. Since $\Delta\sim J_\perp^{\nu/2}$, which relates length scale to temperature scale, the dimension of the energy-density correlation function at $\delta T=0$ can be found from the usual thermodynamic arguments to be $\langle\delta\epsilon(x)\delta(\epsilon)\rangle\sim x^{2-\alpha/\nu}$.

Examination of the equivalent mass term in the fermion problem defines the exponent θ in the lattice theory,

$$\langle\delta\epsilon(x)\delta(\epsilon)\rangle\sim\langle\psi_1^\dagger(x)\psi_2(x)\psi_2^\dagger\psi_1\rangle\equiv x^{-4\theta}\ , \tag{27}$$

and, therefore, $4\theta=2-\alpha/\nu$, which gives $\theta=1-\mu/\pi$, since α and ν are known.[4,5]

It is not really necessary to construct the n-point functions for the continuum limit of the spin chain problem, because the form of the continuum equations of motion together with cutoff prescription, are clearly of the form considered previously.[2] The only departure from published results concerns the appearance of $\psi_1^\dagger\psi_2^\dagger$ as the mass (or $J_\perp$) term, rather than the usual $\psi_1^\dagger\psi_2$ of the massive Thirring problem. Within the framework of the cutoff or lattice theory, this poses no problem for our proof, for we observe that the perturbation expansion of the partition function for $e^{-\beta\mathcal{H}}$ in the mass term is, to all orders, the same for both, provided the sign of the $\rho_1\rho_2$ term is reversed. The solution of one problem therefore gives the solution of the other.

This proof is constructed as follows: Consider the expansion of the partition function in powers of $J_\perp$. Involved in this are products such as

$$T\langle \psi_1^\dagger(x_1)\psi_2^\dagger(x_2)\cdots\psi_1(x_n)\psi_2(x_n)\rangle ,$$

where x_i refers to a space-time (complex-temperature) point, and the average is in the density matrix of the massless theory. Only even n contribute to this average, so that we always remain in the same charge sector when averaging, since charge is conserved in the massless theory. Under the average sign, therefore, $\psi_2 \to \psi_2^\dagger$ changes only the coefficient of $\rho_1\rho_2$ in the Hamiltonian, that is, the sign of J_z in the Hamiltonian of Eq. (7). Thus, in turn, only affects the n-point function through a change in sign of coupling constant. We conclude that the n-point functions in the mass perturbation expansion with the $\psi_1^\dagger\psi_2^\dagger$-type mass terms are identical to the $\psi_1^\dagger\psi_2$ type, provided the sign of the coupling constant is reversed. By definition, this exponent is the $\beta^2/8\pi$ which appears in the mass perturbation theory.[2] Together with Eq. (27), this justifies the result $\theta=\beta^2/8\pi$ used in Secs. III and IV.

There now remains the proof of the scaling ansatz for the continuum theory. This has previously been discussed for the Tomonaga-Luttinger model.[9] Consider the calculation of the correlation function $\langle \psi_1^\dagger(x)\psi_2(x)\psi_2^\dagger\psi_1\rangle_M$ in the massive theory, denoted by the subscript M. The formal perturbation expansion for this quantity involves n-point functions of the type studied by Coleman[2]:

$$T\langle \psi_1^\dagger(x)\psi_2(x)\psi_2^\dagger(x_n)\psi_1(x_n)\cdots\psi_2^\dagger\psi_1\rangle \tag{28}$$

where a cutoff mass equal to the lattice constant must be introduced. Collecting the powers of x, and integrating over $\Pi_n\int d^2x_n$, leads to the statement that the mass perturbation series has the form

$$\langle \psi_1^\dagger(x)\psi_2(x)\psi_2^\dagger\psi_1\rangle = x^{-4\theta}\sum_{n=0}^{\infty} A_n m_0^n (x^{2n})^{1-\theta} , \tag{29}$$

where $\theta=\beta^2/8\pi$, and A_n are coefficients arising from the multiple integrations in the perturbation series. This is a purely formal result, whose only virtue is the dimensional statement that, under scale change $x\to\lambda x$, $m_0\to m_0\lambda^{(2-2\theta)^{-1}}$. This is sufficient to identify $m=m_0^\nu$ with $2\nu=(1-\theta)^{-1}$. Of course this does not prove a gap exists. For that, the solution for the lattice in the continuum limit must be invoked. But this does show that, if a gap exists, it satisfies the same dimensional scaling law as in the spin chain problem.

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